An Introduction to Holomorphic Dynamics

I. Introduction; Normal Families

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This handout is created from the overhead slides used during lectures. Examples and proofs will be done on the board, and are not included.

I.1 Introduction

I.1.1 Discrete dynamical systems

Discrete dynamical systems

General setting:

- X *phase space*;
- $f : X \to X$ function;
- $f^n = f \circ \cdots \circ f$ *iterates* of f;
- study behaviour of $f^n(x)$ as $n \to \infty$.

A remark

Remark. It may very well make sense to have f defined only on a subset of X.

For example, one can study the iteration of *meromorphic* functions $f : \mathbb{C} \to \hat{\mathbb{C}}$, or more general families of functions such as those considered by Adam Epstein and others.

Holomorphic dynamics

- X is a Riemann surface (i.e., a connected one-dimensional complex manifold);
- $f : X \to X$ is a holomorphic function.

Interesting behavior only for

$$X \in \{\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C} \setminus \{0\}, \mathbb{C}/\mathbb{Z}^2\}.$$

Our setting

I.1.1 Standing Assumption. X is either the *complex plane* $\mathbb{C}$, the *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, or the *punctured plane* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

$f : X \to X$ is a *nonconstant* holomorphic function which is *not a conformal automorphism* of X.

Entire functions

Recall that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ which is not a *polynomial* is called a *transcendental entire function*.

I.e.,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k \neq 0$ for infinitely many k and the series converges for all $z \in \mathbb{C}$.

The case where $X = \mathbb{C}$ and f is a transcendental entire function is the one we will have in mind for most of the lectures.

Julia and Fatou sets

The phase space X can be partitioned into two fundamentally different sets:

- The Fatou set is the set where the dynamics is *regular*.

  This is an open set, and the possible types of behaviour are (fairly) well-understood.

- The Julia set is the set where the dynamics is *"chaotic"*.

  The structure and dynamics of the Julia set can be *very complicated*.

I.1.2 An example

The simplest possible case

$$f(z) = z^2.$$

The quadratic family

$$f(z) = z^2 + c, \quad c \in \mathbb{C}.$$

Very complicated behaviour as c varies — gives rise to the *Mandelbrot set*.

I.2 Definition of Julia and Fatou sets

Equicontinuity

Recall that we want to define the Fatou set as the locus of *stable* behaviour. This means that

small perturbations lead to *small changes* in long-term behaviour.

I.2.1 Definition (Equicontinuity). Let A and B be metric spaces. A family $\mathcal{F}$ of functions from A to B is *equicontinuous* in a point $x_0 \in A$ if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall f \in \mathcal{F}\, \forall x \in A : \\ d(x, x_0) < \delta \;\Rightarrow\; d(f(x), f(x_0)) < \varepsilon.$$

Fatou and Julia sets

Let X and $f : X \to X$ be as in our standing assumption.

I.2.2 Definition (Fatou set). A point $z \in X$ belongs to the *Fatou set* $F(f)$ if there is a neighborhood U of z such that the family

$$\{f^n : n \in \mathbb{N}\}$$

is equicontinuous in every point of U *(with respect to the* spherical *metric)*.

I.2.3 Definition (Julia set). The *Julia set* of f is the complement of the Fatou set:

$$J(f) := X \setminus F(f).$$

I.3 Normal families

Locally uniform convergence

Let f_n be a *family* of holomorphic (or meromorphic) functions defined on some open set U.

Recall that we say that (f_n) converges *locally uniformly* to a function f if the sequence converges uniformly on every compact subset of U.

(For example, the sequence $f_n(z) = z/n$ converges locally uniformly to $f(z) = 0$ on $\mathbb{C}$.)

Results from Complex Analysis

I.3.1 Theorem (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$ (where $\mathbb{D}$ is the unit disk). Then*

$$|f'(0)| \leq 1 \quad \textit{and} \quad |f(z)| \leq |z| \quad \textit{for all } z \in \mathbb{D},$$

with equality if and only if f is a rotation.

I.3.2 Theorem (Weierstraß theorem)**.** *If $f_n \to f$ locally uniformly, where f_n and f are holomorphic functions defined on some open set $U \subset \mathbb{C}$, then $f_n' \to f'$ locally uniformly.*

I.3.3 Theorem (Hurwitz theorem)**.** *If $f_n \to f$ locally uniformly, as above, and $f_n(z) \neq 0$ for all z, then either $f \neq 0$ for all z, or f is constant.*

Normality

A family $\mathcal{F}$ of holomorphic or meromorphic functions on U is *normal* (on U) if every sequence of functions in $\mathcal{F}$ contains a locally uniformly convergent subsequence.

We say that $\mathcal{F}$ is normal *in a point* z if z has an open neighborhood on which $\mathcal{F}$ is normal.

Arzelá-Ascoli Theorem

I.3.4 Theorem (Arzelà-Ascoli)**.** *$\mathcal{F}$ is normal if and only if it is equicontinuous in every point of U.*

(In particular, normality is a local property: $\mathcal{F}$ is normal if and only if it is normal in every point of U.)

Hence the Fatou set of a function $f : X \to X$ is the *set of normality* of the family of iterates.

Marty's theorem

The *spherical derivative* of a meromorphic function f in z is

$$f^{\#}(z) := \frac{2|f'(z)|}{1+|f(z)|^2}.$$

I.3.5 Theorem (Marty)**.** *The family $\mathcal{F}$ of meromorphic functions is normal if and only if the spherical derivatives in $\mathcal{F}$ are locally bounded.*

(I.e., every $z_0 \in U$ has a neighborhood N such that $f^{\#}(z)$ is uniformly bounded in N, with the bound independent of $f \in \mathcal{F}$.)

Two theorems of Montel

I.3.6 Theorem (Montel)**.** *A uniformly bounded family of holomorphic functions is normal.*

I.3.7 Theorem (Montel)**.** *Let $a, b, c \in \hat{\mathbb{C}}$. Let $\mathcal{F}$ be a family of meromorphic functions on some open set U which omits the the three values a, b, c.*

(I.e., $f(z) \notin \{a, b, c\}$ for all $f \in \mathcal{F}$ and all z.)

Then $\mathcal{F}$ is normal.

Basic properties

I.3.8 Lemma (Basic properties of Julia and Fatou sets)**.**

- *$F(f)$ is open; $J(f)$ is closed (in X).*
- *$F(f)$ and $J(f)$ are completely invariant; i.e.*

$$z \in F(f) \iff f(z) \in F(f).$$

- *Julia and Fatou sets are preserved under iteration.*

  (That is, $F(f^n) = F(f)$, $J(f^n) = J(f)$.)

Properties of the Julia set

I.3.9 Theorem (Julia set infinite)**.** *The Julia set $J(f)$ contains infinitely many points.*

(Proof for *entire functions*: see course by Rippon and Stallard. Proof for *rational functions*: easy; see e.g. book by Milnor.)

Consequences

I.3.10 Corollary (Backward orbits are dense)**.** *For all points $z_0 \in \hat{\mathbb{C}}$ with at most three exceptions, the closure of the backward orbit*

$$O^-(z_0) := \{w \in X : f^n(w) = z_0 \, \textit{for some} \; n \geq 0\}$$

contains the Julia set $J(f)$.

I.3.11 Corollary (Characterization of $J(f)$)**.** *$J(f)$ is the smallest closed and backward invariant set containing at least three points.*

I.3.12 Corollary (Julia sets with interior)**.** *If $J(f) \neq X$, then $J(f)$ has no interior.*

(I.e., $J(f)$ contains no nonempty open set.)

More consequences

I.3.13 Corollary (Julia set is perfect)**.** *$J(f)$ has no isolated points. In particular, $J(f)$ is unbounded.*

I.3.14 Corollary (Dense orbits)**.** *There exist (uncountably many) points $z \in J(f)$ such that the orbit*

$$O^+(z) := \{f^n(z) : n \geq 0\}$$

is dense in $J(f)$.

Density of repelling periodic points

I.3.15 Definition (Periodic points). A point $z \in X$ with $f^n(z) = z$ is called *periodic*. (The smallest such n is the *period* of z.)

Such a periodic point is called

- *attracting* if $0 < |(f^n)'(z)| < 1$;
- *superattracting* if $|(f^n)'(z)| = 0$;
- *repelling* if $|(f^n)'(z)| > 1$;
- *indifferent* (or "neutral") if $|(f^n)'(z)| = 1$.

I.3.16 Theorem (Density of repelling cycles)**.** *Repelling periodic points are dense in the Julia set.*