Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Problem Sheet 3

Solutions should be handed in on **Monday, February 13th**.

1. Let $V = R^3$, and let $u_1 := (1, -1, 1)$, $u_2 := (3, -5, 1)$ and $u_3 := (2, -1, 3)$.
   (a) Determine $U := \text{span}(u_1, u_2, u_3)$. (In other words, given $(x, y, z) \in V$, determine when $(x, y, z) \in U$.
   (b) For each of $w_1 := (1, -3, -1)$, $w_2 := (2, -3, 1)$. and $w_3 := (3, -8, -2)$, decide whether $w_j \in U$.
   (c) Set $W := \text{span}(w_1, w_2, w_3)$ and show that $U = W$.

2. Let $V := \text{Pol}_3(\mathbb{R})$,
$$U := \{ax^3 + cx + d : a = c\} \quad \text{and}$$
$$W := \{ax^3 + bx^2 + d : 2a + b = 2d\}.$$
   Find a basis for and the dimension of $U, W, U \cap W$ and $U + W$.

3. Let $V := W := \mathbb{R}^2$, and define $\varphi : V \to W$ by
$$\varphi(x, y) := (9y - x, 5y - x).$$
   (a) Show that φ is linear, and find the rank and nullity of φ.
   (b) Find the matrix representation of φ (with respect to the standard basis $((1, 0), (0, 1))$ of $\mathbb{R}^2$).
   (c) Find the matrix representation of φ with respect to the standard basis $((1, 0), (0, 1))$ of V and the basis $((-1, -1), (9, 5))$ of W.
   (d) Find the matrix representation of φ, using the basis $((3, 1), (2, 1))$ of $\mathbb{R}^2$ as a basis for both V and W.

4. Let $V := \text{Pol}_2(\mathbb{R})$ and $W := \text{Pol}_3(\mathbb{R})$. Define a linear map $\varphi : V \to W$ by
$$\varphi(ax^2 + bx + c) := bx^3 + ax^2 + cx + a.$$
   Find the matrix of φ with respect to the basis $(x^2, x, 1)$ of V and the basis $(x^3, x^2, x, 1)$ of W. Also determine the rank and nullity of φ.

5. Consider the complex vector space $V := W := \mathbb{C}^2$ and define
$$\varphi : V \to W; \varphi(z_1, z_2) := (3z_1 - 4iz_2, -3z_2 - 2iz_1).$$
   Find the matrix representation of φ with respect to the basis $((i, 1), (2, -i))$ of $\mathbb{C}^2$.