

**Linear Algebra, Geometry and Groups (MATH244)**  
**Problem Sheet 4**

Solutions should be handed in on **Monday, February 20th**.

1. Let  $V := \mathbb{R}^4$  and  $W := \mathbb{R}^3$ . Define bases

$$B := ((1, 2, 4, 1), (0, 1, 0, -1), (1, 2, 1, 4), (1, -1, 1, -1))$$

of  $V$  and  $C := ((-1, 3, 5), (2, -3, -1), (1, 3, 10))$  of  $W$ .

- (a) Find  $P$ , the change of basis matrix from the standard basis of  $\mathbb{R}^4$  to  $B$ , and  $Q$ , the change of basis matrix from the standard basis of  $\mathbb{R}^3$  to  $C$ .  
 (b) Compute  $Q^{-1}$ .  
 (c) Calculate the matrix of the linear map

$$(x, y, z, w) \mapsto (y + w, z - x, x + y + z + w)$$

with respect to  $B$  and  $C$ .

2. Set  $V := \mathbb{R}^{2 \times 2}$  and  $W := \mathbb{R}^3$ . Define a linear map  $\varphi$  by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (a - d, b + c, 2c).$$

- (a) Find  $A$ , the matrix of  $\varphi$  with respect to the basis

$$B := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of  $V$  and the standard basis of  $\mathbb{R}^3$ .

- (b) Let  $C$  be the basis  $\{(2, 0, 1), (1, 0, 1), (0, 1, 1)\}$  of  $\mathbb{R}^3$ . Calculate the matrix of  $\varphi$  with respect to  $B$  and  $C$ .  
 (c) Find bases  $\tilde{B}$  of  $V$  and  $\tilde{C}$  of  $W$  such that the matrix representation of  $\varphi$  with respect to  $\tilde{B}$  and  $\tilde{C}$  has the standard “partial identity” form.

3. Let  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear map corresponding to reflection in the line  $y = kx$  (where  $k \neq 0$  is a fixed constant). Show that  $v_1 = (1, k)$  and  $v_2 = (-k, 1)$  are eigenvectors of  $\varphi$ . Find a diagonal matrix representing  $\varphi$ .

4. For each of the following linear maps  $\varphi : V \rightarrow V$ , find the eigenvalues and eigenvectors of  $\varphi$ , and hence decide whether  $\varphi$  is diagonalizable.

(a)  $V := \mathbb{R}^2$ ,  $\varphi(x, y) := (-3x - 5y, 2x + 3y)$ .

(b)  $V := \text{Pol}_3(\mathbb{R})$ ,  $\varphi : V \rightarrow V$  is defined by

$$\varphi(a + bx + cx^2 + dx^3) = d + cx + bx^2 + ax^3$$

(c)  $V := \mathbb{C}^2$ ;

$$\varphi(z_1, z_2) := (-3z_1 - 5z_2, 2z_1 + 3z_2).$$

5. Let  $V = M_2(\mathbf{R})$ , the vector space of  $2 \times 2$  matrices with real entries. Define  $\varphi : V \rightarrow V$  by  $\varphi(A) = A^T$ . What are the eigenvalues and eigenvectors of  $\varphi$ ?