Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Problem Sheet 2

Solutions should be handed in on **Monday, February 6th**.

1. In each of the following, A is a subset of the vector space V. Calculate $\text{span}(A)$ and determine whether A is linearly independent. Is A a basis of V?

   (a) $V := \mathbb{R}^3$, $A := \{(1,2,3),(4,5,6),(7,8,9)\}$.

   (b) $V := \mathbb{R}^{2\times 2}$ and

   $$A := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \right\}.$$

   (c) Let V denote $\mathbb{R}^{\mathbb{R}}$, the space of all functions $f : \mathbb{R} \to \mathbb{R}$. For every $a \in \mathbb{R}$, we define a function $f_a \in V$ by

   $$f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise.} \end{cases}$$

   Set $A := \{f_a : a \in \mathbb{R}\}$.

2. Let $V := \mathbb{R}^4$,

   $$U := \{(x,y,z,w) : x + y + z + w = 0\} \quad \text{and}$$
   $$V := \{(x,y,z,w) : x = z, y = -2w\}.$$

   Find a basis for and the dimension of $U, W, U \cap W$ and $U + W$.

3. In each of the following, find the rank and the nullity of φ, and hence decide whether φ is an isomorphism.

   (a) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3; (x,y,z) \mapsto (x - y + z, x + 2y + z, x + z)$.

   (b) $\varphi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}; A \mapsto A^T$.

4. Let V and W be vector spaces, let $A \subset V$, and let $\varphi : V \to W$ be linear. Show that $\varphi(\text{span}(A)) = \text{span}(\varphi(A))$. If φ is an isomorphism, show that A is a basis of V if and only if $\varphi(A)$ is a basis of W.

5. Let V be a vector space and $A \subset V$. Let $\mathcal{W}$ be the set of all subspaces W of V for which $A \subset W$. Prove (from the definition of $\text{span}(A) \subset V$) that

   $$\text{span}(A) = \bigcap_{W \in \mathcal{W}} W.$$