Dynamical properties of a family of entire functions

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Outline

Fatou and Julia sets

Let f be a meromorphic function which is not rational of degree one and denote by $f^n, n \in \mathbb{N}$, the nth iterate of f.

- The *Fatou set*, $F(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that the sequence $\{f^n\}_{n \in \mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of z.
- The complement, $J(f)$, of $F(f)$ is called the *Julia set* of f.

Note: A family of functions is "normal" at a point z if every sequence in it contains a subsequence that is convergent on a neighbourhood of z.

Examples of Julia sets of rational functions

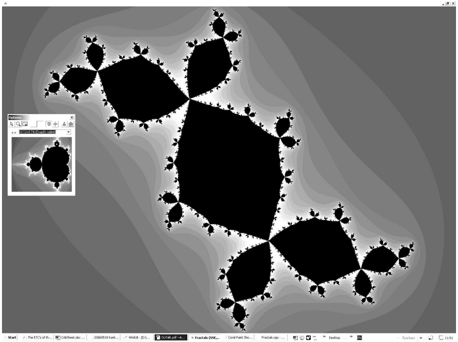


Figure: Douady Rabbit (courtesy of Peter Kankowski)

Examples of Julia sets of rational functions (continued)

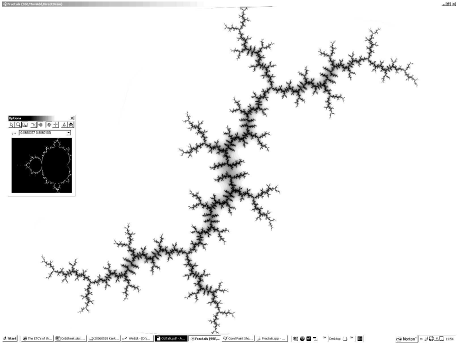


Figure: Dendrite (courtesy of Peter Kankowski)

Some properties of $J(f)$ and $F(f)$

- By definition, for any rational or transcendental function f, $F(f)$ is open and $J(f)$ is closed.
- Both $F(f)$ and $J(f)$ are completely invariant.
- $J(f)$ is a perfect set; that is $J(f)$ is closed, non-empty and contains no isolated points.
- Either $J(f) = \mathbb{C}$ or $J(f)$ has empty interior.
- If $z_0 \in J(f)$ is not an exceptional point, then $J(f) = \overline{\mathcal{O}^-(z_0)}$ where $\mathcal{O}^-(z_0)$ is the set of points $w \in \mathbb{C}$ such that $f^n(w) = z_0$ for some $n \in \mathbb{N}$.

Note: Sketch.

Types of Fatou components

Distinction between transcendental entire functions and polynomials
Common:

- attracting
- parabolic
- Siegel disc

Transcendental entire only:

- wandering
- Baker domain

Note: A "Fatou component" is a maximal open connected subset of $F(f)$.
Note: The framework for categorization of components is due to Fatou and Julia (common) and Baker (transcendental entire).
Note: Fatou's example. $f(z) = z + 1 + e^{-z}$.

Link between order of growth and unbounded Fatou components

In 1981 (J. Austral. Math. Soc.) Baker proved

Theorem

If for transcendental entire f there is an unbounded invariant component of $F(f)$, then the growth of f must exceed order $1/2$, minimal type.

- Is the value $1/2$ sharp?
- That is, can an example be found of a transcendental entire function with order of growth $1/2$ mean type which has an invariant unbounded Fatou component?

Note: There is a formal definition for "Order" and "Type" but I shall not give it here.

The family of functions

Baker demonstrated that the value $1/2$ is indeed sharp by introducing the example

$$f_c(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + c, \qquad \text{for } c \in \mathbb{R}. \tag{1}$$

- Entire $\left(\frac{\sin\sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \dots\right)$,
- Transcendental,
- Order 1/2 (the order of e^{z^a} is a).

Computer experiments

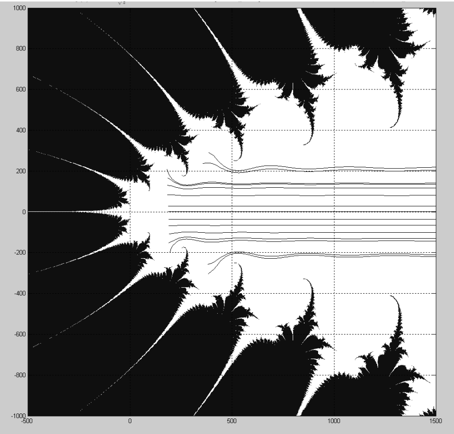


Figure: Julia set of f_c for $c = 6$ and "typical" orbits

Computer experiments (continued)

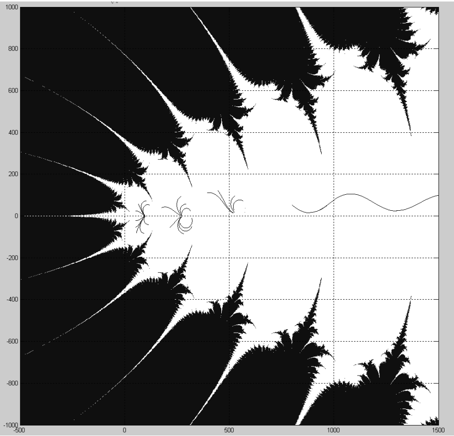


Figure: Julia set of f_c for $c = 0.05$ and "typical" orbits

Baker's original result

Proposition

The function f_c has a Baker domain for sufficiently large c.

Baker constructed a domain D symmetric about the real axis with a (truncated) parabolic boundary. He proved that $f(D) \subset D$, by showing that

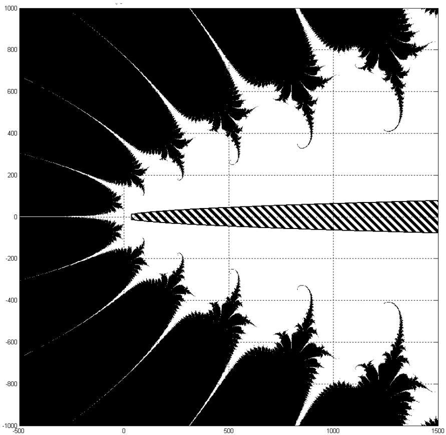
$$\left| \frac{\sin\sqrt{z}}{\sqrt{z}} \right| < \operatorname{dist}(z + c, \partial D),$$

for $z \in D$ and c sufficiently large. The precise criterion is

$$\frac{1}{2}c\left(x + 1 + \frac{1}{2}c\right)^{-1/2} > e|z|^{-1/2}, \qquad \text{for } z = x + iy \in D,$$

which is true when $c > 6$ so f_c has a Baker domain for such c.

Baker's parabola



Did Baker exhaust all possible values of c?

With more care, similar arguments can be used to show that an invariant domain exists for $c > 1$.
However, if $0 < c < 1$, a serious problem arises since ***no*** invariant parabolic domain exists.

- Is 1 a significant constant?
- Does f_c have a Baker domain for smaller values of c?

A new proof strategy would be required for smaller c.

Statement of new Theorem

Theorem

For all $c > 0$*, the function* f_c *defined by*

$$f_c(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + c$$

has an invariant Baker domain U*, symmetric about the real axis and containing the interval* (x_c, ∞) *for some* $x_c > 0$.

Key ideas of proof

We begin by observing that

- there exists some $x_0 \in \mathbb{R}$ such that

$$f(x) > x + \frac{c}{2}$$

  for all $x \geq x_0$, and
- Any curve defined by the interval $[x_1, \infty)$ is invariant, where $x_1 \geq x_0$.

The rest of the proof is concerned with showing that there exists a Fatou component U containing $[x_1, \infty)$ for some $x_1 > x_0$.

Key ideas of proof (continued)

We consider the auxilliary function g_c defined by

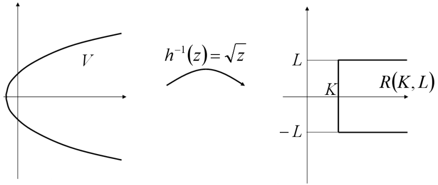
$$g_c(w) = h^{-1} \circ f_c \circ h(w) = \sqrt{f_c(w^2)} = \sqrt{w^2 + \frac{\sin w}{w} + c} = w\sqrt{1 + \frac{\sin w}{w^3} + \frac{c}{w^2}} \qquad (2)$$

which is analytic when $1 + \frac{\sin w}{w^3} + \frac{c}{w^2}$ is away from the origin and negative real axis.
For $K \geq 0$ and $L > 0$ we define the open half-strip R by

$$R(K, L) = \{w : \Re(w) > K, |\Im(w)| < L\}.$$

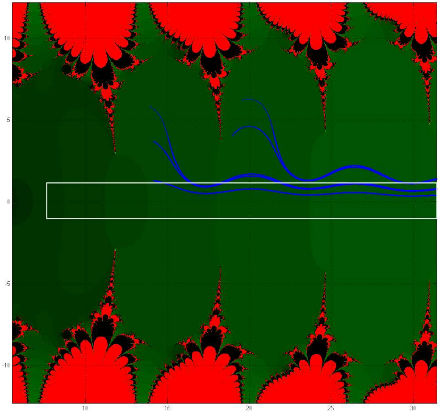
Note: g_c is not meromorphic, so $F(g_c)$ is not defined.

Change of variables - mapping of parabola to strip



Dynamics in the *w*-plane

Although $R(K, L)$ is not invariant for $0 < c < 1$, computer experiments do suggest that orbits omit a large part of $\mathbb{C}$.



Key ideas of proof (continued)

We show that for any $c > 0$ there are two strips $R(K, L)$ and $R(K, 2L)$ such that

$$g_c^n(R(K,L)) \subset R(K,2L), \quad \text{for all } n \in \mathbb{N},$$

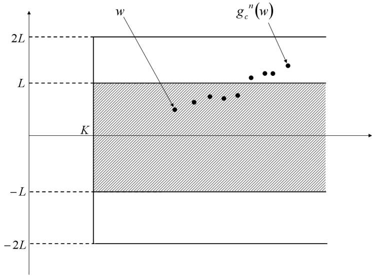


Figure: The sets $R(K, L)$ and $R(K, 2L)$

Key steps to show that $g(R(K,L)) \subset R(K,2L)$

- Expand g_c using the binomial theorem: $g_c(w) = w + \frac{\sin w}{2w^2} + \frac{c}{2w} + B(w)$.
- Real and imaginary parts of the function: $g_c(w) = w + \delta u + i\delta v$, where
  - $\delta u = \Re\left(\frac{\sin w}{2w^2}\right) + \Re\left(\frac{c}{2w}\right) + \Re(B(w))$, and
  - $\delta v = \Im\left(\frac{\sin w}{2w^2}\right) + \Im\left(\frac{c}{2w}\right) + \Im(B(w))$.
- Estimate δu and δv.

Symmetry

Since f_c and g_c are symmetrical in the sense that

$$f_c(\overline{z}) = \overline{f_c(z)} \quad \text{and } g_c(\overline{w}) = \overline{g_c(w)},$$

it suffices to consider $w \in R^+(K, L)$.
Note: Sketch

Key ideas of proof (continued)

It can be shown that

- $\delta u = \frac{c}{2u} + O\left(\frac{1}{u^2}\right)$ and
- $\delta v = \frac{\cos u \sinh v}{2u^2} - \frac{cv}{2u^2} + O\left(\frac{v}{u^3}\right)$, as $|w| \to \infty$ in $R^+(K, L)$

for K sufficiently large.

When $0 < c < 1$ and $w = 2\pi n + iv \in R^+(K, L)$, we have $\delta v > 0$, so $R^+(K, L)$ (and hence $R(K, L)$) is ***not invariant*** under g_c no matter how small L.

Key ideas of proof (continued)

Writing $w_n = u_n + iv_n = g_c^n(w_0)$, from the form of δu and δv we can deduce that

- orbits move to the right by $\delta u_n \approx \frac{c}{2u_n}$, and
- the growth of the imaginary part of the orbit is controlled as the real part increases from u_0 to $u_0 + 2\pi$. In particular,

$$|v_n - v_0| < A\frac{v_0}{u_0}$$

  for every iterate w_n in $R^+(K, 2L)$ with real part lying between u_0 and $u_0 + 2\pi$.

We use this to improve the particular estimate for $v_N - v_0$, where $u_N \approx u_0 + 2\pi$.
Note: Sketch

Key ideas of proof (continued)

In fact, $0 < v_N < v_0$ for all $w_0 \in R^+(K, L) \subset R^+(K, 2L)$.

Below we outline why this is so.

Key ideas of proof (continued)

$$v_N - v_0 = \sum_{n=0}^{N-1} \delta v_n$$
$$= \frac{1}{2} \sum_{n=0}^{N-1} \frac{\sinh v_n \cos u_n}{u_n^2} - \frac{c}{2} \sum_{n=0}^{N-1} \frac{v_n}{u_n^2} + A \sum_{n=0}^{N-1} \frac{v_n}{u_n^3},$$

for some constant A. Now the last sum is much smaller as $u_0 \to \infty$, so the task is to show that

$$\frac{1}{2} \sum_{n=0}^{N-1} \frac{\sinh v_n \cos u_n}{u_n^2} < \frac{c}{2} \sum_{n=0}^{N-1} \frac{v_n}{u_n^2}.$$

Key ideas of proof (continued)

The key problem here is that $\sinh v_n > cv_n$ when $c < 1$, so the $\cos u_n$ term must be exploited to reduce the size of the left-hand side; that is, there is significant cancelation.

Key ideas of proof (continued)

Now

$$\frac{\pi}{12}\frac{v_0}{u_0} < \frac{c}{2}\sum_{n=0}^{N-1}\frac{v_n}{u_n^2},$$

since $N > \frac{4\pi u_0}{3c}$, and

$$\frac{1}{2}\sum_{n=0}^{N-1}\frac{\sinh v_n \cos u_n}{u_n^2} \approx \frac{\sinh v_0}{2u_0^2}\sum_{n=0}^{N-1}\cos u_n + \text{ smaller terms }.$$

So it suffices to show that

$$\sum_{n=0}^{N-1}\cos u_n = O(1) \quad \text{as } u_0 \to \infty.$$

Note: v_0 is very close to v_n.

Key ideas of proof (continued)

We do this by noting that $\sum_{n=0}^{N-1} \delta u_n \cos u_n$ is a Riemann sum for

$$\int_0^{2\pi} \cos u \, du$$

Note: It is crucial that the factor δu_n can be introduced without too much error, since $\delta u_n \approx \frac{c}{2u_n} \approx \frac{2\pi}{N}$.

Generalizing the key steps

If x_1 is sufficiently large (and $x_1 > x_0$), then for any $w_0 \in R(x_1, L)$

- w_n moves to the right and $u_N \approx u_0 + 2\pi$, and
- $0 < v_n < 2L$ for all $n \in \{0, \ldots, N\}$ and $0 < v_N < v_0$.

By induction $w_n \in R(x_1, L)$, for all $n \in \mathbb{N}$.

For $z_0 \in h(R(x_1, L)) \equiv P$ we have $z_n \in h(R(x_1, 2L)) \equiv Q$ for all $n \in \mathbb{N}$.

Concluding the proof

- $\mathbb{C} \setminus Q$ contains more than 3 points, so by Montel's theorem, $P \subset F(f_c)$.
- Since P is connected and unbounded, there exists a single unbounded component, U say, of $F(f_c)$ such that $P \subset U$.
- From above,
  - $[x_1, \infty) \subset P \subset U$ is invariant, and
  - $f_c^n(x) \to \infty$ as $n \to \infty$ for all $x \in [x_1, \infty)$
- Thus ***all*** points in U tend to infinity under f_c, so U is a Baker domain.

This concludes the proof.

Background

For $p \in \mathbb{N}$, we denote by

$$\operatorname{sing}(f^{-p})$$

the set of finite singularities of f^{-p}; that is, the set of points $w \in \mathbb{C}$ such that some branch of f^{-p} cannot be analytically continued through w.

The set $\operatorname{sing}(f^{-1})$ consists of the critical values and finite asymptotic values of f, and we refer to these points as *singular values of* f.

Let γ be a curve starting at zero and tending to ∞. Suppose there exists $\alpha \in \mathbb{C}$ such that $f(z) \to \alpha$ as $z \to \infty$ on γ. Then α is a finite asymptotic value of f.

Bargmann's Theorem (J. Anal. Math.) 2001

Theorem

Let f be an entire function with an invariant Baker domain. Then there exist constants $C > 1$ and $r_0 > 0$ such that:

$$\{z : r/C < |z| < Cr\} \cap sing(f^{-1}) \neq \emptyset, \quad for\ r \geq r_0.$$

Note: Generalized by Rippon and Stallard to meromorphic functions that have a finite number of poles and a p-cycle of Baker domains with $\mathrm{sing}(f^{-1})$ replaced by $\mathrm{sing}(f^{-p})$.

Comment on Bargmann's Theorem

It is natural to ask if Bargmann's result is sharp.

Many known examples of functions satisfying the hypotheses have the property that $\text{sing}(f^{-1})$ meets every annulus of some uniform width.

For example $f(z) = z + e^{-z} + 1$ has an invariant Baker domain with no finite asymptotic values and critical points $\{2n\pi i : n \in \mathbb{Z}\}$ so $\text{sing}(f^{-1}) = \{2n\pi i + 2 : n \in \mathbb{Z}\}$.

f_c has sparsely distributed singular values

Theorem

For the function

$$f_c(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + c, \quad c > 0,$$

the set $\mathbb{C} \setminus sing(f_c^{-1})$ *contains an infinite sequence of nested annuli* $\{A_n\}$ *with* $A_n = \{z : a_n < |z| < a_n + Kn\}$ *where* $\{a_n\}$ *is an increasing sequence which tends to* ∞ *and where* K *is a positive constant.*

This is the first example of a transcendental entire function with such sparsely distributed singular values.
Whereas this does not demonstrate the sharpness of Bargmann's result, it does close the gap somewhat.