Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Problem Sheet 6

Solutions should be handed in on **Monday, March 6th**.
Note that the first two questions were rolled over from the last exercise sheet. If you already handed these in for Sheet 5, there is no need to do so again!

1. Let $V = \mathrm{Pol}_2(\mathbb{R})$ be the vector space of polynomials in x of degree at most 2, with real coefficients. Given that the map $f : V \times V \to \mathbf{R}$ defined by

$$f\big(\vartheta(x), \varphi(x)\big) = \int_0^1 \vartheta(x)\varphi(x)\mathrm{d}x.$$

is a symmetric bilinear form on V, find the matrix of f with respect to the basis $e_1 = x^2, e_2 = x, e_3 = 1$ for V.

[Remember that, when f is any bilinear form on any vector space V, and A is the matrix of f relative to a basis $\{e_1, e_2, \dots e_n\}$, then the i, j-th entry of A will be $f(e_i, e_j)$].

2. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f\big((x_1, y_1), (x_2, y_2)\big) = 2x_1x_2 - 3x_1y_2 + y_1y_2.$$

   (a) Find the matrix A of f relative to the basis (u_1, u_2), where $u_1 = (1, 0), u_2 = (1, 1)$.

   (b) Find the matrix B of f relative to the basis (v_1, v_2), where $v_1 = (2, 1), v_2 = (1, -1)$.

   (c) Find the change of basis matrix P from (u_1, u_2) to (v_1, v_2) and show that $B = P^T AP$.

3. Consider the symmetric 2×2 matrix

$$A := \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A, and hence verify that A has two linearly independent eigenvectors and that these can be chosen to be orthogonal.

(b) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

(c) Consider the symmetric bilinear form

$$f((x_1, y_1), (x_2, y_2)) := 5x_1x_2 + x_1y_2 + y_1x_2 + 2y_1y_2.$$

What is the matrix B of f with respect to the standard basis?

(d) Find an orthogonal change of basis which diagonalizes f.

(e) Use the "matrix method" to diagonalize f.

4. Consider the symmetric bilinear form

$$f((x_1, y_1, z_1), (x_2, y_2, z_2)) := x_1x_2 + y_1y_2 + z_1z_2 - 5x_1z_2 - 5z_1x_2$$

on $\mathbb{R}^3$. Find the rank and signature of f by the following two methods:

(a) Use the "matrix method" to diagonalize f.

(b) Find the eigenvalues of f.

Also find an orthogonal change of basis which diagonalizes f.