Dr. L. Rempe Spring Semester 2006

Linear Algebra, Geometry and Groups (MATH244)
Problem Sheet 1

Solutions should be handed in on **Thursday, February 2nd**.

1. For each of the following, decide whether W is a subspace of V.

   (a) $V := \mathbb{R}^3$, and $W := \{(x, y, z) : 2x - y = z\}$.

   (b) V is $\text{Pol}(\mathbb{R})$, the vector space of polynomials with real coefficients. W consists of all polynomials $f : \mathbb{R} \to \mathbb{R}$ for which $f(0) \cdot f(1) = 0$.

   (c) V is $\mathbb{R}^{3\times3}$, the space of real 3-by-3 matrices. W is the subset of V consisting of *symmetric* matrices; i.e. matrices $A = (a_{ij})_{1\leq i,j\leq 3}$ with $a_{ij} = a_{ji}$ for all i, j.

   (d) V is the *complex* vector space $\mathbb{C}^2$. $W := \mathbb{R}^2$ (i.e., W is the subset of vectors $(z, w) \in \mathbb{C}^2$ for which $\text{Im}(z) = \text{Im}(w) = 0$).

2. In each of the following cases, show that W_1 and W_2 are subspaces of V. Also determine the subspace $W_1 + W_2$ and decide whether $V = W_1 \oplus W_2$.

   (a) V is the space $\text{Pol}(\mathbb{R})$ of polynomials with real coefficients, i.e. functions of the form
   $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$
   W_1 consists of all polynomials with zero constant term, i.e. $a_0 = 0$. W_2 consists of all polynomials with zero linear term, i.e. $a_1 = 0$.

   (b) $V := \mathbb{R}^3$, W_1 is the xz-plane, and W_2 is the y-axis.

   (c) $V := \mathbb{R}^{2\times2}$, W_1 consists of those matrices whose bottom row is 0 0, and W_2 of those whose left column is $\begin{matrix}0\\0\end{matrix}$.

3. In each of the following, is the function F linear? Injective? Surjective? An isomorphism? If F is linear, also compute $\ker(F)$.

   (a) $F : \mathbb{R}^3 \to \mathbb{R}^2; (x, y, z) \mapsto (y - 1, x + y + z)$.

   (b) Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and define
   $$F : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}; f \mapsto f \circ h.$$

   (c) $F : \text{Pol}(\mathbb{R}) \to \mathbb{R}; f \mapsto f'(0)$. (Here $f'(0)$ is the derivative of f at 0, as usual.)

   (d) $F : \mathbb{C} \to \mathbb{C} : x + iy \mapsto x + 2iy$ (considered as a function from the *complex* vector space $\mathbb{C}$ to itself).

4. Let V be a vector space. Using (only) the axioms (V1) to (V8) introduced in the lecture, show that, for all $v \in V$:

   (a) If w and w' are elements of V with $v + w = v + w'$, then $w = w'$. (In particular, the neutral element 0 in (V3) and the inverse element $-v$ in (V4) are unique.)

   (b) $0 \cdot v = 0$.

   (c) $(-1) \cdot v = -v$.

5. Decide whether the following statement is true. (If so, prove it. If not, provide a counterexample.)

   Let V be a vector space and let W be a subspace of V. If V and W are isomorphic, then $W = V$.