CHAPTER 0

Introduction

In mathematics, we often encounter specific objects with certain interesting structures (say that of calculations in the real or complex numbers or the geometry of 3-dimensional space). Upon such encounters, mathematicians frequently take an abstract point of view by formulating the basic properties of these structures as axioms and then studying their consequences. This procedure has several advantages, e.g.

(a) it is not necessary to repeatedly develop the same theory each time we encounter an object with similar properties, but can rather apply general theorems;
(b) we will be able to clearly identify the consequences of our axioms, advancing our understanding of the objects under consideration;
(c) we might discover interesting new mathematics along the way.

This idea, which should be familiar to those who have already encountered rings and fields, will play an important role in this course. We will define *vector spaces*, inspired by the Euclidean spaces $\mathbb{R}^n$, and study some general properties of these spaces and their structure-preserving ("linear") maps. We will then turn our attention to some geometric objects in whose study linear algebra proves very useful. Finally, we will study *groups*, another abstract class of objects which is pervasive in mathematics and beyond.

0.1. Operations

We will be mostly concerned with structures arising from *operations*, such as addition and multiplication (as for real numbers), scalar multiplication (in $\mathbb{R}^n$), etc.

For example, we will be in the following situation: given a set V and "rules" for addition and scalar multiplication, we want to say what it means for V to be a *vector space*.

To do this in a mathematically correct way, we should say what a "rule for addition" actually is. Let us think about this for a moment. Clearly addition is an operation which takes two elements $v, w \in V$ and produces another element (the sum $v + w$).

In other words, $+$ is a *function* from the set of pairs $V \times V$ into V, or in short:

$$+ : V \times V \to V.$$

So, in the following, if we write

"let $+ : V \times V \to V$ be an operation,"

this means that $+$ is some arbitrary rule for addition which is defined for any two elements of V, yielding another element of V.

Another example: what kind of object is the standard division in the real numbers? The answer is simple: we can divide any real number by any *nonzero* real number, so $\div$ is a function

$$\div : \mathbb{R} \times \underbrace{\mathbb{R}^*}_{=\mathbb{R}\setminus\{0\}} \to \mathbb{R}.$$

Of course, for any operation defined in this way, we will still keep writing $v + w$, $x \cdot y$, etc., instead of $+(v, w)$, $\cdot(x, y)$ etc. Also, when we define a product operation $\cdot$, we will often abbreviate ab instead of $a \cdot b$, as usual.