Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Problem Sheet 1

1. (a) Let $V := \mathbb{R}^3$, and $W := \{(x,y,z) : 2x - y = z\}$.

   Claim. W is a subspace of V.

   Proof. We verify the conditions (S1), (S2) and (S3) of the subspace criterion (Proposition 1.1.4). The zero element of $\mathbb{R}^3$, $(0,0,0)$, clearly belongs to W, so (S1) is satisfied. If $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2) \in W$ and $\lambda \in \mathbb{R}$, then

   $$2(x_1 + x_2) - (y_1 + y_2) = (2x_1 - y_1) + (2x_2 - y_2) = z_1 + z_2 \quad \text{and}$$
   $$2(\lambda x_1) - (\lambda y_1) = \lambda(2x_1 - y_1) = \lambda z_1.$$

   Thus $v_1 + v_2$ and λv_1 belong to W by definition. So (S2) and (S3) are satisfied, as required. ■

   (b) Let $V := \text{Pol}(\mathbb{R})$ and $W := \{f \in V : f(0) \cdot f(1) = 0\}$.

   Claim. W is not a subspace of V.

   Proof. Consider the functions $f_1(x) := x - 1$ and $f_2(x) := x$. Then $f_1, f_2 \in W$, but $(f_1 + f_2)(x) = 2x - 1$, and thus

   $$(f_1 + f_2)(0) \cdot (f_1 + f_2)(1) = (-1) \cdot 1 = -1 \neq 0.$$

   So property (S2) is not satisfied. ■

   (c) Let $V := \mathbb{R}^{3\times 3}$ and let $W \subset V$ be the space of *symmetric* matrices.

   Claim. W is a subspace of V.

   Proof. Properties (S1)-(S3) are easily verified. ■

   (d) Let $V := \mathbb{C}^2$ and $W := \mathbb{R}^2$.

   Claim. W is *not* a subspace of V (as a complex vector space).

   Proof. The point $(1,0)$ belongs to W, but $i \cdot (1,0) = (i,0) \notin W$. So condition (S3) is violated. ■

2. In each of the following cases, show that W_1 and W_2 are subspaces of V. Also determine the subspace $W_1 + W_2$ and decide whether $V = W_1 \oplus W_2$.

   (a) Let $V := \text{Pol}(\mathbb{R})$, let W_1 consist of polynomials with zero constant term and W_2 of those with zero linear term.

   Claim. W_1 and W_2 are subspaces of V with $W_1 + W_2 = V$. However, V is not the direct sum of W.

Proof. The subspace criterion is easily verified just as in the first problem, and we omit the calculations here.

To show that $V \subset W_1 + W_2$ (the other inclusion is trivial), let $f : x \mapsto \sum_{k=0}^{n} a_k x^k$ be an arbitrary element of V. Then $f_1 : x \mapsto \sum_{k=1}^{n} a_k x^k$ is an element of W_1, and $f_2 : x \mapsto a_0$ is an element of W_2. Clearly $f = f_1 + f_2$, so $f \in W_1 + W_2$.

To see that V is not the direct sum of W_1 and W_2, we must show that $W_1 \cap W_2 \neq \{0\}$. To this end, it suffices to note that $g : x \mapsto x^3$ is a nonzero element of both W_1 and W_2. ■

(b) Let $V := \mathbb{R}^3$, $W_1 := \{(x, 0, z) : x, z \in \mathbb{R}\}$ and $W_2 := \{0, y, 0) : y \in \mathbb{R}\}$.

Claim. W_1 and W_2 are subspaces of V, and $V = W_1 \oplus W_2$.

Proof. Again, we omit the verification of the subspace criteria for W_1 and W_2.

It is easy to check, as in the first example, that $W_1 + W_2 = \mathbb{R}^3$. Furthermore, clearly $W_1 \cap W_2 = \{0\}$, so $V = W_1 \oplus W_2$. ■

(c) Let $V := \mathbb{R}^{2\times 2}$, let W_1 consists of those matrices whose bottom row is $0\,0$, and let W_2 consist of those whose left column is $\begin{matrix}0\\0\end{matrix}$.

Claim. W_1 and W_2 are subspaces of V, and

$$W_1 + W_2 = \{(a_{ij}) : a_{21} = 0\} \neq V.$$

(In particular, V is not the direct sum of W_1 and W_2.

Proof. Similar as above. ■

3. (a) Claim. the function $F : \mathbb{R}^3 \to \mathbb{R}^2; (x, y, z) \mapsto (y - 1, x + y + z)$ is not linear or injective. However, F is surjective.

Proof. Set $v := (1, 0, 0)$ and $\lambda := 0$. Then

$$F(\lambda \cdot v) = F(0) = (-1, 0) \neq (0, 0) = \lambda \cdot F(v).$$

So F is not linear. Also,

$$F(0, 1, -1) = (0, 0) = F(-1, 1, 0),$$

so F is not injective. To prove that F is surjective, let $(x, y) \in \mathbb{R}^2$. Then

$$F(-1, x + 1, y - x) = (x, y),$$

so $(x, y) \in \mathrm{Im}(F)$.

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let I_h denote the range of h, i.e., $I_h := h(\mathbb{R}) = \{h(x) : x \in \mathbb{R}\}$. We define

$$F : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}; f \mapsto f \circ h.$$

Claim. F is a linear function, and $\ker(F) = \{f : \mathbb{R} \to \mathbb{R}, f(x) = 0 \text{ for all } x \in I_h\}$. In particular, F is injective if and only if h is surjective.

Furthermore, F is surjective if and only if h is injective. In particular, F is an isomorphism if and only if h is bijective.

Proof. Let $f, g \in \mathbb{R}^{\mathbb{R}}$, and let $\lambda \in \mathbb{R}$. Then, for every $x \in \mathbb{R}$,

$$\begin{aligned} (F(f+g))(x) &= (f+g)(h(x)) = \\ &= f(h(x)) + g(h(x)) = (F(f) + F(g))(x) \quad \text{and} \\ (F(\lambda f))(x) &= (\lambda \cdot f)(h(x)) \\ &= \lambda \cdot f(h(x)) = \lambda \cdot (F(f))(x). \end{aligned}$$

So $F(f+g) = F(f) + F(g)$ and $F(\lambda f) = \lambda F(f)$, and thus F is linear.

The claim about the kernel is easily verified. Finally, suppose that h is injective. Let $f \in \mathbb{R}^{\mathbb{R}}$, and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(y) := \begin{cases} f(x) & \text{if } y \in I_h \text{ and } x \text{ is the unique number satisfying } h(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(F(g))(x) = g(h(x)) = f(x)$$

for all $x \in \mathbb{R}$, so F is surjective.

On the other hand, suppose that h is not injective. Then there are $x_1 \neq x_2$ such that $h(x_1) = h(x_2)$. It follows that $f(x_1) = f(x_2)$ for all $f \in \operatorname{Im} F$. Thus e.g. the function $x \mapsto x$ does not belong to $\operatorname{Im} F$, and F is not surjective.

(c) Claim. The function $F : \mathrm{Pol}(\mathbb{R}) \to \mathbb{R}; f \mapsto f'(0)$ is linear and surjective. Its kernel consists of all polynomials with constant linear term, so F is not injective.

Proof. Linearity follows from the standard differentiability rules. If $\lambda \in \mathbb{R}$, then the polynomial $f : x \mapsto \lambda x$ has derivative $f'(0) = \lambda$ at the origin, so F is surjective.

If $f = \sum_{j=0}^{n} a_j x^j$ is an arbitrary element of $\mathrm{Pol}(\mathbb{R})$, then

$$f'(0) = \sum_{j=1}^{n} j a_j 0^{j-1} = a_1,$$

so $f'(0) = 0$ if and only if $a_1 = 0$, as claimed. ■

(d) Claim. The map $F : \mathbb{C} \to \mathbb{C} : x + iy \mapsto x + 2iy$ (considered as a function from the *complex* vector space $\mathbb{C}$ to itself) is not linear, but injective and bijective.

Proof. We have

$$F(i \cdot 1) = F(i) = 2i \neq i = i \cdot F(1),$$

so F is not linear. Injectivity and bijectivity are easy and are left to the reader. ■

4. Let V be a vector space, and let $v \in V$.

(a) Claim. If w and w' are elements of V with $v + w = v + w'$, then $w = w'$. (In particular, the neutral element 0 in (V3) and the inverse element $-v$ in (V4) are unique.)

Proof. We have

$$\begin{aligned} w &\underset{\text{(V3)}}{=} w + 0 \\ &\underset{\text{(V2)}}{=} 0 + w \\ &\underset{\text{(V4)}}{=} (v + (-v)) + w \\ &\underset{\text{(V2)}}{=} (-v + v) + w \\ &\underset{\text{(V1)}}{=} -v + (v + w) \\ &\underset{\text{assumption}}{=} -v + (v + w') \\ &\underset{\text{(V1)}}{=} (-v + v) + w' \\ &\underset{\text{(V2)}}{=} (v + (-v)) + w' \\ &\underset{\text{(V4)}}{=} 0 + w' \\ &\underset{\text{(V2)}}{=} w' + 0 \\ &\underset{\text{(V3)}}{=} w'. \end{aligned}$$

■

(b) Claim. $0 \cdot v = 0$.

Proof. We have

$$\begin{aligned} v + 0 &\underset{\text{(V3)}}{=} v \\ &\underset{\text{(V6)}}{=} 1 \cdot v \\ &= (1 + 0) \cdot v \\ &\underset{\text{(V7)}}{=} 1 \cdot v + 0 \cdot v \\ &\underset{\text{(V6)}}{=} v + 0 \cdot v. \end{aligned}$$

The claim follows from the first part of the exercise.

(c) CLAIM. $(-1) \cdot v = -v$.

Proof. The proof follows the same lines as the previous one. ■

5. CLAIM. There is a vector space V and a subspace $W \neq V$ of V such that V and W are isomorphic.

Proof. Let $V := \text{Pol}(\mathbb{R})$ and let W consist of all maps with zero constant part. The function

$$F : W \to V, f \mapsto f'$$

is an isomorphism between W and V, as is readily verified. ■