An Introduction to Holomorphic Dynamics

II. Properties of Julia and Fatou sets

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This handout is created from the overhead slides used during lectures. Examples and proofs will be done on the board, and are not included.

II.1 Exceptional values

Exceptional values

II.1.1 Definition (Exceptional value). A value $z_0 \in \hat{\mathbb{C}}$ is called *(Fatou) exceptional* if the backward orbit

$$O^-(z_0) = \{w \in X : \exists n \geq 0, f^n(w) = z_0\}$$

is a *finite* set.

Example 1.

- $f(z) = z^2$; $z_0 = 0$.
- $f(z) = \exp(z)$; $z_0 = 0$.

II.1.2 Lemma (Number of exceptional points)**.** *f has at most two exceptional points in $\hat{\mathbb{C}}$.*

Exceptional values

Remark. For rational functions of degree at least two, exceptional values are always in the *Fatou set*.

- A rational map with *one* exceptional value is conjugate to a *polynomial*.
- A rational map with *two* exceptional values is conjugate to $z \mapsto z^m$, $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

Density of backward orbits

We can now reformulate a property of the Julia set which we mentioned already in the previous lecture:

II.1.3 Lemma (Backward orbits)**.** *If z_0 is not a Fatou exceptional value, then*

$$J(f) \subset \overline{O^{-}(z_0)}.$$

If furthermore $z_0 \in J(f)$, then

$$J(f) = \overline{O^{-}(z_0)}.$$

II.2 The Bloch principle

II.2.1 The Bloch principle

Liouville's Theorem

Recall that *Liouville's Theorem* states that a bounded entire function must be constant.

Compare this with the *Removable Singularities Theorem*, which says that an isolated singularity of a bounded holomorphic function is removable.

Also recall that any family of bounded entire functions, with a uniform bound, is normal.

Theorems of Montel and Picard

II.2.1 Theorem (Picard)**.** *Suppose f is meromorphic on a domain U, except at an isolated singularity $z_0 \in U$.*

If f omits three values in the Riemann sphere (e.g., f never takes the values 0, 1 and ∞), then z_0 is a removable singularity.

II.2.2 Theorem (Picard)**.** *Any meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ which omits three values is constant.*

II.2.3 Theorem (Montel)**.** *A family of meromorphic functions which all omit the same three values is normal.*

The Bloch Principle

> A property which implies that an entire (or meromorphic) function on the plane is *constant* should imply that a family of entire (or meromorphic) functions with this property is *normal*.

Of course, this *heuristic principle* isn't true as stated: for a trivial example, consider the property *f omits some collection of three points*.

(There are more interesting examples as well.)

II.2.2 The Zalcman lemma

Zalcman's rescaling lemma

Larry Zalcman formulated a rescaling lemma which makes Bloch's heuristic principle explicit.

II.2.4 Theorem (Zalcman's Lemma)**.** *The family f of meromorphic functions is not normal near a point z_0 if and only if:*

There exists a sequence (f_n) in $\mathcal{F}$, a sequence $z_n \to z_0$, and a sequence of rescaling factors ρ_n with $\rho_n \to 0$ such that the functions

$$z \mapsto f_n(z_n + \rho_n z)$$

converge locally uniformly to a nonconstant meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$.

(Furthermore, f can be chosen with $f^\# \leq 1$ for all $z \in \mathbb{C}$.)

Zalcman's rescaling lemma

Zalcman's lemma has *revolutionized* the study of normal families.

It can not only be used to prove the *equivalence* of results for normal families and global analytic functions, but often also to *prove such results themselves*.

For example: *simple proofs* of Montel's theorem, Picard's theorem, Koebe's theorem, some theorems by Nevanlinna and Ahlfors, ...

Idea of the proof

- If $\mathcal{F}$ is not normal near z_0, then there is a sequence of points z_n and functions $f_n \in \mathcal{F}$ such that the spherical derivative tends to ∞ (by Marty's theorem).
- This gives us a sequence of rescalings of f_n with spherical derivative, say, bounded by 1.
- Again, we can apply Marty's theorem to see that this sequence is normal, and hence extract a convergent subsequence.

Proof of Montel's theorem from Picard's theorem

1. Let $\mathcal{F}$ be a family of functions on U, all of which omit the values $\{0, 1, \infty\}$.
2. If $\mathcal{F}$ is not normal, we can find a *sequence of rescalings* converging to a *nonconstant entire function* f.
3. The limit f must also omit $\{0, 1, \infty\}$ by Hurwitz's theorem.
4. This contradicts Picard's theorem.

Some other instances of Bloch's principle

- Nevanlinna's *deficiency relation*.
- The Ahlfors *five islands theorem*.
- ...

II.3 Density of repelling cycles

Periodic points

- $z \in \mathbb{C}$ is *periodic* if $f^n(z) = z$.
- A periodic point is *attracting* if $|(f^n)'(z)| < 1$.
  (Attracting points are in the Fatou set.)
- A periodic point is *repelling* if $|(f^n)'(z)| > 1$.
  (Repelling points are in the Julia set.)

II.3.1 Theorem (Density of repelling cycles). *Let $f : X \to X$ be nonlinear and nonconstant, as before, where $X \in \{\mathbb{C}, \hat{\mathbb{C}}, \mathbb{C}^*\}$.*

Then repelling periodic points are dense in $J(f)$.

For rational functions, the usual proof uses the *finiteness of nonrepelling cycles*.

Baker's original proof for entire functions uses the *five islands theorem*.

We will give a proof using *Zalcman's lemma*, essentially due to Schwick (with simplifications due to Duval-Berteloot and Bargmann).

II.4 Expansion property of the Julia set

Expansion property of the Julia set

As a consequence of the density of repelling periodic points, we can strengthen a number of properties of the Julia set.

II.4.1 Theorem (Expansion property)**.** *Let $K \subset X$ be a compact set which does not contain any exceptional points.*

If U is an open set with $U \cap J(f) \neq \emptyset$, then there is $n \geq 0$ with

$$K \subset f^n(U).$$

Existence of convergent subsequence

II.4.2 Lemma. *Let $z \in J(f)$. Then z has no neighborhood in which the sequence (f^n) has any uniformly convergent subsequence.*