Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Solutions 2

1. (a) Set $V := \mathbb{R}^3$ and $A := \{(1,2,3),(4,5,6),(7,8,9)\}$.

   CLAIM. A is linearly dependent, and

$$\operatorname{span}(A) = \{(x,y,z) \in \mathbb{R}^3 : z = 2y - x\}.$$

   In particular, A is not a basis of V.

   Proof. We have

$$(7,8,9) = 2(4,5,6) - (1,2,3),$$

   so A is linearly dependent, and thus not a basis of V. Also, it follows that $\operatorname{span}(A) = \{(1,2,3),(4,5,6)\}$.
   Clearly any $(x,y,z) \in \operatorname{span}(A)$ must satisfy $(x,y,z) \in \mathbb{R}^3$ (since all vectors in A have that property). It remains to show that $(x,y,2y-x) \in \operatorname{span}(A)$ for all $x,y \in \mathbb{R}$. Indeed, we have

$$\begin{aligned}(2x-y)(4,5,6) + (4y-5x)(1,2,3) = \\ &(8x - 4y + 4y - 5x, \\ &\ 10x - 5y + 8y - 10x, 12x - 6y + 12y - 15x) = \\ &(3x, 3y, -3x + 6y) = 3(x,y,2y-x).\end{aligned}$$

   So $(x,y,2y-x) \in \operatorname{span}(A)$. ■

   (b) Let $V := \mathbb{R}^{2\times 2}$ and

$$A := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \right\}.$$

   CLAIM. A is a basis of V.

   Proof. We already know that $\dim(V) = 4$, so it is sufficient to check that A is linearly independent. Indeed, suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

   Then we get the system of linear equations

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 &= 0, \\ \lambda_2 + 2\lambda_4 &= 0, \\ \lambda_2 + \lambda_4 &= 0 \quad \text{and} \\ \lambda_1 + \lambda_2 - \lambda_3 &= 0. \end{aligned}$$

The second and third equation, taken together, imply that $\lambda_2 = \lambda_4 = 0$. So we have

$$\begin{aligned} \lambda_1 + \lambda_3 &= 0 \quad \text{and} \\ \lambda_1 - \lambda_3 &= 0. \end{aligned}$$

Thus also $\lambda_1 = \lambda_3 = 0$, as required. ■

(c) Let V denote $\mathbb{R}^{\mathbb{R}}$, the space of all functions $f : \mathbb{R} \to \mathbb{R}$. For every $a \in \mathbb{R}$, we define a function $f_a \in V$ by

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise.} \end{cases}$$

Set $A := \{f_a : a \in \mathbb{R}\}$.

Claim. The set A is linearly independent, and $\operatorname{span}(A)$ consists of all functions $f : \mathbb{R} \to \mathbb{R}$ which are nonzero only at finitely many points. In particular, A is not a basis of V.

Proof. Let $a_1, \dots, a_n$ be different real numbers, and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then

$$f := (\lambda_1 f_{a_1} + \dots + \lambda_n f_{a_n})(x) = \begin{cases} \lambda_j & \text{if } x = a_j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In particular, $f = 0$ (i.e., $f(x) = 0$ for all $x \in \mathbb{R}$) if and only if $\lambda_j = 0$ for all j, which proves that A is linearly independent. Furthermore, f is nonzero only at finitely many points.

Finally, suppose that g is a function which is nonzero at only finitely many points $a_1, \dots, a_n$. Then, by the above,

$$g = g(a_1) \cdot f_{a_1} + \dots + g(a_n) \cdot f_{a_n} \in \operatorname{span}(A).$$

This concludes the proof. ■

2. Let $V := \mathbb{R}^4$, and define subspaces U, W by $U := \{(x, y, z, w) : x + y + z + w = 0\}$ and $W := \{(x, y, z, w) : x = z, y = -2w\}$.

Claim. $B_1 := \{(1, 0, -1, 0), (0, 1, 0, -1), (1, 0, 0, -1)\}$ is a basis for U, and $B_2 := \{(1, 0, 1, 0), (0, -2, 0, 1)\}$ is a basis for W. Furthermore, $B := \{(1, -4, 1, 2)\}$ is a basis for $U \cap W$, and $U + W = \mathbb{R}^4$. (So e.g. the standard basis is the basis of $U + W$.)

In particular, the dimensions of U, W, $U \cap W$ and $U + W$ are 3, 2, 1 and 4, respectively.

Proof. B_1 is linearly independent. Indeed, let $\lambda, \mu, \nu \in \mathbb{R}$ such that

$$\lambda(1, 0, -1, 0) + \mu(0, 1, 0, -1) + \nu(1, 0, 0, -1) = (0, 0, 0, 0),$$

then (comparing the third components), we must have $\lambda = 0$. Comparing the second components, we see that also $\mu = 0$, and so finally also $\nu = 0$, as required.

Also, clearly $B_1 \subset U$. So $\dim U \geq 3$. Since $U \neq \mathbb{R}^4$ (for example, $(1,0,0,0) \notin U$), we see that $\dim U = 3$. In particular, B_1 is a basis of U.

The set B_2 is clearly a linearly independent subset of W. Furthermore, if $(x, y, z, w) \in W$, then

$$(x, y, z, w) = (x, -2w, x, w) = x(1, 0, 1, 0) + w(0, -2, 0, 1) \in \operatorname{span}(B_2),$$

so B_2 is a linearly independent spanning set, and thus a basis, for W.

Now B is a subset of $U \cap W$, and thus $\dim U \cap W \geq 1$. On the other hand, B is a proper subset of the two-dimensional subspace W (for example, $(1, 0, 1, 0)$ is in W but not in $U \cap W$). So we see that $\dim(U \cap W) = 1$, and B is a basis. Finally,

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) = 3 + 2 - 1 = 4,$$

so $U + W = V$, as required. ■

3. (a) Define $\varphi : \mathbb{R}^3 \to \mathbb{R}^3; (x, y, z) \mapsto (x - y + z, x + 2y + z, x + z)$.

   Claim. nullity$(\varphi) = 1$ and rank$(\varphi) = 2$. In particular, φ is not an isomorphism.

   Proof. We claim that $\{(1, 0, -1)\}$ is a basis for $\ker(\varphi)$. Indeed, this vector clearly belongs to $\ker(\varphi)$. On the other hand, if $(x, y, z) \in \ker(\varphi)$, then $x + z = 0$ and $x - y + z = 0$, and thus $x = -z$ and $y = 0$. So $(x, y, z) = x(1, 0, -1)$.
   So nullity$(\varphi) = 1$ and, by the rank and nullity theorem, we have

   $$\operatorname{rank}(\varphi) = \dim(\mathbb{R}^3) - \operatorname{nullity}(\varphi) = 3 - 1 = 2.$$

   (b) Define $\varphi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}; A \mapsto A^T$.

   Claim. nullity$(\varphi) = 0$ and rank$(\varphi) = 4$. In particular, φ is an isomorphism.

   Proof. Clearly $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ is the only matrix whose transpose is $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$. In other words, $\ker(\varphi) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$, so nullity$(\varphi) = 0$. We thus have rank$(\varphi) = \dim(\mathbb{R}^{2\times 2}) - \operatorname{nullity}(\varphi) = 4 - 0 = 4$. ■

4. Let V and W be vector spaces, let $A \subset V$, and let $\varphi : V \to W$ be linear.

   (a) Claim. $\varphi(\operatorname{span}(A)) = \operatorname{span}(\varphi(A))$.

Proof. We have

$$\begin{aligned}\varphi(\mathrm{span}(A)) &= \{\varphi(\lambda_1 v_1 + \cdots + \lambda_n v_n : v_1, \ldots, v_n \in A, \lambda_1, \ldots, \lambda_n \in \mathbb{K}\} \\ &= \{\lambda_1 \varphi(v_1) + \cdots + \lambda_n \varphi(v_n) : v_1, \ldots, v_n \in A, \lambda_1, \ldots, \lambda_n \in \mathbb{K}\} \\ &= \{\lambda_1 w_1 + \cdots + \lambda_n w_n : w_1, \ldots, w_n \in \varphi(A), \lambda_1 \ldots, \lambda_n \in \mathbb{K}\} \\ &= \mathrm{span}(\varphi(A)).\end{aligned}$$

(b) CLAIM. If φ is an isomorphism, then A is a basis of V if and only if $\varphi(A)$ is a basis of W.

Proof. Let A be a basis of V. Then $\varphi(A)$ is a spanning set of W by the first part of the exercise; it remains to show that $\varphi(A)$ is linearly independent. Suppose that there are n different vectors $v_1, \ldots, v_n \in A$ and numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $\lambda\varphi(v_1) + \cdots + \lambda\varphi(v_n) = 0$.

Then by linearity of φ we have $\varphi(\lambda_1 v_1 + \cdots + \lambda_n v_n) = 0$; i.e.

$$\lambda_1 v_1 + \cdots + \lambda_n v_n \in \ker(\varphi).$$

Since φ is injective this means that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. By linear independence of A, it follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, which concludes the proof that $\varphi(A)$ is a basis of W; i.e., the "if" direction of the claim.

The "only if" direction can be reduced to the "if" direction by exchanging the roles of V and W and replacing φ by its inverse φ^{-1}, which is also an isomorphism. ■

5. Let V be a vector space and $A \subset V$. Let $\mathcal{W}$ be the set of all subspaces W of V for which $A \subset W$.

CLAIM. $\mathrm{span}(A) = \bigcap_{W \in \mathcal{W}} W.$

Proof. Let $W \in \mathcal{W}$; i.e., let W be a subspace of V with $A \subset W$. It follows from the subspace criterion that any linear combination of elements in A must also belong to W. In other words, we have $\mathrm{span}(A) \subset W$. By definition, the intersection over all such subspaces W thus contains $\mathrm{span}(A)$:

$$\mathrm{span}(A) \subset \bigcap_{W \in \mathcal{W}} W.$$

For the opposite inclusion, let us show that $\mathrm{span}(A)$ is a subspace of V. Indeed, let $v = \lambda_1 v_1 + \cdots + \mu_n v_n$ and $w = \mu_1 w_1 + \cdots + \mu_n w_n$ be linear combinations of vectors in A. Then clearly $\nu v + \varrho w$ can also be written as such a linear combination; i.e. $\nu v + \varrho w \in \mathrm{span}(A)$ for all $v, w \in \mathrm{span}(A)$. So $\mathrm{span}(A)$ is itself a subspace of V containing A, and thus an element of $\mathcal{W}$. In other words,

$$\mathrm{span}(A) \subset \bigcap_{W \in \mathcal{W}} W \subset \mathrm{span}(A),$$

and we are done. ■