

## Determining a span

Set  $V = \mathbb{R}^3$  and  $v_1 = (1, -2, -1)$ ,  $v_2 := (1, 2, 3)$ ,  $v_3 := (1, -10, -9)$ .

We want to *determine the span of these vectors*. In other words, given  $(x, y, z) \in \mathbb{R}^3$ , when is  $(x, y, z) \in \text{span}(v_1, v_2, v_3)$ ?

This leads to a system of linear equations:

$$\begin{array}{rcccccl} \lambda & + & \mu & + & \nu & = & x \\ -2\lambda & + & 2\mu & - & 10\nu & = & y \\ -\lambda & + & 3\mu & - & 9\nu & = & z. \end{array}$$

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## Reminder: Solving a system of linear equations

In general, we have a system of the form  $Ax = v$  (where  $A$  is an  $m$ -by- $n$  matrix,  $v$  is a given vector in  $\mathbb{R}^m$ , and  $x \in \mathbb{R}^n$  is the solution we are looking for).

We write this equation in matrix form

$$(A \mid v)$$

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$$\begin{aligned}\lambda + \mu + \nu &= x \\ -2\lambda + 2\mu - 10\nu &= y \\ -\lambda + 3\mu - 9\nu &= z.\end{aligned}$$

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In our example:

$$\begin{aligned}\left( \begin{array}{ccc|c} 1 & 1 & 1 & x \\ -2 & 2 & -10 & y \\ -1 & 3 & -9 & z \end{array} \right) &\longrightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 4 & -8 & y + 2x \\ 0 & 4 & -8 & z + x \end{array} \right) \\ &\longrightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 4 & -8 & y + 2x \\ 0 & 0 & 0 & z - (y + x) \end{array} \right)\end{aligned}$$

At this point, we have put our matrix into a form where (disregarding the last column)

The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.

In particular, the bottom rows of the matrix consist of all zeros, and our system of linear equations *has a solution if and only if the entry in the last column corresponding to any zero row is also zero*.

In other words,

$$\text{span}(v_1, v_2, v_3) = \{(x, y, z) \in \mathbb{R}^3 : z = y + x\}.$$

Also, the number of columns which do not contain the leading entry of any row give us the number of “free parameters” in the solution (i.e., the nullity of  $A$ ). The number of other columns — which is the same as the number of nonzero rows — determine the dimension of the image of  $A$ ; i.e., the rank of  $A$ .

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If we want to actually solve the system of linear equations, we can reduce further to echelon form:

$$\begin{aligned} \left( \begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 4 & -8 & y + 2x \\ 0 & 0 & 0 & z - (y + x) \end{array} \right) \\ \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & 3 & (2x - y)/4 \\ 0 & 1 & -2 & (y + 2x)/4 \\ 0 & 0 & 0 & z - (y + x) \end{array} \right). \end{aligned}$$

From this, we can read off the solution (letting the third variable  $\nu$  be arbitrary, as it corresponds to a column which does not contain the leading entry of any row):

$$\lambda = (2x - y)/4 - 3\nu; \quad \mu = (y + 2x)/4 + 2\nu/.$$

Recall that our system represented the equation

$$\lambda v_1 + \mu v_2 + \nu v_3 = (x, y, z),$$

where  $v_1 = (1, -2, -1)$ ,  $v_2 := (1, 2, 3)$ ,  $v_3 := (1, -10, -9)$ . As the number of nonzero rows is only 2, we know that these three vectors are linearly dependent.

Setting  $\nu = 0$  and  $(x, y, z) := v_3$ , we see that

$$v_3 = 3v_1 - 2v_2.$$

### 1.3.1. Definition.

Let  $V$  and  $W$  be  $n$ - and  $m$ -dimensional vector spaces, respectively, and let  $B = \{v_1, \dots, v_n\}$  and  $C = \{w_1, \dots, w_m\}$  be bases of  $V$  resp.  $W$ .

If  $\varphi : V \rightarrow W$  is a linear map, then the matrix

$$A := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

defined by  $\varphi(v_j) = a_{11}w_1 + \dots + a_{m1}w_m$  is called the **matrix representation of  $\varphi$**  with respect to the bases  $B$  and  $C$ .

### 1.3.3. Proposition (Rank of a linear map).

Let  $\varphi : V \rightarrow W$  be a linear map of finite-dimensional vector spaces  $V$  and  $W$  of dimension  $n$  and  $m$ . Let  $A \in \mathbb{K}^{m \times n}$  be a matrix representation of  $\varphi$  with respect to some bases of  $V$  and  $W$ . Then

$$\begin{aligned} \text{rank}(\varphi) &= \text{number of l.i. columns of } A \\ &= \text{number of l.i. rows of } A. \end{aligned}$$

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$$\tilde{v}_j = a_{1j}v_1 + \dots + a_{nj}v_n.$$

The  $n \times n$ -matrix

$$P := \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

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### 1.3.6. Observation.

If  $\varphi : V \rightarrow W$  is linear,  $B$  and  $\tilde{B}$  are bases of  $V$  and  $C$  and  $\tilde{C}$  are bases of  $W$ , then

$$\tilde{A} = Q^{-1}AP,$$

where

- $A$  and  $\tilde{A}$  are the matrices of  $\varphi$  with respect to  $B$  and  $\tilde{B}$ , respectively,
- $P \in \mathbb{K}^{n \times n}$  is the change of basis matrix from  $B$  to  $\tilde{B}$ , and
- $Q \in \mathbb{K}^{m \times m}$  is the change of basis matrix from  $C$  to  $\tilde{C}$ .

For practical purposes this is most useful when we want to convert several matrices to a new basis (otherwise, it might be just as quick to do the calculation directly).

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**1.3.7. Examples.** 1. Consider again  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $\varphi(x, y) = (2y, 3y - x)$ . Let  $B$  be the standard basis of  $\mathbb{R}^2$ , and let  $\tilde{B} = ((2, 1), (1, 1))$  of  $\mathbb{R}^2$ . The change of basis matrix from  $B$  to  $\tilde{B}$  is

$$P = Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Its inverse is

$$Q^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

The matrix of  $\varphi$  with respect to  $B$  is

$$A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}.$$

So we have

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned}$$

(This is the matrix we already computed in an example.)

2. Let  $\varphi : \mathbb{R}^2 \rightarrow \text{Pol}_3(\mathbb{R})$  be defined by

$$\varphi(a, b) = (a - b)x^3 + ax + b.$$

$$B = ((1, 0), (0, 1)), \quad C := (x^3, x^2, x, 1), \quad \tilde{B} := ((1, -1), (-1, 2)), \quad \tilde{C} := ((x^3 - 1, x^2 - 1, x, 1)).$$

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}.$$

We compute

$$Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

So the matrix of  $\varphi$  with respect to  $\tilde{B}$  and  $\tilde{C}$  is

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ 0 & 0 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 \\ 0 & 0 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$