Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Problem Sheet 5

Solutions should be handed in on **Monday, February 27th**.

1. Let $V := \mathbb{R}^3$, and define bases B and C of V by
$$B := ((1,1,1),(0,1,1),(0,1,2)) \quad \text{and} \quad C := ((1,2,2),(1,0,-1),(0,0,1)).$$
Let E denote the standard basis of $\mathbb{R}^3$.
   (a) Compute the change-of-basis matrix P from the E to B, and the change-of-basis matrix Q from E to C.
   (b) Compute the change-of-basis matrix R from B to C (by writing the elements of C as linear combinations of B).
   (c) Verify that $R = P^{-1}Q$.

2. Consider the matrix
$$A = \begin{pmatrix} -1 & -3 & -2 & -2 \\ 1 & 3 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 3 & 4 & 3 & 4 \end{pmatrix}.$$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find an invertible matrix P such that $B = P^{-1}AP$ is an upper triangular matrix.

   Voluntary exercise: find a matrix P such that B is in Jordan normal form.

3. For each of the following matrices A, find an invertible matrix P such that $B = P^{-1}AP$ is in Jordan normal form.
   (a) $A = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 2 \end{pmatrix}$.
   (b) $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$.
   (c) $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \\ 1 & -1 & 1 \end{pmatrix}$.

4. Let $V = \mathrm{Pol}_2(\mathbb{R})$ be the vector space of polynomials in x of degree at most 2, with real coefficients. Given that the map $f : V \times V \to \mathbf{R}$ defined by

$$f\big(\vartheta(x), \varphi(x)\big) = \int_0^1 \vartheta(x)\varphi(x)\mathrm{d}x.$$

is a symmetric bilinear form on V, find the matrix of f with respect to the basis $e_1 = x^2, e_2 = x, e_3 = 1$ for V.

[Remember that, when f is any bilinear form on any vector space V, and A is the matrix of f relative to a basis $\{e_1, e_2, \dots e_n\}$, then the i, j-th entry of A will be $f(e_i, e_j)$].

5. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f\big((x_1, y_1), (x_2, y_2)\big) = 2x_1x_2 - 3x_1y_2 + y_1y_2.$$

   (a) Find the matrix A of f relative to the basis (u_1, u_2), where $u_1 = (1, 0), u_2 = (1, 1)$.
   (b) Find the matrix B of f relative to the basis (v_1, v_2), where $v_1 = (2, 1), v_2 = (1, -1)$.
   (c) Find the change of basis matrix P from (u_1, u_2) to (v_1, v_2) and show that $B = P^T AP$.