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Linear Algebra, Geometry and Groups (MATH244)
Solutions 4

1. Let $V := \mathbb{R}^4$ and $W := \mathbb{R}^3$. Define bases

$$B := ((1,2,4,1),(0,1,0,-1),(1,2,1,4),(1,-1,1,-1))$$

of V and $C := ((-1,3,5),(2,-3,-1),(1,3,10))$ of W.

(a) To find P, the change of basis matrix from the standard basis of $\mathbb{R}^4$ to B, and Q, the change of basis matrix from the standard basis of $\mathbb{R}^3$ to C, all we have to do is write the elements of B and C as the column vectors of P resp. Q. So we have

$$P = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 4 & 0 & 1 & 1 \\ 1 & -1 & 4 & -1 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -3 & 3 \\ 5 & -1 & 10 \end{pmatrix}.$$

(b) To compute Q^{-1}, we write Q and I next to each other and perform simultaneous row transformations on both matrices:

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -3 & 3 & 0 & 1 & 0 \\ 5 & -1 & 10 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 6 & 3 & 1 & 0 \\ 0 & 9 & 15 & 5 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & \frac{1}{3} & 0 \\ 0 & 9 & 15 & 5 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & -3 & -4 & -3 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 1 & \frac{-1}{3} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} -1 & 2 & 0 & \frac{-1}{3} & -1 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{-5}{3} & \frac{-5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} & 1 & \frac{-1}{3} \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 3 & \frac{7}{3} & -1 \\ 0 & 1 & 0 & \frac{-5}{3} & \frac{-5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} & 1 & \frac{-1}{3} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -\frac{7}{3} & 1 \\ 0 & 1 & 0 & \frac{-5}{3} & \frac{-5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} & 1 & \frac{-1}{3} \end{array}\right)$$

In other words, we have

$$Q^{-1} = \frac{1}{3}\begin{pmatrix} -9 & -7 & 3 \\ -5 & -5 & 2 \\ 4 & 3 & -1 \end{pmatrix}.$$

(c) To calculate the matrix A of the linear map

$$\varphi(x,y,z,w) = (y+w, z-x, x+y+z+w)$$

with respect to B and C, we first write down the matrix with respect to the standard bases:

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We now compute

$$A = Q^{-1}MP = \frac{1}{3}\begin{pmatrix} -9 & -7 & 3 \\ -5 & -5 & 2 \\ 4 & 3 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 4 & 0 & 1 & 1 \\ 1 & -1 & 4 & -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -9 & -7 & 3 \\ -5 & -5 & 2 \\ 4 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 6 & -2 \\ 3 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -24 & 0 & -30 & 18 \\ 14 & 0 & 14 & 10 \\ 13 & 0 & 16 & -8 \end{pmatrix}.$$

2. Set $V := \mathbb{R}^{2\times 2}$ and $W := \mathbb{R}^3$. Define a linear map φ by

$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} := (a-d, b+c, 2c).$$

(a) To find A, the matrix of φ with respect to the basis

$$B := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of V and the standard basis of $\mathbb{R}^3$, we have to apply φ to the basis vectors in B, and put the results as the columns of A:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

(b) Let C be the basis $\{(2,0,1),(1,0,1),(0,1,1)\}$ of $\mathbb{R}^3$. To calculate M, the matrix of φ with respect to B and C, we let Q be the change-of-basis matrix from the standard basis of $\mathbb{R}^3$ to C:

$$Q = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and compute its inverse:

$$Q^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Alternatively, we could have computed Q^{-1} directly as the change-of-basis matrix from C to the standard basis, by writing the standard basis vectors as linear combinations of vectors in C. With the given vectors, this is possible by inspection.

Then we have

$$M = Q^{-1}A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

(Again, we could have found this matrix by inspection, by writing the columns of A with respect to the new basis. However, in more general situations, we would have to solve four systems of linear equations, so this would be more work than the method given above!)

(c) Finally, to find bases $\widetilde{B}$ of V and $\widetilde{C}$ of W such that the matrix representation of φ with respect to $\widetilde{B}$ and $\widetilde{C}$ has the standard "partial identity" form, we proceed as described in the lecture. We begin by finding the kernel of φ. In this case, we can write down the kernel directly:

$$\ker(\varphi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (0, 0,) \right\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = d, b = c = 0 \right\}.$$

So the kernel is one-dimensional; a basis is given by

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

We now extend this to a basis of V; say the basis

$$B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

(These four vectors are clearly linearly independent, and hence a basis of V.)

Now we need to compute the images of the first three basis vectors under φ:

$$\varphi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1, 0, 0); \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0, 1, 0); \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (0, 1, 1).$$

Our "recipe" now tells us to extend these vectors to a basis of W. However, these vectors already form a basis of $W = \mathbb{R}^3$, so there is nothing to extend. Our desired basis of $\mathbb{R}^3$ thus is

$$C = ((1, 0, 0), (0, 1, 0), (0, 1, 1)).$$

Finally, let us check that the matrix of φ with respect to these bases has the desired form:

$$\begin{aligned}\varphi\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} &= 1\cdot(1,0,0)+0\cdot(0,1,0)+0\cdot(0,1,1);\\ \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} &= 0\cdot(1,0,0)+1\cdot(0,1,0)+0\cdot(0,1,1);\\ \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} &= 0\cdot(1,0,0)+0\cdot(0,1,0)+1\cdot(0,1,1);\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} &= 0\cdot(1,0,0)+0\cdot(0,1,0)+0\cdot(0,1,1).\end{aligned}$$

So the matrix is indeed

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix},$$

as desired.

3. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map corresponding to reflection in the line $y = kx$ (where $k \neq 0$ is a fixed constant), and set $v_1 = (1, k)$ and $v_2 = (-k, 1)$.

   CLAIM. The vectors v_1 and v_2 are eigenvectors of φ, with eigenvalues 1 and -1, respectively.

   Proof. The vector v_1 lies on the line $y = kx$, so we have $\varphi(v_1) = v_1$, and v_1 is an eigenvector with eigenvalue -1.

   The vector v_2 is perpendicular to the line $y = kx$, so its reflection is given by $\varphi(v_2) = -v_2$. So v_2 is an eigenvector with eigenvalue -1. ■

   In particular, the diagonal matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

   represents φ with respect to the basis (v_1, v_2).

4. (a) Let $V := \mathbb{R}^2$, $\varphi(x, y) := (-3x - 5y, 2x + 3y)$. To find the eigenvalues of φ, we write φ in matrix form:

$$\varphi(x,y) = \underbrace{\begin{pmatrix}-3 & -5\\ 2 & 3\end{pmatrix}}_{=:A}\begin{pmatrix}x\\ y\end{pmatrix}$$

   and find the characteristic polynomial of A:

$$\det(\lambda I - A) = \begin{vmatrix}\lambda+3 & 5\\ -2 & \lambda-3\end{vmatrix} = \lambda^2 - 9 + 10 = \lambda^2 + 1.$$

   Since $\lambda^2 + 1 = 0$ has no real solutions, the matrix A has no eigenvectors in $\mathbb{R}^2$. In particular, it is not diagonalizable (over R).

(b) Let $V := \mathrm{Pol}_3(\mathbb{R})$, $\varphi : V \to V$ be defined by

$$\varphi(a + bx + cx^2 + dx^3) = d + cx + bx^2 + ax^3.$$

The matrix of φ with respect to the basis $(1, x, x^2, x^3)$ is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

To find the eigenvalues of A (and thus of φ), we again find the characteristic polynomial:

$$\begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} + \begin{vmatrix} 0 & 0 & -1 \\ \lambda & -1 & 0 \\ -1 & \lambda & 0 \end{vmatrix}$$
$$= \lambda^2(\lambda^2 - 1) - (\lambda^2 - 1) = (\lambda^2 - 1)^2 = (\lambda - 1)^2(\lambda + 1)^2.$$

So the eigenvalues are 1 and -1 (both are repeated).

Remark. Because φ is of particularly simple form, we could have instead solved directly for

$$\varphi(a + bx + cx^2 + dx^3) = \lambda a + \lambda bx + \lambda cx^2 + \lambda dx^3,$$

and seen that we must have $\lambda^2 = 1$. In general, however, the above method is preferable.

To find the eigenvectors of φ with eigenvalue 1, we need to find all vectors $v = a + bx + cx^2 + dx^3$ with $\varphi(v) = v$:

$$\varphi(v) = v \quad \iff \quad a = d \text{ and } b = c.$$

Similarly, the eigenvectors of φ with eigenvalue -1 are given by the vectors $v = a + bc + cx^2 + dx^3$ with $a = -d$ and $b = -c$. In particular, we can find a basis of $\mathrm{Pol}_3(\mathbb{R})$ consisting of eigenvectors of φ, e.g.

$$B = (x^3 + 1, x^2 + x, x^3 - 1, x^2 - x).$$

Thus φ is diagonalizable.

(c) We now consider the map

$$\varphi(z_1, z_2) := (-3z_1 - 5z_2, 2z_1 + 3z_2)$$

from the first part of this exercise as a linear function $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$. In this setting, the characteristic polynomial x^2+1 *can* be factorized as $(x-i)(x+i)$, so the eigenvalues are $\pm i$. Since we have two distinct eigenvalues, the map φ is diagonalizable.

To find the eigenvectors with eigenvalue i, we solve for

$$\varphi(z_1, z_2) = (iz_1, iz_2):$$

$$\begin{aligned}\begin{pmatrix} i+3 & 5 \\ -2 & i-3 \end{pmatrix} &\longrightarrow \begin{pmatrix} i+3 & 5 \\ -10 & 5i-15 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} i+3 & 5 \\ (i+3)(3-i)-10 & 0 \end{pmatrix} = \begin{pmatrix} i+3 & 5 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

So the eigenvectors with eigenvalue i are the vectors (z_1, z_2) with $(i+3)z_1 = -5z_2$; a basis for this eigenspace is given by

$$v_1 = (-5, 3+i).$$

Similarly, a basis for the eigenspace of $-i$ is given by

$$v_2 = (-5, 3-i).$$

5. Let $V = M_2(\mathbf{R})$, the vector space of 2×2 matrices with real entries. Define $\varphi : V \to V$ by $\varphi(A) = A^T$.

   To find the eigenvalues/-vectors of A, we need to find the solutions of

$$A^T = \lambda A.$$

   For any such A, we would have

$$A = (A^T)^T = (\lambda A)^T = \lambda A^T = \lambda^2 A,$$

   so that $\lambda^2 = 1$; i.e., the only possible eigenvalues are $\lambda = 1$ and $\lambda = -1$.

   If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$, then the equation

$$A^T = A$$

   is satisfied if and only if $b = c$. So

$$\mathrm{Eig}(1) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : b = c\}.$$

   A basis for this space is clearly given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In order to have

$$A^T = -A,$$

we must have $a = d = 0$ and $b = -c$, so that a basis for $\mathrm{Eig}(-1)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Overall, we have four linearly independent eigenvectors, so that φ is diagonalizable.