Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Solutions

1.

$$B := ((1,1,1),(0,1,1),(0,1,2)) \quad \text{and} \quad C := ((1,2,2),(1,0,-1),(0,0,1)).$$

(a) The change-of-basis matrices from the standard basis are obtained by putting the basis vectors as columns:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$
$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & -1 & 1 \end{pmatrix}.$$

(b) We have

$$\begin{aligned} (1,2,2) &= 1\cdot(1,1,1) + 1\cdot(0,1,1) + 0\cdot(0,1,2), \\ (1,0,-1) &= 1\cdot(1,1,1) + 0\cdot(0,1,1) + (-1)\cdot(0,1,2) \quad \text{and} \\ (0,0,1) &= 0\cdot(1,1,1) + (-1)\cdot(0,1,1) + 1\cdot(0,1,2). \end{aligned}$$

(These are easily found by inspection, or alternatively by solving three systems of linear equations.) Thus the change-of-basis matrix R is

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(c) The inverse of P is easily computed as

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

so we have

$$P^{-1}Q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = R.$$

2. We consider the matrix

$$A = \begin{pmatrix} -1 & -3 & -2 & -2 \\ 1 & 3 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 3 & 4 & 3 & 4 \end{pmatrix}.$$

(a) We need to compute $\det(\lambda I - A)$. This is a very large matrix, so we help ourselves first by adding the second row twice to the first:

$$\begin{vmatrix} \lambda+1 & 3 & 2 & 2 \\ -1 & \lambda-3 & -1 & -1 \\ 1 & 1 & \lambda-1 & 1 \\ -3 & -4 & -3 & \lambda-4 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 2\lambda-3 & 0 & 0 \\ -1 & \lambda-3 & -1 & -1 \\ 1 & 1 & \lambda-1 & 1 \\ -3 & -4 & -3 & \lambda-4 \end{vmatrix}.$$

It may also help to subtract the third column from the fourth, giving

$$\begin{vmatrix} \lambda-1 & 2\lambda-3 & 0 & 0 \\ -1 & \lambda-3 & -1 & 0 \\ 1 & 1 & \lambda-1 & 2-\lambda \\ -3 & -4 & -3 & \lambda-1 \end{vmatrix}.$$

Now let us calculate the determinant: after some more calculations, we should get

$$\det(\lambda I - A) = (\lambda-2)^3(\lambda-1).$$

So the eigenvalues of A are 1 and 2, of multiplicities 1 and 3, respectively.

A simple calculation shows that for each eigenvalue, the eigenspace is 1-dimensional: $(1,0,0,-1)$ is an eigenvector with eigenvalue 1, and $(-1,1,-1,1)$ is one with eigenvalue 2.

(b) We begin by extending the eigenvectors of A to a basis of $\mathbb{R}^4$; e.g.

$$(1,0,0,-1), (-1,1,-1,1), (0,0,1,0), (0,0,0,1).$$

The change-of-basis matrix for this basis is

$$P_1 \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

Its inverse is calculated to be

$$P_1^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$A_1 = P_1^{-1} A P_1 = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

It remains to triangulate the matrix

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

Its sole eigenvalue is 2 (as expected), and an eigenvector is given by $(0, 1)$. We set

$$P_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(note that $P_2^{-1} = P_2$). Then

$$P_2^{-1} A_1 P_2 = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The overall change-of-basis matrix is

$$P = P_1 P_2.$$

(We can compute a matrix P which puts A into Jordan form by solving the equations $Av_2 = 2v_2 + (-1, 1, -1, 1)$ and $Av_3 = 2v_3 + v_2$. Solutions are given e.g. by $v_2 = (-1, 0, 0, 2)$ and $v_3 = (1, 0, 1, -2)$, so a change-of-basis-matrix is given by

$$P = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 1 & 2 & -2 \end{pmatrix}.$$

Its inverse is

$$P-1 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

and

$$P^{-1} A P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

is in Jordan normal form.)

3. (a) Computing the eigenvalues and eigenvectors will produce three linearly independent eigenvectors, and thus the matrix

$$P = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 1 & -11 & \end{pmatrix}$$

of eigenvectors will transform A into diagonal form:

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b) $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$. The characteristic polynomial is

$$\begin{vmatrix} \lambda - 1 & -1 & 1 \\ 0 & \lambda - 2 & 0 \\ -1 & 1 & \lambda - 3 \end{vmatrix} = (\lambda - 2) \begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda^2 - 4\lambda + 3 + 1) = (\lambda - 2)^3.$$

So the only eigenvalue is $\lambda = 2$. To find eigenvectors, we solve for $(A-2I)v = v$, obtaining after a simple calculation that all eigenvectors are of the form $(\lambda, \lambda+\mu, \mu)$ with $\lambda, \mu \in \mathbb{R}$. So we have two linearly independent eigenvectors, and we must have a Jordan normal form with two blocks, i.e.

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This means that we need to find v_1, v_2, v_3 such that

$$Av_1 = 2v_1, \quad Av_2 = v_1 + 2v_2, \quad Av_3 = 2v_3.$$

There are two ways of doing this:

i. In the lectures, we discussed that we can pick v_2 to be *any* non-eigenvector; e.g. $v_2 = (1, 0, 0)$. Then

$$v_1 := Av_2 - 2v_2 = (-1, 0, 1)$$

is an eigenvector, as expected. We can complete the basis by picking another eigenvector v_3 which is linearly independent to v_1; obtaining e.g.

$$P = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

ii. If we do not remember this trick, we can still find a basis by solving the above equations one after the other: since v_1 has to be an eigenvector, we must have $v_1 = (\lambda, \lambda + \mu, \mu)$. We now need to find a solution to the equation $(A - 2I)v_2 = v_1$. It turns out that this is only possible if $\lambda + \mu = 0$, in which case the general solution is given by vectors of the form

$$(x, y, y - x - \lambda).$$

Choosing e.g. $x = y = 0$ and $\lambda = 1$, and again picking another eigenvector, we obtain a change-of-basis matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

(c) The characteristic polynomial is

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & -2 & -3 \\ 2 & \lambda - 4 & -2 \\ -1 & 1 & \lambda - 1 \end{vmatrix} &= -1 \begin{vmatrix} -2 & -3 \\ \lambda - 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} \lambda - 1 & -3 \\ 2 & -2 \end{vmatrix} + (\lambda - 1) \begin{vmatrix} \lambda - 1 & -2 \\ 2 & \lambda - 4 \end{vmatrix} \\ &= -(4 + 3(\lambda - 4)) - (-2(\lambda - 1) + 6) + (\lambda - 1)(\lambda^2 - 5\lambda + 4 + 4) \\ &= -4 - 3\lambda + 12 + 2\lambda - 2 - 6 + \lambda^3 - 6\lambda^2 + 13\lambda - 8 \\ &= \lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3. \end{aligned}$$

So the only eigenvalue is 2. Solving the equation $(A - 2I)v = 0$, we see that the only eigenvectors are of the form $(x, 2x, -x)$ with $x \in \mathbb{R}$. So the Jordan normal form must consist of a single block: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Translating this back into equations for our basis vectors, we are looking for v_1, v_2, v_3 such that

$$Av_1 = 2v_1, \quad Av_2 = 2v_2 + v_1, \quad Av_3 = 2v_3 + v_2.$$

Again, there are several methods we could use:

i. We set $v_1 = (-1, 2, -1)$ and solve first for v_2, and then for v_3. This will give us e.g. the change-of-basis matrix $P = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$.

ii. We take any vector v_3 for which $v_2 := Av_3 - 2v_3$ is not an eigenvector; e.g. $v_3 = (0, 1, 0)$. We get

$$v_2 := Av_3 - 2v_3 = (2, 2, -1)$$

and

$$v_1 := Av_2 - 2v_2 = (-1, -2, 1).$$

iii. Finally, if we don't remember the first two "tricks", we can again solve the above equations in full generality: we must have $v_1 = (\lambda, 2\lambda, -\lambda)$, and want $(A - 2I)v_2 = v_1$. This turns out to be soluble for every λ, yielding the general solution

$$v_2 = (\mu, 2\lambda + 2\mu, -\lambda - \mu).$$

Now we solve for v_3, and get the general solution

$$v_3 = (\nu, 2\mu + 2\nu + 3\lambda, -2\lambda - \mu - \nu).$$

We can choose arbitrary values for μ and ν (and any nonzero value for λ). Setting $\lambda = 1$ and $\mu = \nu = 0$, we get

$$v_1 = (1, 2, -1), \quad v_2 = (0, 2, -1), \quad v_3 = (0, 3, -2).$$

Thus a suitable change-of-basis matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 3 \\ -1 & -1 & -2 \end{pmatrix}.$$