1.5.1. Definition.

Let V be a (real) vector space. A bilinear form *on V is an operation $f : V \times V \to \mathbb{R}$ (i.e., $f(v, w) \in \mathbb{R}$ for all $v, w \in V$) such that*

BL1 $f(\lambda v_1 + \mu v_2, w) = \lambda f(v_1, w) + \mu f(v_2, w)$ *and*

BL2 $f(v, \lambda w_1 + \mu w_2) = \lambda f(v, w_1) + \mu f(v, w_2)$.

A bilinear form is symmetric *if $f(x, y) = f(y, x)$ for all $x, y \in V$.*

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1.5.3. Definition.

Let V be a real vector space of dimension n, and let f be a bilinear form on V. If $B = (v_1, \ldots, v_n)$ is a basis of V, then the matrix

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

defined by $a_{ij} := f(v_i, v_j)$ is called the matrix representation of f *with respect to B.*

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1.5.5. Proposition (Change of Basis)**.**
Let $A \in \mathbb{R}^{n\times n}$.

(a) If A represents a bilinear form f with respect to a basis C_1, then the matrix of f wrt another basis C_2 is given by

$$B = P^T AP,$$

where P is the change-of-basis matrix from C_1 to C_2.

(b) If A represents a linear map with respect to a basis C_1, then the matrix of f wrt another basis C_2 is given by

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1.5.6. Theorem (Sylvester's Law of Inertia).

(a) *Let* A *be a symmetric* $n \times n$*-matrix. Then there is an orthogonal change-of-basis matrix* P *such that* $B = P^T AP$ *is a diagonal matrix.*

If P *is* any *invertible* $n\times n$*-matrix for which* $B = P^T AP$ *is diagonal, then the number* p *of positive and the number* n *of negative diagonal entries of* B *depend only on* A*. The number* $p+n$ *is called the* rank *of* A*, and the number* $p-n$ *is called the* signature *of* A.

(b) *Let* f *be a symmetric bilinear form on a finite-dimensional vector space* V*. Then there is a basis* C *of* V *such that the matrix* A *of* f *wrt* C *is diagonal. The number* p *of positive and the number* n *of negative diagonal entries of* A *depend only on* f*. The number* $p + n$ *is called the* rank *of* A*, and the number* $p - n$ *is called the* signature *of* A.