Towers of Finite Type Complex Analytic Maps

by

Adam Lawrence Epstein

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1993

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Date

Chair of Examining Committee

Date

Executive Officer

Fred Gardiner

Linda Keen

John Velling

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Towers of Finite Type Complex Analytic Maps

by

Adam Lawrence Epstein

Adviser: Professor Dennis Sullivan

An analytic map $f : W \to X$ of complex 1-manifolds is said to be of *finite type* if X is compact and f is an even cover near all but finitely many singular values in X; when $W \subseteq X$, the iterates of f constitute a one-generator dynamical system. We extend the three core principles of rational dynamics,

1. The density of repelling periodic points in the Julia set,

2. The standard classification of periodic components of the Fatou set,

3. The nonexistence of wandering components,

to finite type maps. Our results apply more generally to countably generated *towers* constructed inductively from finite type maps. Such towers can be *geometric limits* of sequences of one-generator systems.

Acknowledgements

It is difficult to contemplate how this project could have come to fruition without the assistance of numerous individuals. I extend heartfelt thanks to my adviser Dennis Sullivan for his inspiration and guidance over the past several years. The force of his mathematical character began to shape this work long before he and I met; although many specific instances come to mind, I am at a loss to enumerate them. I have profited additionally from interaction with Mitsuhiro Shishikura, Mikhail Lyubich, and Linda Keen in complex dynamics, and with Fred Gardiner in Teichmüller theory. Special thanks go out to Steve Kerckhoff for introducing me to geometric limits of Kleinian groups, and to Curt McMullen for his suggestion that I develop such a theory for rational maps. I am further indebted to John Velling for sharing his visceral understanding of potential theory. My friends and classmates helped to buoy my spirits more than they can imagine, and I wish them all well in their respective endeavors. In particular, I thank the Meister, Oscar Riquelme, for his hospitality and services in my eleventh hour of need, and Tim Gendron for his tolerance and good cheer when sharing an office with me would have tried anyone's patience. I will greatly miss the special atmosphere of the Mathematics Department at the Graduate Center, and will always remember Józef Dodziuk, Nils Tongring, and especially Reeva Goldsmith, for their generosity and warmth. Most of all, I thank my parents, Robert and Devora Epstein, for their unceasing support, both material and emotional, throughout the tortuous progress of this dissertation. Finally, I wish to express my gratitude to the National Science Foundation, the Alfred P. Sloan Foundation, and the Einstein Chair at CUNY for underwriting this research.

"Paris vaut bien une messe." - attribué à Henri de Navarre

Introduction

Our goal in this thesis and its sequels [?, ?] is to elucidate a new entry in Sullivan's "dictionary" between Kleinian groups and rational maps. Both subjects abound with questions concerning the relation between algebraic and quasiconformal deformations, and the role of parabolic bifurcations in the study of boundary phenomena in parameter space. In the theory of Kleinian groups, such issues are naturally addressed in the language of algebraic and geometric convergence. The space of representations of a fixed abstract group into $PSL_2\mathbf{C}$ can be topologized in terms of convergence of generators; when $\rho_k \to \rho$, we say that the image groups converge *algebraically.* On the other hand, the space of all subgroups carries a natural topology of *geometric convergence*, that is, convergence of subgroups in the sense of the Hausdorff metric. This compact topology is the appropriate arena for the discussion of Kleinian groups as dynamical systems. Given an algebraically convergent sequence we may pass to a geometrically convergent subsequence and compare the respective limits. The geometric limit always contains the algebraic limit, but is often larger; the additional dynamics encodes a good deal of the structure of the asymptotic geography of parameter space.

Some years ago, McMullen asked whether the natural complex analytic product structure on the space of quasi-Blaschke products, rational maps possessing two fully invariant attracting domains, extends continuously to the closure in the parameter space of degree d rational maps. This was known to be true for $d = 2$ [?], but believed to be false for $d \geq 3$. There were two grounds for this suspicion. Douady and Hubbard [?] had shown that for $d \geq 3$, the process of straightening polynomial-like maps is discontinuous. Moreover, Kerckhoff and Thurston [?] had shown that the natural product structure of the space of genus 2 quasi-Fuchsian groups does not extend continuously to the space of Möbius representations of the underlying abstract group. The latter argument involves a comparison of algebraic and geometric limits, as one quotient surface degenerates through pinching or twisting, while the other remains fixed in Teichmüller space. When the parallel degeneration is performed in a different "Bers slice", the new complex structure reflecting the change of slice may spill into the regions uniformizing the degenerated surface, possibly altering its moduli. A specific example where such a change occurs is then constructed and verified.

The Douady-Hubbard counterexample is similar in spirit: discontinuity occurs at a map possessing a parabolic point, and is exhibited by means of their technique of "Ecalle Cylinders". As was well undertood classically, parabolic fixed points give rise to local quotient cylinders. Douady showed how to construct cylinders for appropriate perturbations of a parabolic map f; in an appropriate sense, these cylinders converge to the quotient cylinders for f. High iterates of the perturbations may converge to a second, transcendental generator, defined on the parabolic basin and commuting with f; the two-generator system is a useful caricature of the dynamics of the perturbations. This approach yields a "first-order" understanding of the asymptotic structure of parameter space near parabolic bifurcations. Prototypical examples are Douady and Lavaurs' description of the limiting shape of "elephants" advancing into the main cusp of the Mandelbrot set, and Lavaurs' proof of non-local connectivity for the connectedness locus of degree 3 polynomials [?]. A refined analysis [?] leads to the expected negative resolution of McMullen's question concerning quasi-Blaschke products. Douady's construction may often be iterated: Shishikura's proof [?] that the boundary of the Mandelbrot set has Hausdorff dimension 2 employs three-generator systems in a similar fashion. McMullen observed that this method could be viewed as a rational maps analogue to the use of geometric limits in [?], and proposed the development of such a theory.

The sequel [?] to this work will put forward a framework for investigating geometric convergence in the setting of rational maps. The local nature of convergence of analytic maps and the transcendentality of limits force a sheaf-theoretic formulation for general conformal dynamical systems. A *conformal dynamical system* on a complex 1-manifold X is a collection of nowhere locally constant analytic maps from open subsets to X, closed under restriction and composition, and containing the identity; a sequence of systems $\mathcal{F}_k$ converges *geometrically* to the system $\mathcal{F}$, and we write $\mathcal{F}_k \to \mathcal{F}$, when:

1. If $f_{k_\ell} \in \mathcal{F}_{k_\ell}$ and $f_{k_\ell} \to g$, then $g \in \mathcal{G}$,
2. If $g \in \mathcal{G}$ there exist $f_k \in \mathcal{F}_k$ with $f_k \to g$.

By $f_k \to g$ we mean local uniform convergence of maps on convergent domains, that is, any compact subset of $\text{dom}(g)$ lies in $\text{dom}(f_k)$ for large k and vice-versa. The space of conformal dynamical systems on X is compact; the subspace of systems which are themselves closed under local uniform convergence is Hausdorff.

The geometric limit of a convergent sequence of 1-generator systems $\langle f_k \rangle$ associated to maps f_k consists of maps g expressible as local uniform limits $g = \lim_{k\to\infty} f_k^{n_k}$, where the n_k are non-negative and possibly unbounded. There is a Structure Theorem for such geometric limits under the assumption of *algebraic* convergence, that is local uniform convergence $f_k \to f$, where f is a map of finite type: an analytic map $f : W \to X$ is said to be of *finite type* if X is compact and f evenly covers neighborhoods of all but finitely many *singular values*, e.g. any rational map. The Structure Theorem, together with a converse Realization Theorem, yields a purely synthetic characterization of the conformal dynamical systems arising as geometric limits of 1-generator systems with finite type algebraic limit. Such systems are *towers* assembled through inductive application of the Ecalle cylinder construction.

A tower of height 1 is simply a system $\langle f \rangle$ where f is an analytic map on a complex 1-manifold X; the tower construction terminates here unless f possesses parabolic cycles. By the Fatou Flower Theorem, each such cycle has an associated cluster of quotient cylinders, one for each petal. The *interior* cylinders are quotients of the various parabolic basins; the *exterior* cylinders are local quotients of the intervening zones. We compactify the cylinders to spheres by adjoining "North and South poles", and consider isomorphisms, respecting these poles, from interior cylinders to exterior cylinders. In the case of interest, these *transit maps* respect each cluster and obey simple combinatorial rules of *admissibility*.

A transit map Φ lifts to a isomorphism between the universal covers of the interior and exterior cylinders. Each lift $\tilde{\Phi}$ gives rise to a transcendental analytic map $g = \chi \circ \tilde{\Phi} \circ \varpi$ from the corresponding parabolic basin to X. Here, ϖ is the small-orbit projection from the basin to the universal cover of the interior cylinder, and χ is the reverse projection from the universal cover of the exterior cylinder to X. By construction, g commutes with f. The smallest conformal dynamical system $\mathcal{F}$ containing f and the maps g for the various lifts of Φ is a tower of height 2.

The Julia and Fatou sets of f project down to the exterior cylinders, and there is an induced map E from the image of the parabolic basins to the interior cylinders. If f is a map of finite type, then E inherits this property. The map $\Phi \cup E$ on the collection of cylinders determines a new 1-generator system which may itself possess parabolic cycles. In this case the above construction may be repeated, yielding a height 2 tower on the cylinders and, upon lifting, a height 3 tower on X. Continuing in this fashion, one may asssemble towers of any finite height; an infinite height tower is the ascending union of a compatible sequence of finite height towers of unbounded height. The *base* of a tower $\mathcal{F}$ is the map f from which the construction originated; if f is a finite type map, then $\mathcal{F}$ is said to be a tower of finite type. A tower is *admissible* if all transit maps in the construction obey the admissibility rules..

The Structure Theorem asserts that a geometric limit of 1-generator systems with finite type algebraic limit f is an admissible *augmented* tower $\mathcal{G}$ with base f: that is, a tower $\mathcal{F}$ with a bit of additional dynamics supported on the smallest invariant set $\Xi(\mathcal{F})$ containing the lifts to X of all rotation domains of maps arising in the construction of $\mathcal{F}$. The Realization Theorem asserts that, up to this negligible extra dynamics, every admissible tower arises as a geometric limit of such an algebraically convergent sequence. The proof of the Structure Theorem has two main ingredients. On general grounds, the geometric limit of an algebraically convergent sequence possesses a natural *hierarchical* structure. An inductive argument, making use of the extension to the general case of the discussion in [?, ?, ?] for simple parabolic cycles, yields an admissible augmented tower forming the "initial" part of the geometric limit; this discussion will appear in [?]. Any "transfinite" dynamics beyond this augmented tower lives on Fatou domains of infinite height relative to the underlying tower: here, the *height* of an open set U is the least $n \in \mathbf{N} \cup \{\infty\}$ such that U supports maps added at each stage prior to n in the tower construction. We show that infinite height domains must wander, but that finite type towers have no wandering domains; the augmented tower therefore accounts for the entire geometric limit.

The bulk of this thesis consists of the extension, to finite type maps and more generally finite type towers, of the basic structure theory of rational maps:

1. The density of repelling periodic points in the Julia set;
2. The standard classification of periodic components of the Fatou set;
3. The nonexistence of wandering domains.

The general philosophy is to extend results first to finite type maps and then, through fairly simple inductive arguments, to towers. Enough of the Fatou-Julia theory carries over to address the first two points without any major difficulty. New proofs required for the *typical* maps which overflow their domain are often simpler and yield more information. Our treatment of the third point follows Sullivan's use of quasiconformal deformations [?]. As the finite dimensionality of the algebraic deformation space of rational maps has no direct analogue in the general setting, we must construct a finite dimensional parameter space out of "pure thought" and the raw material of Teichmüller theory. Our main result is the following:

Central Finiteness Theorem *Let $\mathcal{F}$ be a finite type tower. Then $Teich^+(\mathcal{F})$ is a finite dimensional complex manifold.*

Here $Teich^+(\mathcal{F})$ consists of isotopy classes of $\mathcal{F}$-invariant complex structures supported on the domain of the base.

In Chapter 1, we develop the necessary background from Riemann surface theory to discuss finite type maps and effect our constructions later on. We present the classical tools in the context of hyperbolic geometry, in particular, a simple but crucial metric comparison principle. After a brief discussion of ideal boundaries and the theory of quasiconformal maps, we indulge in an admittedly idiosyncratic "functorially correct" presentation of Teichmüller Theory. The cornerstone of our constructions is the existence of Möbius equivariant extensions of circle homeomorphisms, as applied in [?] to obtain conformally natural isotopies. These considerations are applied to prove the first version of the Injection Principle. The chapter ends with a discussion of the Contraction Principle, stated and proved in the language of quadratic differentials, for analytic maps between Teichmüller spaces.

In Chapter 2, we pursue a painstaking but ultimately soft discussion of iterated maps, hierarchical conformal dynamical systems, the Ecalle cylinder construction, and finally towers. The main result is that infinite height Fatou components must wander. For use in the next chapter, we introduce the class of complete maps and towers.

In Chapter 3, we extend the Fatou-Julia theory to finite type maps and towers. Our main tool is a covering property of finite type and associated maps near their domain boundaries. We prove the density of repelling points in the Julia set and the classification of fixed Fatou components, and conclude with a discussion of the measure alternative for strongly geometrically finite maps.

In Chapter 4, we take up the discussion of quasiconformal deformations and the construction of finite dimensional parameter spaces. Dynamical versions of the Injection and Contraction Principles lead to a proof of the Central Finiteness Theorem; along the way, we establish the conformal rigidity of geometrically finite maps. The No Wandering Domains Theorem follows, and we deduce the nonexistence of infinite height Fatou components for finite type towers.

Contents

List of Figures

List of Figures

Chapter 1

Foundations

1.1 Riemann Surfaces

We begin by fixing some terminology and notation. A *2-manifold* is a Hausdorff space X equipped with a maximal atlas of coordinate neighborhoods $\psi_\alpha : W_\alpha \to \mathbf{R}^2 = \mathbf{C}$; a *surface* is a connected 2-manifold. The choice of a a sub-atlas with holomorphic overlaps specifies a *complex structure* on X. A *complex* 1-*manifold* is a 2-manifold with a choice of complex structure; when other complex structures are considered, we shall term this choice the *fiducial* structure. Every complex 1-manifold has a mirror-image X^*. A *Riemann surface* is a connected complex 1-manifold. A map $f : X \to Y$ between complex 1-manifolds is *analytic* if its expression in all local coordinate systems in holomorphic.

Let $f : X \to Y$ be an an analytic map of complex 1-manidolds. We shall say that an open set $V \subseteq Y$ is *evenly covered* if $f^{-1}(V)$ is homeomorphic to the product of V with a discrete space, possibly empty. For connected X

and Y, we say that f is a *covering space* when every point of Y has an evenly covered neighborhood. On the other hand, a point of Y is a *singular value* if no neighborhood is evenly covered.

Recall that $x \in X$ is a *critical point* when the derivative Df_x vanishes. The images of such points constitute the set of *critical values* $C(f)$. We say that $y \in Y$ is an *asymptotic value* if $f \circ \gamma(t) \to y$ along some path γ in X tending to infinity, and write $A(f)$ for the set of such values. Then $C(f) \cup A(f) \subseteq S(f)$ by simple path-lifting arguments. Note as well that any boundary point of the image $f(X)$ is automatically a singular value, and thus $S(f_{|U}) = S(f)$ for open dense $U \subseteq X$.

Consider analytic maps $f : V \to Y$, $g : W \to Z$, where X, Y, and Z are complex 1-manifolds, $V \subseteq X$, and $W \subseteq Y$. By elementary covering space theory,

$$S(g \circ f) = S(g_{|W \cap f(V)}) \cup g(S(f)). \tag{1.1}$$

In particular, $S(g) \subseteq S(g \circ f)$ when $f(V)$ is open and dense in W.

There is a one-to-one correspondence between equivalence classes of pointed covering spaces and conjugacy classes of subgroups of the fundamental group. In particular, there is a simply connected *universal covering space* $\tilde{X}$, and $X \cong \tilde{X}/\pi_1(X)$.

Uniformization Theorem. *Up to conformal equivalence, there are three simply connected Riemann surfaces: the disc* $\boldsymbol{\Delta}$, *The plane* $\mathbf{C}$, *and the sphere* $\hat{\mathbf{C}}$.

It follows easily from the Uniformization Theorem every Riemann surface is second countable; this fact was established eariler by Rado.

Lemma 1 *Let $\{f_{\alpha\beta} : X_\alpha \to X_\beta : \alpha \leq \beta\}$ be a direct system of covering maps of Riemann surfaces. Assume that at least one X_α has a non-abelian fundamental group. Then $\lim_{\to} X_\alpha$ exists as a Riemann surface X_∞. Moreover, if $\pi_1(X_\infty)$ is finitely generated, then the maps $f_{\alpha\beta}$ are eventually bijective.*

Proof: See [37]. □

The underlying smooth structure on a complex 1-manifold determines a σ-finite Lebesgue measure class. Nonconstant analytic maps are locally invertible away from the isolated *critical points* where the derivative vanishes, so measure 0 is preserved under images and pre-images. Moreover, we may discuss the tensor bundles $\kappa^{m,n} = \kappa^m \otimes \overline{\kappa}^n$, where $\kappa = T^*X$ is the canonical line bundle. The type (m,n) tensors, measurable sections of $\kappa^{m,n}$ form complex linear spaces $M(X : \kappa^{m,n})$; each space carries a topology of local uniform convergence. The tensor product on sections gives

$$Ten(X) = \bigoplus_{m,n} M(X : \kappa^{m,n})$$

the structure of a bigraded algebra over the ring $M(X : \kappa^{0,0})$ of complex-valued measurable functions, complex conjugation interchanging $M(X : \kappa^{m,n})$ and $M(X : \kappa^{n,m})$. A tensor $\tau \in M(X : \kappa^{m,n})$ vanishing on a set of measure 0 has a multiplicative inverse $\frac{1}{\tau} \in M(X : \kappa^{-m,-n})$. Furthermore, for $\tau \in M(X : \kappa^{m,n})$ with $m + n = 2\ell$, $|\tau| = \sqrt{\tau\overline{\tau}} \in M(X : \kappa^{\ell,\ell})$ is well-defined.

An analytic map $f : X \to Y$ induces a pull-back homomorphism of graded algebras $f^* : Ten(Y) \to Ten(X)$ given in local coordinates by

$$f^* \tau(z) dz^m d\bar{z}^n = \tau(f(z)) f'(z)^m \overline{f'(z)}^n dz^m d\bar{z}^n.$$

As the line bundles $\kappa^{m,0}$ are holomorphic, we may speak of type $(m, 0)$ holomorphic and meromorphic tensors and their local vanishing orders $\mathrm{ord}_z \tau$, where $\mathrm{ord}_z \tau = -k$ at a pole of order k; the pole is *simple* if $k = 1$. By convention, $\mathrm{ord}_z \tau = -\infty$ if τ is not meromorphic at z. If τ is meromorphic at $f(z)$ then $f^*\tau$ is meromorphic at z, the local orders related by the formula

$$ord_z f^* \tau = d \ \mathrm{ord}_{f(z)} \tau + m(d-1) \tag{1.2}$$

where $d = \deg_z f$. Conversely, if $f^*\tau$ is meromorphic at z, then τ is holomorphic in a punctured neighborhood of $f(z)$; we claim that τ is meromorphic at $f(z)$. As f^* is an algebra homomorphism, we may assume via 4.2 that τ is a function and $f^*\tau$ is holomorphic. Then τ is locally bounded, so $f(z)$ is a removable singularity. Consequently, 1.2 remains valid at points of order $-\infty$.

Metric Comparisons

Each of the three simply connected Riemann surfaces carries a constant curvature geometry. In terms of their densities relative to the standard coordinate on $\mathbf{C}$, these metrics are:

- Spherical metric on $\hat{\mathbf{C}}$: $\rho(z) = \frac{2}{1+|z|^2}|dz|$,

- Flat metric on $\mathbf{C}$: $\rho(z) = |dz|$,
- Hyperbolic metric on $\mathbf{\Delta}$: $\rho(z) = \frac{2}{1-|z|^2}|dz|$.

The hyperbolic metric on $\mathbf{\Delta}$ is invariant under the conformal automorphism group, and consequently descends to the *Poincaré* metric d_X on any Riemann surface X it covers. Such Riemann surfaces are termed *hyperbolic*, and represent the typical case. Inspection of $Aut(\mathbf{C})$ and $Aut(\hat{\mathbf{C}})$ reveals that, up to conformal equivalence, the non-hyperbolic Riemann surfaces are precisely $\hat{\mathbf{C}}$, $\mathbf{C}$, the punctured plane $\mathbf{C}^*$, and the tori; all but the first have universal cover $\mathbf{C}$, and hence carry flat metrics. We shall call a complex 1-manifold *hyperbolic* if all of its components are hyperbolic Riemann surfaces, its infinitessimal Poincaré metric specified componentwise.

A key property of the Poincaré metric is its behavior under analytic maps. This is the content of Schwarz' Lemma as expressed by Pick.

Schwarz' Lemma. *Let $f : X \to Y$ be an analytic map of hyperbolic complex 1-manifolds. Then the derivative of f with respect to the X-Poincaré metric in the domain and Y-Poincaré metric in the range is everywhere less than or equal to 1 in modulus; consequently, analytic maps do not increase Poincaré distance. Moreover, if equality holds at one point it holds everywhere, hence f is a local isometry and a covering space.*

For hyperbolic subsets U and V of a complex 1-manifold X, we denote $\eta_V^U : U \cap V \to \mathbf{R}$ the ratio of the infinitessimal Poincaré metric on U to that on V. As a consequence of Schwarz' Lemma we have the fundamental metric comparison:

Proposition 1 *Let X be a hyperbolic Riemann surface, $K \subseteq X$ closed, $U = X - K$. Then*

A. $\eta_X^U(x) \geq \alpha(d_X(x,K))$ *for* $d_X(x,K)$ *sufficiently small, where* $\alpha(t) \sim \frac{1}{t|\log t|}$ *as* $t \to 0$.

B. $\eta_X^U(x) \leq \beta(d_X(x,K))$ *for* $d_X(x,K)$ *sufficiently large, where* $\beta(t) \to 1$ *as* $t \to \infty$.

Proof: We may assume U connected. Fixing $x \in U$, choose $y \in K$ with $d_X(x,y) = d_x(x,k) = R(x)$. Let $\pi : (\mathbf{\Delta}, 0) \to (X, y)$ be a universal cover, $\tilde{x} \in \pi^{-1}(x)$ with $d_X(\tilde{x}, 0) = R(x)$, $\tilde{U}$ the connected component of $\pi^{-1}(U)$ containing $\tilde{x}$. Note that $|\tilde{x}| = r(x)$ where $R(x) = \log\frac{1+r(x)}{1-r(x)}$. Successive applications of the Schwarz-Pick Lemma yield

$$\eta_X^U(x) = \eta_{\mathbf{\Delta}}^{\tilde{U}}(\tilde{x}) \geq \eta_{\mathbf{\Delta}}^{\mathbf{\Delta}^*}(\tilde{x}) = \frac{1-|r(x)|^2}{2r(x)|\log r(x)|} \sim \frac{1}{R(x)|\log R(x)|},$$

hence the estimate in **A.**, and, writing $B(x) = \{z \in \mathbf{\Delta} : d(\tilde{x}, z) < R(x)\}$,

$$\eta_X^U(x) = \eta_{\mathbf{\Delta}}^{\tilde{U}}(\tilde{x}) \leq \eta_{\mathbf{\Delta}}^{B(x)}(\tilde{x}) = \eta_{\mathbf{\Delta}}^{\mathbf{\Delta}_{r(x)}}(0) = \frac{1}{r(x)},$$

whence **B.** □

Corollary 1 *Let X be a hyperbolic Riemann surface, $S \subseteq X$ closed and discrete, $U \subset\subset X$ open, C a component of ∂U. Assume C is not a single point. Then* $\eta_{X-S}^U(x) \to \infty$ *as* $x \to C$.

Proof: Fixing $\epsilon > 0$, consider

$$N_\epsilon = \{x \in X - S : d_{X-S}(x, C - S) < \epsilon\}.$$

By definition, N_ϵ is a punctured neighborhood of x for $x \in C - S$. For each of the finitely many $y \in C \cap S$, choose a conformal disc $D \subseteq X$ with $y \in D$, $D^* = D - \{y\} \subseteq X - S$, and coordinate $\psi : (D, y) \to (\mathbf{\Delta}, 0)$. By assumption, C intersects $\Gamma_r = \{x \in D : |\psi(x)| = r\}$ for sufficiently small r, hence as $x \to y$,

$$d_{X-S}(x, C - S) \leq d_{D^*}(x, D^* \cap C) \leq \frac{1}{2}\ell_{\mathbf{\Delta}^*}(\Gamma_{|\psi(x)|}) = \frac{\pi}{|\log|\psi(x)||} \to 0.$$

It follows that N_ϵ contains a punctured neighborhood of every point of $C \cap S$, hence $N_\epsilon \cup C$ is a neighborhood of C, and thus $d_{X-S}(x, C-S) \to 0$ as $x \to C$; by Lemma 1, $\eta^U_{X-S}(x) \to \infty$. □

Definition. Let X and Y be complex 1-manifolds. A family F of analytic maps $f : X \to Y$ is *normal* if every sequence of maps in F has a subsequence which either converges locally uniformly, or else tends locally uniformly to infinity, on any component of X.

Montel's Theorem. *Let X and Y be complex 1-manifolds. If Y is hyperbolic then the family of analytic maps $f : X \to Y$ is normal.*

Picard's Theorem. *Let X be a Riemann surface, $f : \mathbf{\Delta}^* \to X$ analytic with an essential singularity at 0. Then X is the sphere or a torus, and f*

assumes every value in X, with up to two exceptions when $X = \hat{\mathbf{C}}$, in any neighborhood of 0.

Proof: Consider the maps $f_k(z) = f(\frac{z}{2^k})$ defined on punctured discs increasing to $\mathbf{C}^*$. Suppose f omits $F \subseteq X$ with $X - F$ as above. In view of Montel's Theorem, the f_k form a normal family on annuli increasing to $\mathbf{C}^*$, hence some subsequence converges normally on $\mathbf{C}^*$ to $g \in \mathcal{O}(\mathbf{C}^*, X)$. Moreover, $g \equiv x \in F$ or $g : \mathbf{C}^* \to X - F$, hence constant. Thus, $z = 0$ is a removable singularity. □

1.2 Intrinsic Boundaries

Definition. A complex 1-manifold X is *unpunctured* if every analytic embedding $j : \mathbf{\Delta}^* \hookrightarrow X$ extends to $\mathbf{\Delta}$.

Let W and X be complex 1-manifolds, $j : W \hookrightarrow X$ an analytic embedding. We shall refer to the isolated points of $X - j(W)$ as *punctures*; these form a countable set, empty if W is unpunctured. Given a complex 1-manifold X, we construct a complex 1-manifold X^+ by glueing in a disc along every embedding $\mathbf{\Delta}^* \hookrightarrow X$. The canonical injection $j : X \hookrightarrow X^+$ is analytic, and $X^+ - j(X)$ is discrete.

It follows from Alexander Duality that a closed totally disconnected subset of a topological surface has connected complement. See [31] for a geometric treatment, [17] for a discussion from the viewpoint of dimension theory. We will only require this fact in the case of a countable set.

Lemma 2 *Let X be a complex 1-manifold. There exist an unpunctured complex 1-manifold $\hat{X}$ and an analytic embedding $\iota : X \to \hat{X}$, unique up to conformal equivalence, such that $\hat{X} - \iota(X)$ is countable.*

Proof: See revisions. □

By Riemann's Theorem on removable singularities, every analytic embedding $W \hookrightarrow X$ extends to an analytic embedding $\hat{W} \hookrightarrow \hat{X}$. For any analytic map $f : W \to X$ there is a largest subset of $\hat{W}$ to which f extends analytically as a map into $\hat{X}$. We denote this extension $\hat{f}$. If Y is unpunctured and $W \subseteq Y$, we may regard $\hat{f}$ as defined on a subset of Y.

A *bordered complex 1-manifold* is a real 2-manifold with boundary $X \cup \beta$ equipped with a maximal atlas of charts in the closed upper-half plane, where the overlap maps are holomorphic in the open upper-half plane and real-analytic on the real line. Note that the interior of a bordered Riemann surface with non-empty border is necessarily hyperbolic. We define the mirror image bordered 1-manifold $\star X \cup \beta$ as in the unbordered case; we may *double* along β to form the complex 1-manifold $\mathbf{D}_\beta X$ with anticonformal involution $\star_X$.

The structure of a bordered complex 1-manifold enables us to discuss analytic maps. A non-constant analytic map $f : X \cup \beta X \to Y \cup \beta Y$ of bordered Riemann surfaces must map X into Y. Conversely, it follows from Schwarz' Reflection Principle that an analytic map $f : X \to Y$ such that $f(x_k)$ tends to βY for any sequence x_k tending to βX extends analytically to a map betwen the doubles. In particular, if f is injective then so is the extension to βX. Consequently, we may glue any two bordered complex 1-manifolds $X \cup \beta_1 X$, $X \cup \beta_2 X$ along their common interior to form a new bordered manifold. The bordered complex 1-manifolds with interior X thus form a direct system. It follows that the direct limit is a bordered complex 1-manifold $X \cup \partial^I X$ and we refer to its border as the *ideal boundary* of X; the *double* of X is the complex 1-manifold $\mathbf{D}X$ obtained by doubling along $\partial^I X$.

Lemma 3 *Let X be a hyperbolic Riemann surface. A choice of universal cover $p : \Delta \to X$ with deck group Γ determines an analytic isomorphism*

$$(\overline{\Delta} - \Lambda(\Gamma))/\Gamma \xrightarrow{\cong} X \cup \partial^I X.$$

Proof: By definition of the limit set, p extends to a universal covering $\overline{\Delta} - \Lambda(\Gamma) \to (\overline{\Delta} - \Lambda(\Gamma))/\Lambda(\Gamma)$ of bordered surfaces, and thus $\overline{\Delta} - \Lambda(\Gamma)$ injects canonically into $X \cup \partial^I X$. Conversely, p extends to a universal cover $\tilde{X}^I \to X \cup \partial^I X$. As $\overline{\Delta}$ is compact, any sequence tending to the border of $\tilde{X}^I$ must tend to S^1, hence $\tilde{X}^I \hookrightarrow \Delta \cup \mathbf{S}^1$ analytically. The image of the border must lie in $\mathbf{S}^1 - \Lambda(\Gamma)$, and thus $X \cup \partial^I X \hookrightarrow (\overline{\Delta} - \Lambda(\Gamma))/\Gamma$. □

Intuitively, the removal of a closed totally disconnected set should not create any new ideal boundary.

Lemma 4 *Let E be a closed totally disconnected subset of a complex 1-msnifold X. Then the inclusion $X - E \hookrightarrow X$ extends continuously to an analytic injection $(X - E) \cup \partial^I (X - E) \hookrightarrow X \cup \partial^I X$.*

Proof: Let $U \cup \beta$ be a half-disc neighborhood of a point in $\partial^I(X - E)$. It suffices to show that glueing β to X along U produces a bordered complex 1-manifold with interior X; we need only show that a sequence in U tending to β cannot accumulate in X. Let Y be the end compactification of X. Then

$$L = \{y \in Y : \ x_k \to y \text{ for some } x_k \in U \text{ tending to } \beta\}$$

is a compact connected set. Furthermore, if L has two or more points, then $L \cap X$ contains a non-degenerate continuum. As $L \cap X$ lies in the totally disconnected set E, it follows that L is a single point y, necessarily an end of X. □

Many facts about compact Riemann surfaces are valid more generally for surfaces with empty ideal boundary. We may often reduce to the latter case with the aid of the following:

Lemma 5 *Let X be a complex 1-manifold. Then* $\mathbf{D}X$ *has empty ideal boundary.*

Proof: Assume $\partial^I \mathbf{D}X \neq \emptyset$. Let $\alpha = \star_{\mathbf{D}X}$, β the symmetric extension of $\star_X$. Then α and β are anticonformal involutions of $\mathbf{DD}X$, and $\alpha \circ \beta$ is a conformal automorphism interchanging $\mathbf{D}X$ and $\star\mathbf{D}X$. The fixed point sets of α and β are the real-analytic 1-manifolds $A = \partial^I X$ and $B = \overline{\partial^I X} \cup \star\overline{\partial^I X}$, the closure taken in $\mathbf{DD}X$. As $\alpha \circ \beta$ fixes any point in $A \cap B$, it follows that $A \cap B$ is discrete. On the other hand, $A \subseteq B$ by the definition of $\partial^I X$, and we have a contradiction as B cannot be discrete. □

Alternatively, if $\partial^I X \neq \emptyset$ then $\Lambda(\Gamma)$ is a Cantor set; by Lemma 4, the covering space $\hat{\mathbf{C}} - \Lambda(\Gamma)$ of $\mathbf{D}X$ has empty ideal boundary, and we may conclude that $\partial^I \mathbf{D}X = \emptyset$.

1.3 Quasiconformal Maps

We assume familiarity with the basic theory of quasiconformal mapping in the plane, specifically:

- The geometric definition in terms of modules of quadrilaterals, the compactness of uniformly quasiconformal families, and extension to the ideal boundary;
- The conformality of 1-quasiconformal maps, and Bers' Lemma on extensions by the identity;
- The almost everywhere differentiability, existence and properties of complex dilatations, and the Ahlfors-Bers Theorem.

See [23] for details.

Recall that for a locally compact topological space X, composition is continuous in the space of self-maps $C(X)$ equipped with the compact-open topology. A homotopy can be viewed as a path in $C(X)$. Consider the group of self-homeomorphisms $Homeo(X) \subseteq C(X)$ in the induced topology. If X is compact, or locally compact and locally connected, $Homeo(X)$ is a topological group [4]. It follows from elementary point-set topology that

a continuous bijection of a compact Hausdorff space is a homeomorphism. However, there exist rather well-behaved non-compact spaces for which this is false; for finite-dimensional manifolds, the assertion may be recovered via the Jordan-Brouwer theorems.

Let X be a complex 1-manifold. The subgroup $QC(X) \subseteq Homeo(X)$ consisting of quasiconformal self-homeomorphisms of X is a topological group. We will consider various subgroups $\mathcal{Q}(X)$ and their respective identity path components $\mathcal{Q}_0(X)$. Recalling that every $\phi \in QC(X)$ extends continuously to the ideal boundary of X, we consider, for closed $A \subseteq X \cup \partial^I X$,

$$QC(X, A) = \{\phi \in QC(X) : \phi_{|A} = Id_A\}.$$

Given analytic $f : X \to Y$ and a closed set $E \subseteq Y$ containing $S(f)$, we may *lift* any $\phi \in QC_0(Y, E \cup \partial^I Y)$ componentwise on $Y - E$. In view of Bers' Lemma, the extension by the identity is a quasiconformal homeomorphism $f^! \phi \in QC_0(X, (X - f^{-1}(E)) \cup \partial^I X)$.

We will make great use of Douady and Earle's construction of Möbius equivariant extensions to the disc of circle homeomorphisms. As shown in [10], the existence of such extensions is used to produce Möbius equivariant isotopies to Id_Δ of quasiconformal maps with boundary extension Id_{S^1}. We shall comment more on such *functorial isotopies* in the revisions.

Consider a quasiconformal map $\phi \in QC(X)$ preserving an open subset $U \subseteq X$ and fixing ∂U. In many applications it is necessary to pass back and forth between isotopies rel ∂U and isotopies rel $\partial^I U$. By work of Earle and

McMullen [10], a bounded isotopy rel $\partial^I U$ extends to an isotopy rel ∂U; we include their proof below. Through harmonic measure considerations, they show conversely that a homotopy rel ∂U extends to an homotopy rel $\partial^I U$. Strictly speaking, they prove these assertions for bounded plane domains, but the general case follows by easy covering space arguments. We present a more geometric proof of the weaker assertion equating the existence of such isotopies.

Let X be a Riemann surface, U_k a sequence of connected open subsets with base points u_k. Suppose U is open and connected, and $u_k \to u \in U$ with U. We say (U_k, u_k) converges to (U, u) *in the sense of Carathéodory* if every compact connected subset of U lies in U_k for k sufficiently large and conversely. Let W be a Riemann surface, $f_k : W \to X$ analytic with non-constant limit f; by Rouché's Theorem and local compactness, $(f_k(W), f_k(w))$ converges to $(f(W), f(w))$ for any choice of $w \in W$. Carathéo dory proved the converse for Riemann maps of plane domains. We require a generalization to covering spaces. It is easy to see that if (U_k, u_k) converges to (U, u) with U hyperbolic, then U_k is hyperbolic for large k. Consequently, any sequence of universal covers $p_k : (\Delta, 0) \to (U_k, u_k)$ has a convergent subsequence.

Lemma 6 *Let X be a Riemann surface, U_k connected open subsets with base points u_k. Assume (U_k, u_k) converges to in the sense of Carathéodory to (U, u) with U hyperbolic. Then the limit of any convergent sequence of universal* *covers* $p_k : (\mathbf{\Delta}, 0) \to (U_k, u_k)$ *is a universal cover* $p : (\mathbf{\Delta}, 0) \to (U, u)$.

Proof: Let $D \subset\subset U$ be simply connected, B a component of $p^{-1}(D)$. By assumption, p_k evenly covers D for large k. Fixing $w \in B$, choose analytic $g_k : D \to \mathbf{\Delta}$ with $p_k \circ g_k = Id_D$ and $g_k(p_k(w)) = w$. Let g be the limit of a convergent subsequence. Then $p \circ g = Id_D$ and $g(p(w)) = w$. As B is connected, $p_{|B} = g^{-1} : B \to D$ is a homeomorphism; thus p evenly covers D. $\square$

Lemma 7 *Let* X *be a complex* 1*-manifold,* $\phi \in QC(X)$ *fixing a closed subset* A *containing* $\partial^I X$. *Then* $\phi \in QC_0(X, A)$ *if and only if* $\phi_{|U} \in QC_0(U, \partial^I U)$ *for every component* U *of* $X - A$.

Proof: We may assume without loss of generality that X is a hyperbolic Riemann surface; in view of Lemma 5 we may further assume $\partial^I X = \emptyset$. Every $\phi \in QC_0(X, A)$ preserves complementary components. Let $F_k \subseteq A$ be finite sets with limit A. By Lemma 4, $U_k = X - F_k$ is a hyperbolic Riemann surface with $\partial^I U_k = \emptyset$. Fix a component U of $X - A$, a point $x \in U$, and universal covers $p_k : (\mathbf{\Delta}, 0) \to (U_k, x)$. As U_k has empty ideal boundary, $\phi_{|U_k}$ has a lift $\psi_k \in QC(\mathbf{\Delta}, \mathbf{S}^1)$. The ψ_k are uniformly quasiconformal, so there is a subsequence ψ_{k_ℓ} converging uniformly on $\overline{\mathbf{\Delta}}$ to some $\psi \in QC(\mathbf{\Delta}, \mathbf{S}^1)$. In view of Lemma 6, we may assume that p_{k_ℓ} converges to a universal cover $p : (\mathbf{\Delta}, 0) \to (U, x)$. Then $\phi \circ p = p \circ \psi$ by the continuity of composition; consequently, $\phi_{|U} \in QC_0(U, \partial^I U)$.

Conversely, let $\phi : X \to X$ be a homeomorphism with $\phi_{|U} \in QC_0(U, \partial^I U)$ for every component U of $X - A$. Fix isotopies $\Xi(\phi_{|U})_t$ to the identity through maps of uniformly bounded dilatation, and define bijections $\Phi_t : X \to X$ by

$$\Phi_t(x) = \begin{cases} \Xi(\phi_{|U})_t & \text{for } x \text{ in a component } U \text{ of } X - A \\ x & \text{for } x \in \mathbf{A} \end{cases}$$

Points of $X - A$ move a uniformly bounded Poincaré distance under the isotopy. In view of Proposition 1, each Φ_t is continuous, so $\Phi_t \in QC(X, A)$ by Bers' Lemma. Consequently, $\phi \in QC_0(X, A)$. □

The existence of functorial isotopies allows the following reduction:

Lemma 8 *Let X be a Riemann surface, $A \subseteq X \cup \partial^I X$. Suppose $\phi \in QC(X)$ with $\mathbf{D}\phi \in QC_0(\mathbf{D}X, A \cup A^\star)$. Then $\phi \in QC_0(X, A)$.*

Proof: In view of Lemma 7, $\mathbf{D}\phi_{|U} \in QC_0(U, \partial^I U)$ for every component U of $\mathbf{D}X - A \cup A^\star$. Moreover, $\mathbf{D}\phi_{|U}$ commutes with the anti-conformal involution $\star_{|U}$ for any U intersecting $\partial^I X$. Using functorial isotopies in the argument above, we may construct an isotopy rel $A \cup A^\star$ preserving $\partial^I X$. □

A general surface homeomorphism isotopic to the identity rel each closed set in an ascending sequence need not be isotopic to the identity rel the limit. Fortunately, quasiconformal maps are better behaved. We shall prove the following folklore lemma in the revisions:

Lemma 9 *Let X be a Riemann surface, $E \subseteq X$ closed, ϕ an orientation-preserving homeomorphism fixing E pointwise. Then ϕ fixes each component of $X - E$.*

Proposition 2 *Let X be a complex 1-manifold, $A_k \subseteq X \cup \partial^I X$ closed subsets with limit A, $\phi_k \in QC_0(X, A_k)$ uniformly quasiconformal with limit ϕ. Then $\phi \in QC_0(X, A)$.*

Proof: We may assume without loss of generality that X is a hyperbolic Riemann surface, and that $\partial^I X = \emptyset$. By hypothesis, $\phi_{|A} = Id_A$. Fix finite $F_k \subseteq A_k$ with $F_k \to A$; then $U_k = X - F_k$ is a hyperbolic Riemann surface with $\partial^I U_k = \emptyset$. In view of Lemma 9, ϕ preserves components of $X - A$. As in the proof of Lemma 7, we conclude that $\phi_{|U} \in QC_0(U, \partial^I U)$ for every component U, hence $\phi \in QC_0(X, A)$. □

Corollary 2 *Let X be a complex 1-manifold, $A_\alpha \subseteq X \cup \partial^I X$ a direct system of closed subsets with limit A. Then $QC_0(X, A) = \bigcap_\alpha QC_0(X, A_\alpha)$.*

Proof: Clearly, $QC_0(X, A) \subseteq \bigcap_\alpha QC_0(X, A_\alpha)$. Let $x_k \in A = \overline{\bigcup_\alpha A_\alpha}$ be a dense sequence, $F_k = \{1, \ldots, k\}$. Each F_k lies in some A_α, and $F_k \to A$. By Proposition 2,

$$\bigcap_\alpha QC_0(X, A_\alpha) \subseteq \bigcap_{k=1}^{\infty} QC_0(X, F_k) \subseteq QC_0(X, A) \quad \square$$

1.4 Teichmüller Spaces

Teichmüller Theory studies the geometry of spaces of complex structures on a fixed topological object. Let X be a complex 1-manifold. The quantity

$$d(c_1, c_2) = \frac{1}{2} K_{c_1, c_2}(Id_X)$$

defines the distance between complex structures c_1 and c_2 on X. A complex structure c at finite distance from the fiducial structure is said to be *bounded.* The bounded structures form the metric space $\mathcal{C}(X)$. The Ahlfors-Bers bijection between $\mathcal{C}(X)$ and the unit ball in $L^\infty(X : \kappa^{-1,1})$ with its "hyperbolic metric"

$$\rho(\mu_1, \mu_2) = ??$$

is an isometry. By means of this identification, $\mathcal{C}(X)$ becomes a complex Banach manifold.

For closed $E \subseteq X$, let

$$Teich(X, E) = \mathcal{C}(X)/QC_0(X, E \cup \partial^I X)$$

as a topological space, $Teich(X) = Teich(X, \emptyset)$. By the compactness of uniformly quasiconformal maps and the fact that 1-quasiconformal maps are conformal, the distance on complex structures descends to *Teichmüller* metrics on the spaces $Teich(X, E)$. The fiducial structure specifies a base point in each Teichmüller space. While it is somewhat more elegant to discuss, in base point free terms, the Teichmüller space of a quasiconformal surface, the global nature of quasiconformality creates certain complications in the non-compact case. The presence of base-points in our formulation necessitates certain naturality considerations. For quasiconformal $\phi : X \to X_1$, the induced translation $\phi^* : \mathcal{C}(X_1) \to \mathcal{C}(X)$ is an isometry, descending to an isometric *allowable bijection*

$$\phi^\# : Teich(X_1, \phi(E)) \to Teich(X, E)$$

for each choice of closed $E \subseteq X$. The allowable self-bijections constitute the *modular group* $Mod(X, E)$ acting on $Teich(X, E)$.

The primative operations on complex structures give rise to continuous *structure* maps between the various Teichmüller spaces. Let X be a complex 1-manifold, $F \subseteq E$ closed subsets. As $QC_0(X, F \cup \partial^I X) \subseteq QC_0(X, E \cup \partial^I X)$, the identity on $\mathcal{C}(X)$ descends to a surjective *forgetful* map

$$Teich(X, E) \to Teich(X, F).$$

Let $f : W \to X$ be analytic, with W open in Y. For $c \in \mathcal{C}(Y)$, there is a unique $f^{\#}c \in \mathcal{C}(X)$ agreeing with $f^{*}c$ on W and the fiducial structure on $X - W$. Suppose the image of f intersects $S(f)$ in a set of measure 0, and let $E \subseteq X$ be a closed set containing $S(f)$. If $c_1, c_2 \in \mathcal{C}(Y)$, and $c_1 = \phi^{*}c_2$ with $\phi \in QC_0(Y, E \cup \partial^I Y)$, then $f^{\#}c_1 = (f^{!}\phi)^{*} f^{\#}c_2$. Consequently:

Lemma 10 *Let X and Y be complex 1-manifolds, $W \subseteq Y$ open. An analytic map $f : W \to X$ whose image intersects $S(f)$ in a set of measure 0 induces a continuous injection*

$$Teich(Y, E) \xrightarrow{f^{\#}} Teich(X, (X - W) \cup f^{-1}(E)).$$

The above constructions are functorial in that base points and compositions are respected. Furthermore, the forgetful maps

$$Teich(X, E) \to Teich(X, F)$$

are natural in the sense that

$$\begin{array}{ccc} Teich(X,E) & \stackrel{\phi^\#}{\longleftarrow} & Teich(X_1,\phi(E)) \\ \downarrow & & \downarrow \\ Teich(X,F) & \stackrel{\phi^\#}{\longleftarrow} & Teich(X_1,\phi(F)) \end{array}$$

commutes for every quasiconformal $\phi : X \to X_1$.

The maps $Teich(Y,E) \xrightarrow{f^\#} Teich(X, f^{-1}(E))$ are natural in the sense that

$$\begin{array}{ccc} Teich(Y,E) & \stackrel{\phi^\#}{\longleftarrow} & Teich(Y_1,\phi(E))) \\ f^\# \downarrow & & \downarrow f_1^\# \\ Teich(X,f^{-1}(E)) & \stackrel{\varphi^\#}{\longleftarrow} & Teich(X_1,f_1^{-1}(\phi(E))) \end{array}$$

commutes for every pair of quasiconformal maps $\phi : Y \to Y_1$, $\varphi : X \to X_1$, where ϕ is conformal on $Y - W$ and $f_1 \circ \varphi = \phi \circ f$.

Injectivity Principle

Our dynamical applications will involve various infinite processes on Teichmüller spaces. A direct system $\{E_\alpha\}$ of closed subsets of X with limit E determines an inverse system of forgetful maps and a corresponding canonical map

$$Teich(X,E) \to \varprojlim Teich(X,E_\alpha).$$

The injectivity of such maps will be a key ingredient of the construction in Chapter 3 of functorial deformation spaces of conformal dynamical systems. We first establish a simple property of quotients of group actions.

Lemma 11 *Let G a group acting without fixed points on a set X, $\{\Gamma_\alpha\}$ an inverse system of subgroups with intersection Γ. Then there is a canonical injection*

$$X/\Gamma \hookrightarrow \varprojlim X/\Gamma_\alpha$$

Proof: The quotient maps $X \to X/\Gamma_\alpha$ induce a canonical map

$$j : X \to \varprojlim X/H_\alpha.$$

If $j(x) = j(y)$, then $y = \gamma_\alpha(x)$ for some $\gamma_\alpha \in \Gamma_\alpha$. By hypothesis, the γ_α are equal, hence $y = \gamma(x)$ where $\gamma \in \Gamma$. Consequently, the induced map

$$X/\Gamma \to \varprojlim X/\Gamma_\alpha$$

is injective. □

Proposition 3 *Let X be a complex 1-manifold, $\{E_\alpha\}$ a direct system of closed subsets with limit E. Then the canonical map*

$$Teich(X, E) \to \varprojlim Teich(X, E_\alpha)$$

is injective.

Proof: Without loss of generality, we may assume X is hyperbolic, hence $QC_0(X, \partial^I X)$ acts on $\mathcal{C}(X)$ without fixed points. Injectivity follows from Lemma 11 and Corollary 2. □

Lemma 12 A. *Let $\{X_\alpha\}$ be a direct system of covering maps of Riemann surfaces. If $X_\infty = \varinjlim X_\alpha$ is a Riemann surface then*

$$\varprojlim Teich(X_\alpha) \cong Teich(X_\infty)$$

canonically; otherwise, $\varprojlim Teich(X_\alpha)$ *limit is either a point or homeomorphic to* **H**.

B. Let Σ be a semigroup of fixed point free self-coverings of a Riemann surface X, and denote by $Teich(X)^\Sigma$ the fixed point set of the induced action of Σ on $Teich(X)$. If X/Σ is a Riemann surface, then $Teich(X)^\Sigma$ is canonically homeomorphic to $Teich(X/\Sigma)$; otherwise, $Teich(X)^\Sigma$ is either a point or homeomorphic to $\mathbf{H}$.

Proof: See revisions. □

Thus, $\lim_{\leftarrow} Teich(X_\alpha)$ and $Teich(X)^\Sigma$ are always contractible complex Banach manifolds. In the special cases where the direct limit or quotient is not a Riemann surface, the homeomorphism to $\mathbf{H}$ is well-defined up to post-composition with a Möbius transformation; the revisions will contain an invariant formulation in terms of the Teichmüller space of a foliated annulus.

For measurable $A \subseteq X$ with $\partial A \subseteq E$, let $\mathcal{C}^A(X)$ consist of all structures $c \in \mathcal{C}(X)$ agreeing with the fiducial structure on $X - A$, and consider the space

$$Teich^A(X, E) = \mathcal{C}^A(X)/QC_0(X, E \cup \partial^I X).$$

By definition, $Teich^A(X, E)$ is a point when A has measure 0.

Lemma 13 *Let X be a complex 1-manifold, $E \subseteq X$ closed, $A \subseteq X$ with $\partial A \subseteq E$.*

A. $Teich^A(X, E)$ is canonically homeomorphic to $Teich(A, A \cap E)$ for open A.

B. $Teich^A(X, E)$ *is canonically homeomorphic to* $\mathcal{C}^A(X) \cong \mathcal{L}(A)$ *when* $A \subseteq E$.

The inclusion $\mathcal{C}^A(X) \hookrightarrow \mathcal{C}(X)$ determines a canonical continuous map $Teich(X, E) \to Teich^A(X, E)$. If A is open, the restriction map $\mathcal{C}(X) \to \mathcal{C}(A)$ determines a homeomorphism $Teich^A(X, E) \to Teich(A, A \cap E)$, while for $A \subseteq \partial E$, $Teich^A(X, E) \cong \mathcal{C}^A(X) \cong \mathcal{L}(A)$.

Consequently,

$$Teich(X, E) \xrightarrow{\cong} Teich(X - E) \times \mathcal{C}^E(X).$$

Lemma 14 *Let* X *be a complex 1-manifold,* $E \subseteq X$ *closed,* $\{A_k\}$ *a countable partition of* X *into measurable sets with* $\partial A_k \subseteq E$. *Then the canonical map*

$$Teich(X, E) \hookrightarrow \prod_{k=1}^{\infty} Teich^{A_k}(X, E)$$

is injective. If all but finitely many A_k *have measure 0, the canonical map is a homeomorphism.*

Proof: By the Ahlfors-Bers Theorem,

$$\mathcal{C}(X) \hookrightarrow \prod_{k=1}^{\infty} \mathcal{C}^{A_k}(X),$$

and

$$\mathcal{C}(X) \xrightarrow{\cong} \prod_{k=1}^{\infty} \mathcal{C}^{A_k}(X)$$

when all but finitely many A_k have measure 0. □

Complex Structure

Through the work of Ahlfors and Bers, the Teichmüller space of any Riemann surface carries a natural complex Banach manifold structure. This structure is crucial to our dynamical applications, though we will use it non-trivially only in the case of finite type surfaces. It is reasonable to inquire whether the more general spaces $Teich(X, E)$ are complex Banach manifolds. We will establish this fact through a brief recapitulation of the classical construction; see [13] and [32] for further details.

For any complex 1-manifold X, the identification $\mathcal{C}(X) \cong L^\infty(X : \kappa^{-1,1})_1$ turns $\mathcal{C}(X)$ into a complex Banach manifold; the complex structure is natural in the sense that allowable bijections $\mathcal{C}(X_1) \to \mathcal{C}(X)$ are analytic. The tangent space at the base point is $L^\infty(X : \kappa^{-1,1})$, canonically dual to $L^1(X : \kappa^{2,0})$ via the fundamental pairing

$$\langle q, \mu \rangle = \int_Y q\mu.$$

In this sense, we regard $L^1(X : \kappa^{2,0})$ as the cotangent space at the base point.

The complex structure on the Teichmüller space of a Riemann surface arises from considerations in the theory of univalent functions. A locally injective analytic map between domains on $\hat{\mathbf{C}}$ has a holomorphic *Schwarzian derivative*

$$\mathcal{S}f(z) = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2.$$

By virtue of the Cayley identity $\mathcal{S}(fg) = (\mathcal{S}f \circ g)g'^2 + \mathcal{S}g$, the Schwarzian derivative transforms under analytic coordinate changes as a type $(2,0)$-

tensor, or *quadratic differential*; moreover, $\mathcal{S}f = \mathcal{S}(M \circ f)$ for Möbius M. Furthermore, for injective f defined in the lower half-plane $\mathbf{L}$,

$$\sup_{z \in \mathbf{L}} \left| \frac{\mathcal{S}f(z)}{y^2} \right| < \frac{3}{2}$$

by work of Nehari.

The Poincaré metric on a hyperbolic Riemann surface Y determines an area form $\lambda \in M(Y : \kappa^{1,1})$. The complex linear injection

$$\alpha : M(Y^* : \kappa^{2,0}) \hookrightarrow M(Y : \kappa^{-1,1})$$

sending a quadratic differential q on the mirror image Y^* to the *harmonic* Beltrami differential $\frac{*q}{\lambda}$ on Y, induces an L^∞ norm on $M(Y^* : \kappa^{2,0})$; the holomorphic quadratic differentials of finite norm form a Banach space $B(Y^*)$. A covering space $\pi : Z \to Y$ determines an isometric inclusion $\pi^* B(Z^*) \hookrightarrow B(Y^*)$; if $\pi : \mathbf{H} \to Y$ is a universal cover, the image consists of the quadratic differentials on $\mathbf{L}$ which are pull-back invariant under the cover group.

A complex structure $c \in \mathcal{C}(\mathbf{H})$ extends by the fiducial structure on $\mathbf{L}$ to $\hat{c} \in \mathcal{C}(\hat{\mathbf{C}})$, and $\mathcal{S}\phi_{|\mathbf{L}}$, where $\phi : (\hat{\mathbf{C}}, \hat{c}) \to \hat{\mathbf{C}}$ is conformal, is a well-defined element of the ball $B(\mathbf{L})_{\frac{1}{2}}$. By virtue of the analytic parameter dependence of solutions to the Beltrami equation, the assignment $c \rightsquigarrow \mathcal{S}\phi_{|\mathbf{L}}$ determines an analytic map $\mathcal{C}(\mathbf{H}) \to B(\mathbf{L})$. We obtain corresponding maps $\mathcal{C}(Y) \to B(Y^*)$ through uniformization:

$$\begin{array}{ccc} \mathcal{C}(Y) & \hookrightarrow & \mathcal{C}(\mathbf{H}) \\ \downarrow & & \downarrow \\ B(Y^*) & \hookrightarrow & B(\mathbf{L}) \end{array}$$

By work of Bers, $\mathcal{C}(Y) \to B(Y^*)$ is a submersion, descending to an injective map

$$\beta : Teich(Y) \to B(Y^*)$$

with image in the ball $B(Y^*)_{\frac{3}{2}}$; in fact, the image is open by work of Ahlfors. Moreover, the image contains the ball $B(Y^*)_{\frac{1}{2}}$, and somewhat remarkably

$$B(Y^*)_{\frac{1}{2}} \xrightarrow{\alpha} L^\infty(Y : \kappa^{2,0})_1 \cong \mathcal{C}(Y)$$

is a analytic cross-section. We thereby obtain an analytic chart at the base point of $Teich(Y)$. By means of allowable bijections, we obtain charts at every point, and a standard computation reveals that the overlap maps are holomorphic. Thus, $Teich(Y)$ is a complex Banach manifold, the complex structure natural in the sense that allowable bijections are analytic. In view of the universality of the constants $\frac{1}{2}$ and $\frac{3}{2}$, the above considerations apply equally well to disconnected Y. Consequently, if E is a closed subset of a complex 1-manifold X, the canonical splitting

$$Teich(X, E) \cong Teich(X - E) \times \mathcal{L}(E)$$

determines a natural complex Banach manifold structure on $Teich(X, E)$.

Contraction Principle

Quadratic differentials enter more geometrically in the discussion of the infinitessimal structure of Teichmüller space. A quadratic differential $q \in M(X : \kappa^{2,0})$ determines an area form $|q| \in M(X : \kappa^{1,1})$ and, if q vanishes on a set of measure 0, a linefield $\frac{\bar{q}}{|q|} \in M(X : \kappa^{-1,1})$. Two quadratic differentials determine the same area form if and only if their quotient is a.e of modulus 1, the same linefield if and only if the quotient is a.e. positive.

The area form associated to a quadratic differential integrates to a measure on X. The total mass $||q|| = \int_X |q|$ gives a norm on $M(X : \kappa^{2,0})$ and the finite norm quadratic differentials form a Banach space $L^1(X : \kappa^{2,0})$. For closed $E \subseteq X$, let $Q(X, E)$ be the linear subspace of finite norm quadratic differentials which are holomorphic on $X - E$. By the mean value property, the local sup-norms on $X - E$ are bounded in terms of the L^1 norm [32]. Thus $Q(X, E)$ is a Banach space.

The tangent space at the base point is canonically dual to $Q(Y)$ via the fundamental pairing; moreover, If $\pi : Z \to Y$ is a covering space, the co-derivative of $\pi^\# : Teich(Y) \hookrightarrow Teich(Z)$ at the base point is $\pi_* : Q(Z) \to Q(Y)$. The cotangent space at the base point is canonically identified with $Q(X, E)$. Moreover, the co-derivative of

$$f^\# : Teich(Y, E) \hookrightarrow Teich(X, f^{-1}(E)),$$

where $f : X \to Y$ is analytic, $f^{-1}(S(f))$ has measure 0, and $S(f) \subseteq E$, is $f_* : Q(X, f^{-1}(E)) \to Q(Y, E)$. For $F \subseteq E$, the co-derivative of the forgetful map $Teich(X, E) \to Teich(X, F)$ is the isometric inclusion $Q(X, F) \hookrightarrow Q(X, E)$. The injectivity of the canonical map $Teich(X, E_\alpha) \to \lim_{\leftarrow} Teich(X, E)$ arising from a direct system of closed sets E_α with limit E has the infinitessimal analogue

$$Q(X,E) = \overline{\bigcup_{\alpha} Q(X,E_\alpha)}$$

which follows from an approximation theorem of Bers [13].

The norm of a quadratic differential is preserved under pull-back by analytic isomorphisms. More generally, let $f : X \to Y$ be an analytic map of Riemann surfaces. Up to a set of measure 0, $Y - S(f)$ is filled by countably many simply connected open sets: for example, each unit of a countable pants decomposition splits canonically into a pair of geodesic hexagons; alternatively, use any of the standard constructions of fundamental domains [13]. Moreover, if $Y - S(f)$ is connected, it lies entirely in the image of f, and thus $\|f^*q\| = (\deg f)\|q\|$ for $q \in M(X : \kappa^{2,0})$.

If $f : X \to Y$ is an analytic map of complex 1-manifolds, and $S(f)$ has measure 0, there is also a *push-forward* operator: for $q \in L^1(Z : \kappa^{2,0})$ where $Z \supseteq X$, the absolutely convergent sum

$$f_* q = \sum_g g^* q$$

over local inverse branches to f in $Y - S(f)$ defines a quadratic differential in $L^1(Y : \kappa^{2,0})$, and $\|f_* q\| \le \|q\|$. Moreover,

$$f_* : Q(Z,E) \to Q(Y, S(f) \cup \overline{f(E)}).$$

If $Y - S(f)$ is connected and $\deg f < \infty$, then

$$f_* f^* q = \sum_g g^* f^* q = \sum_g (f \circ g)^* q = \sum_g g^* q = (\deg f) q$$

for $q \in L^1(Y : \kappa^{2,0})$.

Lemma 15 *Let X be a Riemann surface, q_1, q_2 measurable quadratic differentials on X.*

A. *If q_1 and q_2 are L^1 and $||q_1|| + ||q_2|| = ||q_1 + q_2||$, then q_1,q_2, and $q_1 + q_2$ determine the same measurable linefield.*

B. *Suppose q_1 and q_2 are holomorphic on $X - E$, where $E \subseteq X$ is closed measure 0 set with connected complement. If q_1 and q_2 determine the same measurable linefield, then q_1 is a positive scalar multiple of q_2.*

Proof:

[A.] If not, the argument of the measurable function $h = \frac{q_1}{q_2}$ is uniformly bounded away from 0 on some positive measure set A. Thus,

$$\int_A |q_1 + q_2| < \int_A |q_1| + \int_A |q_2|$$

and $||q_1|| + ||q_2|| < ||q_1 + q_2||$.

[B.] By assumption, the measurable function $h = \frac{q_1}{q_2}$ satisfies

$$\frac{\overline{h(z)}}{|h(z)|} = 1 \text{ for a.e. } z \in W.$$

Consequently, h is almost everywhere real and positive. As h is meromorphic on $W - E$, it follows that h is a positive constant. □

The coherence expressed in Lemma 15 gives rise to a weak contraction principle sufficient to establish the non-existence of invariant quadratic differentials in the dynamical setting of Chapter 3. The latter principle will enter in the proof of the central finiteness theorem in two essential ways.

Lemma 16 *Let X and Y be Riemann surfaces, $f : X \to Y$ analytic, $S(f)$ totally disconnected of measure* 0. *Assume $q \in Q(X, E)$ is not identically zero. Then $||f_* q|| = ||q||$ if and only if $f^* f_* q = (\deg f) q$; in particular,* $\deg f < \infty$.

Proof: Suppose $||f_* q|| = ||q||$. Then

$$\sum_g ||g^* q|| = ||(f_* q)_{|U}|| = ||\sum_g g^* q||,$$

on any simply connected open $U \subseteq Y - S(f)$, the sums ranging over local inverses to f. By the first part of Lemma 15, $f_* q$ and any $g^* q$ determine the same linefield on U. For inverse branches g_1 and g_2, both $g_1^* q$ and $g_2^* q$ are holomorphic on the complement of $g_1^{-1}(E) \cup g_2^{-1}(E)$, hence positive scalar multiples by the second part of the Lemma; thus $f_* q$ and any $g^* q$ are positive scalar multiples on U. Consequently, $f^* f_* q$ is a locally constant positive multiple of q on $X - f^{-1}(S(f))$. But $f^{-1}(S(f))$ is a totally disconected measure 0 set. Thus, q and $f^* f_* q$ are positive scalar multiples. The conclusion and converse follow as $||f^* f_* q|| = (\deg f)||f_* q||$. □

Lemma 16 is a precursor to more powerful contraction principles for iteration on Teichmüller space - for example, in Thurston's topological characterization of rational maps among branched covers [9], McMullen's approach to the geometrization of 3-manifolds [27, 28], and Sullivan's work on renormalization [39]. See [29] for an account relating these three.

Quadratic differentials arise in the solution of the extremal problem of finding the most efficient quasiconformal map in a given homotopy class. By compactness, for quasiconformal $\phi : X \longrightarrow Y$ and closed $E \subseteq X$, the infinum

$$K[\phi] = \inf_{\psi \in [\phi]_E} K(\phi)$$

is always achieved by some ψ in the class of ϕ rel E. By contrast, the uniqueness and geometric description of extremals are rather deep issues.

A quasiconformal homeomorphism $\phi : X \rightarrow X_1$ of Riemann surfaces is a *Teichmüller map* if its Beltrami differential is a scalar multiple of the line-field determined by some L^1 holomorphic quadratic differential on X. More generally, if $E \subseteq X$ is a closed measure 0 set with connected complement, we will say ϕ is a Teichmüller map *rel E* when

$$\mu_\phi = t\frac{\overline{q}}{|q|}$$

where $q \in Q(X, E)$ is non-zero, and $0 < t < 1$; by Lemma 15, ϕ determines q uniquely up to a postive scalar.

Proceeding from Grötzsch's extremal length inequality [13], Teichmüller proved:

Teichmüller's Theorems: *Every homotopy class of quasiconformal maps of finite type Riemann surfaces has a unique representative of minimal dilatation. Moreover, the extremal map is either conformal or a Teichmüller map.*

Further work of Hamilton, Reich, and Strebel established that Teichmüller maps, when they exist, are always unique extremals [13]. However, extremals need not be unique. Strebel formulated a rather general criterion for unique extremality.

Definition. A quasiconformal map $\phi : X \to Y$ *relaxes at infinity* if there exist compact $L \subseteq X$ and quasiconformal $\varphi \in [\phi]$ with $K(\varphi_{|X-L}) < K(\phi)$.

Note that φ is allowed to have larger dilatation than ϕ in L.

Strebel's Frame Mapping Theorem: The conclusion of Teichmüller's Existence and Uniqueness Theorems hold for every a homotopy class of quasiconformal maps admitting representive that relaxes at infinity.

See [13] for proofs.

Furthermore, if $q \in Q(X, E)$ and $z \in E$ is isolated, then q is meromorphic at z with at worst a simple pole; see [32] for the relevant computation. Quadratic differentials in $Q(X, E)$ with E countable admit an important description.

Lemma 17 *Let X be a complex 1-manifold, $E \subseteq X$ a countable closed set, $q \in Q(X, E)$. Every isolated point of E where q fails to be holomorphic is a simple pole; every point where q fails to be meromorphic is a limit of simple poles.*

Proof: We may assume without loss of generality that q is not holomorphic at any point of E. By the Cantor-Bendixson Theorem, the isolated points, simple poles as above, are dense in E. □

Chapter 2

Conformal Dynamical Systems

2.1 Basic Notions

For complex manifolds X and Y, we denote $\mathcal{O}^*(X;Y)$ the space of nowhere locally constant analytic maps $f : X \to Y$ in the compact-open topology.

Definition. A *conformal dynamical system* $\mathcal{F}$ on a complex 1-manifold X is an assignment $U \rightsquigarrow \mathcal{F}[U] \subseteq \mathcal{O}^*(U;X)$ for non-empty open $U \subseteq X$ satisfying:

- $Id_U \in \mathcal{F}[U]$ for every U;
- If $V \subseteq U$ and $f \in \mathcal{F}[U]$, then $f_{|V} \in \mathcal{F}[V]$;
- If U is the disjoint union of open sets U_α, $f \in \mathcal{O}^*(U;X)$, and each $f_{|U_\alpha} \in \mathcal{F}[U_\alpha]$, then $f \in \mathcal{F}[U]$.
- If $f \in \mathcal{F}[U]$ and $g \in \mathcal{F}[f(U)]$, then $g \circ f \in \mathcal{F}[U]$.

We will freely identify a system $\mathcal{F}$ with the set

$$\coprod_{U \subseteq X} \mathcal{F}[U] \subseteq \coprod_{U \subseteq X} \mathcal{O}^*(U;X).$$

For $x \in X$, the sets $\mathcal{O}^*(U; X)$ with U ranging over connected neighborhoods of x form a direct system under restriction of domain. We refer to elements of the direct limit as *germs* at x, and denote $\mathcal{F}_x$ the collection of germs at x of elements of a presheaf $\mathcal{F}$. A subset of $\mathcal{F}[U]$ or $\mathcal{F}_x$ consisting only of the identity is termed *trivial.*

Definition. Let $\mathcal{F}$ be a conformal dynamical system on X. A connected open set $U \subseteq X$ is *simple* for $\mathcal{F}$ if every $f \in \mathcal{F}[V]$, with $V \subseteq U$ open and connected, has an extension in $\mathcal{F}[U]$.

Equivalently, U is simple whenever every germ at a point of U has an extension in $\mathcal{F}[U]$. Such extensions are of course unique.

Let $\mathcal{F}$ and $\mathcal{G}$ be conformal dynamical systems on X We say $\mathcal{F}$ is a *subsystem* of $\mathcal{G}$, and write $\mathcal{F} \subseteq \mathcal{G}$, when $\mathcal{F}[U] \subseteq \mathcal{G}[U]$ for open $U \subseteq X$. For any collection Γ of conformal dynamical systems on X,

$$U \rightsquigarrow \mathcal{F}[U] = \cap_{\mathcal{G} \in \Gamma} \mathcal{G}[U]$$

defines a conformal dynamical system $\mathcal{F}$. Thus there is a smallest conformal dynamical system $\langle F \rangle$ containing a given collection F of analytic maps from open subsets to X.

Examples:

1. Iterated maps. Let $W \subseteq X$ be open, $f : W \to X$ analytic; we say f is an *analytic map on* X. The system $\langle f \rangle$ on X consists of the iterates f^n, for $n \geq 0$, where defined.

2. Iterated Correspondences. Let $P : X \times X \to Y$ an analytic map to an auxiliary complex 1-manifold Y, $y \in Y$. Then there is a smallest conformal dynamical system $\mathcal{P}$ on X containing every analytic local solution $w = \psi(z)$ of the equation $P(z, w) = y$. See [5] for a look at algebraic correspondences on $\hat{\mathbf{C}}$.

3. Groups of Möbius transformations. A subgroup G of $\mathrm{PSL}_2\mathbf{C}$ determines a conformal dynamical system $\langle G \rangle$ on $\hat{\mathbf{C}}$.

4. Flows. Let v be a holomorphic vector field on an open set $W \subseteq X$, ϕ^t the time t flow where defined. The forward semi-flow $\mathcal{F}_v$ on X consists of all ϕ^t with $t \geq 0$; note that $\mathcal{F}_v = \mathcal{F}_{cv}$ for $c > 0$, while $\mathcal{F}_{-v}$ is the backward semi-flow.

5. Complex codimension 1 transversely holomorphic foliations. Such a structure on a real $n+2$ dimensional manifold M is determined by a system of coordinate charts $\psi_\alpha : W_\alpha \to \mathbf{C}$ with transition maps $\eta_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$ belonging to the pseudogroup of local homeomorphisms of $\mathbf{R}^n \times \mathbf{C}$ of the form $\eta(x, z) = (g(x, z), h(z))$, with g continuous and h holomorphic. The associated *holonomy* system $\mathcal{H}$ on $\mathbf{C}$ is the smallest conformal dynamical system containing these $\eta_{\alpha\beta}$. See [15] for a dynamical study of such foliations.

Due to the generality of the definition, it is an esay matter to construct highly pathological conformal dynamical systems of no intrinsic interest. We must impose more structure to obtain a workable theory. The first four examples satisfy the Continuation Condition:

- If $U = \bigcup_\alpha U_\alpha$, $f \in \mathcal{O}^*(U; X)$, and each $f_{|U_\alpha} \in \mathcal{F}[U_\alpha]$, then $f \in \mathcal{F}[U]$.

When this condition holds, we say that $\mathcal{F}$ is a *sheaf.* In this case, for each $f \in \mathcal{F}[V]$ with $V \subseteq X$ open and connected, there is a largest connected open $U \supseteq V$ such that f is the restriction of a map in $\mathcal{F}[U]$. This *maximal continuation* of f in $\mathcal{F}$ is unique.

We express further properties of conformal dynamical systems in terms of a natural ordering on elements. For $g \in \mathcal{F}[U]$ and $h \in \mathcal{F}[V]$ we write $g \preceq_{\mathcal{F}} h$ when g has an extension $\check{g} \in \mathcal{F}[U \cup V]$ such that $h = \alpha \circ \check{g}$ for some $\alpha \in \mathcal{F}[g(V)]$; we generally abbreviate this to $g \preceq h$ when no confusion is possible. This relation on $\mathcal{F}$ is transitive on any single $\mathcal{F}[U]$. Note however that if $\mathcal{F}$ is a sheaf then $g \preceq h \preceq g$ for any g and h with disjoint domains; thus, $\preceq$ is not generally transitive on the entire system. Examples 1, 3, and 4 satisfy the following Cancellation Condition:

- If $g, h \in \mathcal{F}$ then $g \preceq h$, $h \preceq g$, or both.

Definition. A conformal dynamical system $\mathcal{F}$ meeting the continuation and cancellation conditions is said to be *tight.*

In view of the continuation condition, it suffices to verify cancellation for maps with overlapping connected domains. Let $\mathcal{F}$ be a tight system,

$g, h \in \mathcal{F}[U]$. If $g \preceq h \preceq g$ and $h = \alpha \circ g$, $g = \beta \circ h$, where $\alpha \in \mathcal{F}[g(U)]$, $\beta \in \mathcal{F}[h(U)]$, then $\alpha \circ \beta = Id_{g(U)}$ and $\beta \circ \alpha = Id_{h(U)}$; that is, $\alpha \preceq Id_{g(U)}$ and $\beta \preceq Id_{h(U)}$.

Definition. A tight system $\mathcal{F}$ is *directed* if no $\mathcal{F}[U]$ has a nontrivial element g with $g \preceq Id_U$.

Lemma 18 *Let $\mathcal{F}$ be a tight system on X, $U \subseteq X$ open and connected, $g \in \mathcal{F}[U]$. If $g(U)$ is simple then so is U. The converse holds when $\mathcal{F}$ is directed.*

Proof: Let $V \subseteq U$ be open and connected, $h \in \mathcal{F}[V]$. If $h \preceq g$ then h extends to some $\check{h} \in \mathcal{F}[U]$. On the other hand, if $g \preceq h$ then $h = \alpha \circ g$ for some $\alpha \in \mathcal{F}[g(V)]$, and α extends to some $\check{\alpha} \in \mathcal{F}[g(U)]$. Then $\check{h} = \check{\alpha} \circ g \in \mathcal{F}[U]$ and $\check{h}_{|V} = h$.

Conversely, let $V \subseteq g(U)$ be open and connected, $h \in \mathcal{F}[g(V)]$. Fix a component W of $g^{-1}(V)$. Then $h \circ g_{|W}$ has an extension $\alpha \in \mathcal{F}[U]$. If $\alpha \preceq g$ then $\alpha_{|W} \preceq g_{|W} \preceq \alpha_{|W}$. As $\mathcal{F}$ is directed, it follows that $\alpha = g$, hence $h = Id_V$ which extends to $Id_{g(U)} \in \mathcal{F}[g(U)]$. Otherwise, $g \preceq \alpha$, hence $\alpha = \check{h} \circ g$ for some $\check{h} \in \mathcal{F}[g(U)]$, and $\check{h}_{|V} = h$. □

We say $x \in X$ is a *fixed point* of $\mathcal{F}$ if the semigroup

$$Fix_{\mathcal{F}}(x) = \{f \in \mathcal{F}_x : \ F(f) = x\}$$

is nontrivial. The fixed points of the system $\langle f \rangle$ are the *periodic* points of f; the period of x is the least positive p with $f^p(x) = x$.

Definition. Let $\mathcal{F}$ be a conformal dynamical system on X. A subset A of X is:

- *forward invariant* if $f(x) \in A$ whenever $x \in A$ and $f \in \mathcal{F}_x$,
- *backward invariant* if $x \in A$ whenever $f(x) \in A$ and $f \in \mathcal{F}_x$,
- *invariant* if both forward and backward invariant.

Let Ω be open and forward invariant with simple components. Then $\mathcal{F}$ acts in the obvious way on the components U: we write f_*U for the component containing $f(U)$. We say $f \in \mathcal{F}[U]$ *fixes* U when $f_*U = U$. Each component U has an associated semigroup

$$Fix_{\mathcal{F}}(U) = \{f \in \mathcal{F}[U] : f_*U = U\}.$$

Assume further that $\mathcal{F}$ is directed. Then for each component U of Ω, the maps $g : f_*U \to g_*f_*U$ with $f \in \mathcal{F}[U]$ and $g \in \mathcal{F}[f_*U]$ constitute a direct system in the category of Riemann surfaces.

Definition. In this setting, U is:

- *escaping* if $\mathcal{F}[g_*U]$ is trivial for some $g \in \mathcal{F}[U]$,
- *final* if for every $g \in \mathcal{F}[U]$ there exists $h \in Fix_{\mathcal{F}}(U)$ with $g \preceq h$.

A non-escaping component U is

- *meandering* if no g_*U with $g \in \mathcal{F}[U]$ is final,

- *wandering* if $g_*U \neq h_*U$ for distinct $g, h \in \mathcal{F}[U]$.

By definition, every component either escapes, meanders, or eventually maps to a final component; wandering is a special case of meandering.

Lemma 19 *Let $\mathcal{F}$ be a directed system, Ω open and forward invariant with simple components. Let U be a component of Ω, $f \in \mathcal{F}[U]$. Then*

- *U escapes if and only if f_*U escapes,*
- *If U is final then f_*U is final,*
- *U meanders if and only if f_*U meanders,*
- *If U wanders then f_*U wanders.*

Proof: See revisions.

In the systems of interest, the set of wandering components will also be backward invariant.

Definition. A tight system $\mathcal{F}$ is *hierarchical* if for any maps $g \in \mathcal{F}[U]$ and $h \in \mathcal{F}[V]$ with V, $h(V) \subseteq U$, there exists $\alpha \in \mathcal{F}[g(V)]$ with $\alpha \circ g_{|V} = g \circ h$. A hierarchical system is *abelian* if $g \circ h_{|V} = h \circ g_{|V}$ for any $g, h \subseteq \mathcal{F}[U]$ with V, $g(V)$, $h(V) \subseteq U$.

It is easily verified that iterated maps, semi-flows, and groups of Möbius transformations determine hierarchical systems; the the first two give abelian systems. Let $\mathcal{F}$ be a hierarchical system, Ω a forward invariant open set with simple components U. For $\alpha \in \mathcal{F}[U]$ and $g \in Fix_{\mathcal{F}}(U)$, let $\alpha_* g$ be the unique map in $\mathcal{F}[\alpha_* U]$ with $(\alpha_* g) \circ \alpha = \alpha \circ g$. Then $\alpha_* : Fix_{\mathcal{F}}(U) \to Fix_{\mathcal{F}}(\alpha_* U)$ is a homomorphism of semigroups; there is an analogous action for germs.

By convention, we denote $W(f)$ the domain of an analytic map, when not otherwise specified, and write $\partial(f)$ for the domain boundary. If $\mathcal{F}$ is a conformal dynamical system on X, we similarly denote $W(\mathcal{F})$ the union of the open sets $U \subseteq X$ for which $\mathcal{F}[U]$ is nontrivial.

Definition. Let $\mathcal{F}$ be a conformal dynamical system on X. A map $f \in \mathcal{F}[W(\mathcal{F})]$ such that $f_{|V} \neq Id_V$ on each component V of $W(\mathcal{F})$, and $f \preceq g$ for every nontrivial $g \in \mathcal{F}$ with connected domain is said to be a *base* of f.

Definition. Let $\mathcal{F}$ and $\mathcal{G}$ be conformal dynamical systems on X. We say $\mathcal{G}$ is an *enrichment* of $\mathcal{F}$, and write $\mathcal{F} \trianglelefteq \mathcal{G}$, when:

- $\mathcal{F} \subseteq \mathcal{G}$,
- For connected open U and V with $V \subseteq U$ and $g \in \mathcal{G}[U] - \mathcal{F}[U]$, every $f \in \mathcal{F}[V]$ has an extension $\check{f} \in \mathcal{F}[U]$, and $f \preceq_{\mathcal{G}} g$.

It is not hard to see that the relation of enrichment defines a partial order on the set of conformal dynamical systems on X. Observe that if $\mathcal{F} \trianglelefteq \mathcal{G}$, and U is a connected open set for which $\mathcal{G}[U] \neq \mathcal{F}[U]$ then U is simple for $\mathcal{F}$. It follows by an easy induction that if $\mathcal{F}$ is tight with base f, then $\langle f \rangle \trianglelefteq \mathcal{F}$. We say $\mathcal{F}$ is a *proper* enrichment of f if for each connected open U with

$\mathcal{F}[U] \neq \langle f \rangle$ there exists distinct $m, n \geq 0$ such that $f^m(U)$ and $f^n(U)$ lie in the same component of X.

Definition. Let $\mathcal{F}$ be a conformal dynamical system on X. Then

$$Z_1 \rightsquigarrow_{\mathcal{F}} Z_2 \Leftrightarrow \text{ there exist } x \in Z_1 \text{ and nontrivial } g \in \mathcal{F}_x \text{ with } g(x) \in Z_2$$

defines a relation on the set of components of X; we say that Z_1 is a *predecessor* of Z_2, and that Z_2 is a *successor* of Z_1. The system $\mathcal{F}$ is *trivial* on Z if Z has no successor. If $Z \rightsquigarrow_{\mathcal{F}} Z$ we say that *essential* for $\mathcal{F}$. We say $\mathcal{F}$ is *mixing* if $Z_1 \rightsquigarrow_{\mathcal{F}} Z_2$ for any pair of components with Z_1 nontrivial.

Observe that if $\mathcal{F}$ is a sheaf and every maximally continued element has dense image, then $\rightsquigarrow_{\mathcal{F}}$ is transitive. Similarly, an analytic map f determines a relation

$$Z_1 \rightsquigarrow_f Z_2 \Leftrightarrow \begin{array}{l} \text{there exist } x \in Z_1 \text{ and } m > 0 \text{ with} \\ x \in W(f^m) \text{ and } f^m(x) \in Z_2, \end{array}$$

If no iterate of f is locally the identity, then $\langle f \rangle$ is directed, and the relations $\rightsquigarrow_f$ and $\rightsquigarrow_{\langle f \rangle}$ agree; we shall say simply that f is directed.

Lemma 20 *Let f be an analytic map on X, $\mathcal{F}$ a proper enrichment of f. Assume $\mathcal{F}$ is directed, and that every component of X has an essential succesor under $\mathcal{F}$. Then the same is true of f.*

Proof: See revisions.

Let $\mathcal{F}$ be a conformal dynamical system on X. For any collection $\mathcal{Z}$ of components of X, there is an induced system of returns $\mathcal{F}^{\mathcal{Z}}$ on $\bigcup \mathcal{Z}$: for open $U \subseteq \bigcup \mathcal{Z}$,

$$\mathcal{F}^{\mathcal{Z}}[U] = \{g \in \mathcal{F}[U] : g(U) \subseteq \bigcup \mathcal{Z}\}.$$

If $\mathcal{F}$ is tight, directed, hierarchical, or abelian, then the same is true of $\mathcal{F}^{\mathcal{Z}}$.

Let f be an analytic map on X. Note that $\mathcal{Z}$ is essential if and only if

$$W^{\mathcal{Z}} = \bigcup \mathcal{Z} \cap \bigcup_{n=1}^{\infty} \{W(f^n) \cap f^{-n}(\bigcup \mathcal{Z})\}$$

is nonempty. For each $x \in W^{\mathcal{Z}}$ there is a least positive $n(x)$ with $f^{n(x)}(x) \in \bigcup \mathcal{Z}$, and $f^{\mathcal{Z}}(x) = f^{n(x)}(x)$ defines an analytic map $f^{\mathcal{Z}} : W^{\mathcal{Z}} \to \bigcup \mathcal{Z}$. It is easily checked that $\mathcal{F}^{\mathcal{Z}} = \langle f^{\mathcal{Z}} \rangle$. More generally, if $\mathcal{F}$ is a directed system with base f and $\mathcal{Z}$ is essential, then $f^{\mathcal{Z}}$ is the base of $\mathcal{F}^{\mathcal{Z}}$.

2.2 Fatou-Julia Theory

Classically, a point $x \in \hat{\mathbb{C}}$ belongs to the Fatou or Julia set of a rational map f according to whether or not there exists a neighborhood of x on which the iterates f^n form a normal family. To extend these notions to more general conformal dynamical systems, we must allow for the possibility that the elements of $\mathcal{F}_x$ have no common domain of definition.

Definition. Let $\mathcal{F}$ be a conformal dynamical system on X. The *Fatou set* $\Omega(\mathcal{F})$ consists of all points of X possessing a simple neighborhood U for which $\mathcal{F}[U]$ is a normal family. The complement of $\Omega(\mathcal{F})$ is the *Julia set* $J(\mathcal{F})$.

The Julia set of any conformal dynamical system is a closed set. Moreover, if $\mathcal{F}$ is a sheaf then every component U of $\Omega(\mathcal{F})$ is simple, and $\mathcal{F}[U]$ is normal.

Lemma 21 *Let $\mathcal{F}$ and $\mathcal{G}$ be conformal dynamical systems, $\mathcal{F} \trianglelefteq \mathcal{G}$. Then $J(\mathcal{F}) \subseteq J(\mathcal{G})$.*

If f is an analytic map on X, we define $\Omega(f)$ to be the set of points of x possessing a neighborhood U for which either

- $U \subseteq W(f^n) - W(f^{n+1})$ for some $n \geq 0$, or
- $U \subseteq W_\infty(f)$ and the family $\{f^n_{|U} : n \geq 0\}$ is normal,

and accordingly, $J(f) = X - \Omega(f)$. The Fatou and Julia sets of a directed map f agree with those of the system $\langle f \rangle$. Note that in our setting, the Julia set of an entire map as a system on $\hat{\mathbf{C}}$ contains the point at infinity. In accordance with the usual convention for meromorphic maps [2], $J(f)$ contains $\partial(f^n)$ for every positive n. Moreover, $\Omega(f)$ contains $f^{-n}(X - \overline{W})$ for $n \geq 0$. In view of Montel's Theorem, a self-map of a hyperbolic complex 1-manifold has empty Julia set. More generally, for any analytic map f on a complex 1-manifold, consider the largest open subset on which f is a self-map, namely the interior $W_\infty(f)$ of $X_+(f) = \bigcap_{n=1}^\infty W(f^n)$. We write $\Omega_+(f) = X_+(f) \cap \Omega(f)$ and $J_+(f) = X_+(f) \cap J(f)$.

Definition. An analytic map f on a complex 1-manifold X is *typical* if $W_\infty(f)$ is hyperbolic, *exceptional* otherwise.

This dichotomy was noted by Radström [34]. Clearly, $W_\infty(f) \subseteq \Omega(f)$ for typical f, and by the remarks above we may conclude:

Lemma 22 *Let f be a typical analytic map on a complex 1-manifold. Then $J(f) = \overline{\bigcup_{n=1}^{\infty} \partial(f^n)}$.*

Rational maps are exceptional, as are entire maps and *Radström maps*: self-maps of $\mathbf{C}^*$ with an essential singularity at ∞ and either an essential singularity or pole at 0. Affine toral endomorphisms are also exceptional. Observe that if f is an exceptional map, then so is $\hat{f}$. In view of the Riemann-Hurwitz formula ??, we conclude:

Lemma 23 *Let f be an exceptional map on a Riemann surface. Up to conformal conjugacy, $\hat{f}$ is either a rational, entire, or Radström map on the sphere, or an affine toral endomorphism.*

Definition. An exceptional analytic map f on a Riemann surface X is *elementary* if $\#J(f) \leq 2$.

Such a map is either a translation of the torus, or extends on removing at most two singularities to a Möbius transformation of the sphere. By definition, $\Omega(f)$ is hyperbolic for non-elementary f.

The *exceptional part* of an analytic map f on X consists of all essential components Z for which $f^{\{Z\}}$ is exceptional. In view of Liouville's Theorem, the exceptional part is forward invariant. As $Z \cap W_\infty(f)$ is connected for Z

in the exceptional part, each such Z has a unique successor under f. Consequently, the exceptional part splits into disjoint cycles. We will similarly speak of the *elementary part* of f.

Lemma 24 *Let $\mathcal{F}$ be a directed system admitting a base element with empty elementary part. Then for any open U and $g \in \mathcal{F}[U]$, the family*

$$\{h \in \mathcal{F}[U] : h \preceq g\}$$

is normal.

Proof: See revisions. □

Lemma 25 *Let $\mathcal{G}$ be a directed system. Assume that $\mathcal{G}$ is a proper enrichment of a base element f with empty elementary part, and suppose $\mathcal{F} \trianglelefteq \mathcal{G}$. Then $\mathcal{G}_x = \mathcal{F}_x$ for $x \in J(\mathcal{F})$.*

Proof: Let U be a connected open set containing x, $g \in \mathcal{G}[U] - \mathcal{F}[U]$. As $\mathcal{G}$ is an enrichment of $\mathcal{F}$, U is simple for $\mathcal{F}$ and $h \preceq_{\mathcal{G}} g$ for all $h \in \mathcal{F}[U]$. By Lemma 24, $\mathcal{F}[U]$ is normal, and thus $U \subseteq \Omega(\mathcal{F})$. □

Proposition 4 *Let $\mathcal{F}$ be a directed system with base f. Assume that $\mathcal{F}$ is a proper enrichment of f and that f has no elementary returns. Then $J(\mathcal{F})$ is invariant under $\mathcal{F}$.*

Proof: Fix a point x, a connected open set U containing x, and $g \in \mathcal{F}[U]$. By Lemma 18, U is simple if and only if $g(U)$ is simple. As

$$\mathcal{F}[U] = \{h \in \mathcal{F}[U] : h \preceq g\} \cup \{\alpha \circ g : \alpha \in \mathcal{F}[g(U)]\},$$

if $\mathcal{F}[U]$ is normal then so is $\mathcal{F}[g(U)]$; the converse follows by Lemma 24. Thus, $x \in J(\mathcal{F})$ if and only if $g(x) \in J(\mathcal{F})$. □

Fatou Components

Let f be an analytic map on a complex 1-manifold. The *eigenvalue* $\rho(x)$ of a fixed point x of f is the conformal invariant given in local coordinates by $\rho(x) = f'(x)$; the eigenvalue of a periodic point is its eigenvalue as a fixed point of f^p, where p is the period of f. A periodic point x is *superattracting* if $\rho(x) = 0$, *attracting* if $0 < |\rho(x)| < 1$, *indifferent* if $|\rho(x)| = 1$, and repelling if $|\rho(x)| > 1$. In the indifferent case, write $\rho(x) = e^{2\pi i\theta}$. We say x is *linearizable* when f is locally analytically conjugate to a rotation. In the linearizable case, f is locally of finite order if θ is rational, and we call x a *Siegel point* when θ is irrational. In the non-linearizable case, we say that x is *parabolic* when θ is rational, and a *Cremer point* otherwise. There are subtle Diophantine criteria for linearizability in the irrational case; see [30] for discussion and references.

It is not hard to see that attracting, superattracting, linearizable indifferent periodic points belong to $\Omega(f)$, and that repelling points lie in $J(f)$. We shall see below that parabolic and Cremer points also belong to $J(f)$.

For non-elementary rational f, the fixed components of $\Omega(f)$ are of five types:

1. *Superattracting Domain*: f has a superattracting fixed point $x \in U$,

and $f^n_{|U} \to x$.

2. *Attracting Domain*: f has an attracting fixed point $x \in U$ and $f^n_{|U} \to x$.

3. *Parabolic Domain*: f has an parabolic fixed point $x \in \partial U$ and $f^n_{|U} \to x$.

4. *Siegel Disc*: $f_{|U}$ is analytically conjugate to an irrational rotation of the disc.

5. *Hermann Ring*: $f_{|U}$ is analytically conjugate to an irrational rotation of an annulus.

More generally, let f be an analytic map on X. Periodic components of $\Omega(f)$ lie in essential components of X, and are hyperbolic as long as f has empty elementary part. The classification of periodic components thus entails the study of self-maps of hyperbolic Riemann surfaces.

Proofs of the following two standard results can be found in [38] and [30].

Lemma 26 *Let U be a hyperbolic Riemann surface, $f : U \to U$ analytic, and suppose some orbit is bounded. Then one of the following mutually exclusive possibilities holds:*

- *f has a unique attracting or superattracting fixed point $x \in U$, and $f^n \to x$;*
- *f is analytically conjugate to an irrational rotation of a disc, punctured disc, or finite annulus;*

• *f is a bijection of finite order.*

It remains to discuss the case in which some orbit tends to infinity in W. Note that a path $\gamma : [0,1] \to U$ with $f(\gamma(1)) = \gamma(0)$ extends by the relation $f(\tilde{\gamma}(t+1)) = \tilde{\gamma}(t)$ to a *forward invariant* path $\tilde{\gamma} : [0,\infty) \to U$. Recall the Snail Lemma:

Lemma 27 *Let f be an analytic map on X with fixed point x, and suppose that x is the limit of some forward invariant path. Then $|\rho(x)| < 1$ or $\rho(x) = 1$.*

Lemma 28 *Let $f : W \to X$ be an analytic map on X, and let U be fixed component of $\Omega(f)$. Suppose that $f^n_{|U}$ tends to infinity in U, but not in W. Then U is a parabolic domain.*

Proof: Let $\gamma : [0,1] \to U$ be a rectifiable path with $f(\gamma(1)) = \gamma(0)$; extend to a forward invariant path $\tilde{\gamma} : [0,\infty) \to U$ and consider

$$L = \{x \in X : \tilde{\gamma}(t_k) \to x \text{ for some } t_k \to \infty\}.$$

By the local compactness of X, if L consists of more than one point then no point is isolated.

Suppose $\tilde{\gamma}(t_k) \to x$ for some $t_k \to \infty$. Write $t_k = n_k + s_k$ with $0 \leq s_k < 1$, and let $w_k = \gamma(s_k)$. As $\gamma([0,1])$ is compact and $f^{n_k}(w_k) \to x$, it follows that $f^{n_k}_{|U} \to x$. In particular,

$$\lim_{k\to\infty} f^{n_k}(\gamma(0)) = x = \lim_{k\to\infty} f^{n_k}(\gamma(1)),$$

so any point in $W \cap L$ is fixed by f and therefore isolated. By assumption, $W \cap L \neq \emptyset$, and it follows that $f^n_{|U}$ converges to a fixed point x which, by the Snail Lemma, must be parabolic. □

If some orbit tends to infinity in W and $f^n_{|U}$ converges to a point in $\partial(f)$, shall refer to U as a *Baker Domain*; see [] for examples arising from entire maps. This is the only other possibility If X is compact and $\partial(f)$ is totally disconnected. In the remaining case, we say that U is an *Exotic Domain.*

2.3 Parabolic Enrichments

Definition. Let f be an analytic map on X.

- A *linearizing coordinate* is an analytic map $\varpi : B \to \mathbf{C}$ such that $B \subseteq W(f)$ and $\varpi \circ f = \tau \circ \varpi$ for some linear map $\tau(z) = \lambda z$.

- A linearizing parameter is an analytic map $\chi : V \to X$ where $V \subseteq \mathbf{C}$ and $\chi \circ \tau = f \circ \chi$.

By Schwarz' Lemma, every linearizing coordinate or parameter is associated to an attracting or repelling fixed point with eigenvalue λ.

Definition. Let f be an analytic map on X; denote $\tau : \mathbf{C} \to \mathbf{C}$ the unit translation.

- A *Fatou coordinate* is an analytic map $\varpi : B \to \mathbf{C}$ such that $B \subseteq W(f)$, $\bigcup_{n=0}^{\infty} \tau^{-n}(\varpi(B)) = \mathbf{C}$, and $\varpi \circ f = \tau \circ \varpi$.

- A *Fatou parameter* is an analytic map $\chi : V \to X$ where $V \subseteq \mathbf{C}$, $\bigcup_{n=0}^{\infty} \tau^n(V) = \mathbf{C}$, and $\chi \circ \tau = f \circ \chi$.

As discussed in [30], a parabolic fixed point x has associated attracting and repelling *petals* whose union punctured neighborhood of x; the petals have associated Fatou coordinates and parameters. These assertions constitute Fatou's Flower Theorem. The relevant calculations will be reproduced in the revisions.

Definition. Let f be an analytic map on X. We say $\varpi : \mathcal{B} \to \mathbf{C}$ is a *global* linearizing or Fatou coordinate if $\mathcal{B}$ is maximal. Similarly, $\chi : \mathcal{V} \to X$ is a *global* linearizing or Fatou parameter if $\mathcal{V}$ is maximal.

These are all well-defined up to homothety or translation. In the revisions we shall discuss the functorial construction of source and target planes $\S^{\pm}$.

Definition. Let f be an analytic map on X. The *post-singular set* $PS(f)$ is the smallest forward invariant set containing $S(f)$; we denote its closure $\mathcal{P}S(f)$.

Lemma 29 *Let f be an analytic map on a Riemann surface X. Assume that f has no removable singularities. Then $X - \mathcal{P}S9f)$ is hyperbolic unless f is elementary or conjugate to $z \rightsquigarrow z^{\pm}n$ for some $n \geq 1$.*

Proof: See revisions.

Lemma 30 *Let* $\varpi : \mathcal{B} \to \mathbf{C}$ *be a global linearizing or Fatou coordinate for an analytic map* f *on* X. *Then* $S(\varpi)$ *is the smallest closed* τ^{-1} *invariant subset of* $\mathbf{C}$ *containing* $\varpi(S(f))$. *Similarly, if* $\chi : \mathcal{V} \to X$ *is a global Fatou parameter then* $S(\chi) \subseteq \mathcal{PS}(f)$.

Proof: See revisions.

More precisely, we will show that $S(\varpi_{|B})$, where B is the component of $\mathcal{B}$ fixed by f, is the smallest closed τ^{-1} invariant set containing $\varpi(S(f_{|B})$. It follows that a fixed attracting or parabolic basin for a analytic map with empty elementary part contains a singular value of f.

Let $\chi : \mathcal{V} \to X$ be a global Fatou parameter assoctiated to a repelling petal of a fixed point of eigenvalue 1. In view of the Fatou Flower Theorem, $\mathcal{V}$ contains a left half-plane. Thus there is a unique component which is unbounded to the left. This *main* component is backward invariant under τ, and every other component is disjoint from its translates. Further discussion will appear in the revisions.

Lemma 31 *Let* $\chi : \mathcal{V} \to X$ *be a global linearizing or Fatou parameter for an analytic map* f *on* X. *Consider the component* Z *of* X *containing* $\chi(V)$, *where* V *is the main component of* $\mathcal{V}$.

- *If* $f^{\{Z\}}$ *is exceptional then* $V = \mathcal{V} = \mathbf{C}$. *Furthermore,* χ *has an essential singularity at* ∞ *unless* $f^{\{Z\}}$ *is elementary.*
- *If* $f^{\{Z\}}$ *is typical then* $\partial(\chi) = \bigcup_{n=1}^{\infty} \tau^n(\chi^{-1}(\partial(f^n)) \cup \{\infty\}$.

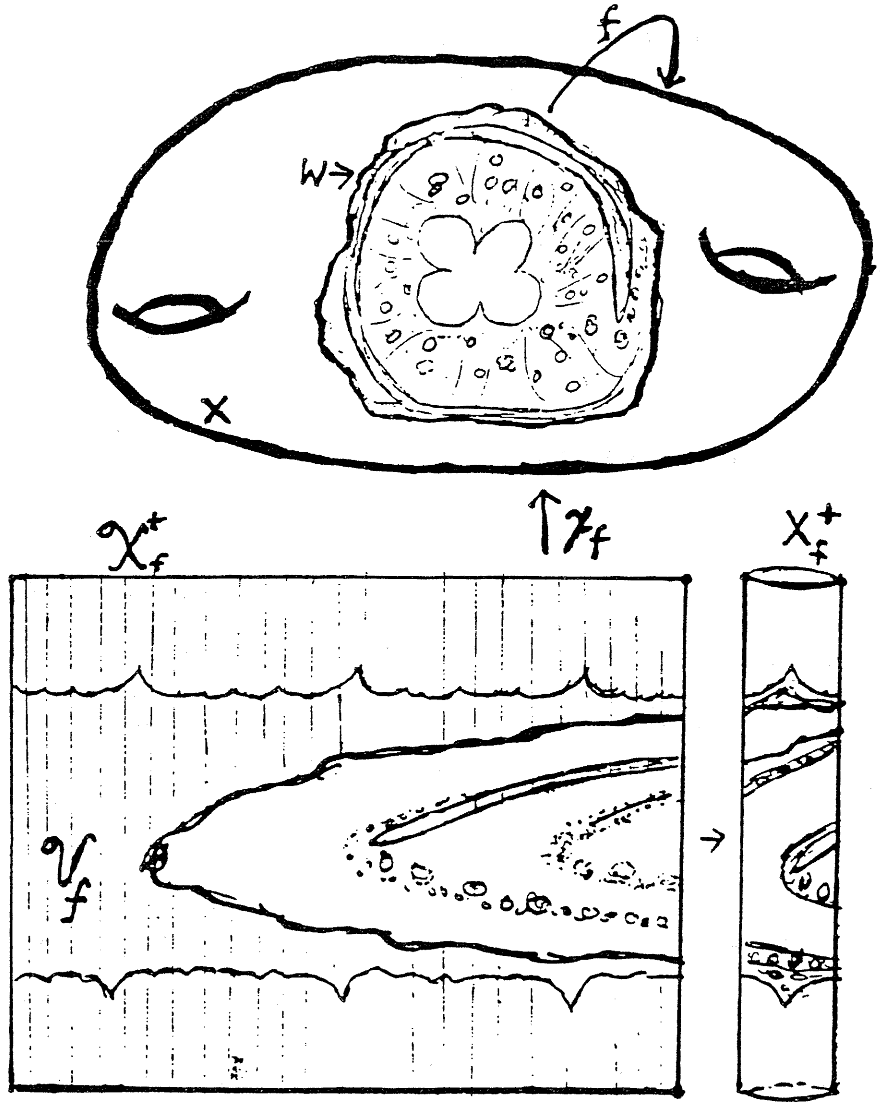

Figure 2.1: Global Fatou parameter associated to a repelling petal

Proof: See revisions.

More generally, if x has period m, then the coordinates and parameters associated to the minimal iterate fixing x give rise to maps ϖ and χ semi-conjugating f and σ, where σ cyclicly permutes m copies of the plane, and the return map is *tau*. We denote $\varpi_f : \mathcal{B}_f \to \mathcal{X}_f^-$ and $\chi : \mathcal{V}_f \to \mathcal{X}_f^+$ the maps so obtained on taking together all parabolic cycles and the canonical global Fatou coordinates and parameters associated to their attracting and repelling petals. We write $\tilde{W}_f = \chi_f^{-1}(\mathcal{B}_f)$. The quotients $X_f^\pm = \mathcal{X}_f^\pm/\sigma$ are cylinders. We denote $\pi_f : \mathcal{B}_f \to X_f^-$ the induced projection. On removing the punctures, we obtain spheres $\hat{X}_f^\pm$ with *poles* which we may label 0 and ∞. There is an induced map $E_f : W_f \longrightarrow \hat{X}_f^-$, where $W_f \subseteq \hat{X}_f^+$ and the poles. For notational ease, we shall drop the subscript f on the maps ϖ_f, χ_f, and π_f when the context is clear.

Definition. Let f be an analytic map on a complex 1-manifold. Assume that f has at least one parabolic cycle, and let $\mathcal{A}$ be a nonempty collection of components of $\hat{X}_f^-$. A *transit map* for f is a analytic map $\Phi : \bigcup \mathcal{A} \to \hat{X}_f^+$ which is a pole preserving isomorphism on each component.

A choice of transit map Φ determines an analytic map $\Phi \cup E$ on the complex 1-manifold $\hat{X}_f^- \cup \hat{X}_f^+$.

Lemma 32 *Let f be an analytic map with empty elementary part, Φ a transit map for f. Then $\Phi \cup E$ is typical.*

Proof: See revisions. □

Consider the following homotopy conditions on a map $f : W \to \mathbf{C}^*$, where $W \subseteq \mathbf{C}^*$ is open:

1. For any homotopically trivial closed curve γ in W, the curve $f \circ \gamma$ is homotopically trivial on $\mathbf{C}^*$.

2. For any homotopically nontrivial closed curve γ in W, the curve $f \circ \gamma$ is homotopically nontrivial on $\mathbf{C}^*$.

We may easily adapt these conditions to apply to maps on unions of spheres with labelled poles.

Lemma 33 *Let f be an analytic map on X, with parabolic cycles. Then $E : W_f \to \hat{X}_f^+$ meets the first and second homotopy conditions.*

Let f be an analytic map on X, where X consists of cylinders. A component U of $\Omega(f)$ may be *equatorial*, that is, homotopically nontrivial on X. We may extend the discussion to the case where some or all of the punctures of X have been filled. An equatorial component containing a pole is is *polar.*

Lemma 34 *Let X be a union of spheres with labeled poles, f an analytic map on X, and U a periodic component of $\Omega(f)$. Assume that U is equatorial. Then U is not a parabolic domain; moreover, if U is a superattracting or attracting domain then U is polar.*

Proof: We may assume f fixes U. Let $\gamma : \mathbf{S}^1 \to U^*$ be homotopically nontrivial on X^*. As f is positive, the curves $\gamma_n = f^n \circ \gamma$ are all homotopically nontrivial. On the other hand, if U is a parabolic domain then the γ_n eventually lie in a contractible petal. Similarly, if U is a superattracting or attracting domain, the associated fixed point must be a pole of X. □

Definition. Let f be an analytic map on a complex 1-manifold. Assume that f has at least one parabolic cycle. Let $\mathcal{H}$ be a conformal dynamical system on $\hat{X}_f^- \cup \hat{X}_f^+$ with base element $\Phi \cup E$, for some choice of transit map Φ. Assume further that any $\alpha \in \mathcal{H}[U]$, where $U \subseteq \hat{X}_f^-$ is open and $h(U) \subseteq \hat{X}_f^+$, may be expressed as $\alpha = E \circ \beta$ for some $\beta \in \mathcal{H}$. Suppose further that every map in $\mathcal{H}$ meets the first homotopy condition.

We define a conformal dynamical system $f * \mathcal{H}$ on X by

- For $U \not\subseteq \mathcal{B}_f$, $f * \mathcal{H}[U] = \langle f \rangle [U]$;
- For $U \subseteq \mathcal{B}_f$, $f * \mathcal{H}[U]$ consists of all $g = \chi \circ \tilde{\alpha} \varpi_{|U}$, where $\tilde{\alpha} : \varpi(U) \to \tilde{W}_f$ is a lift of some $\alpha \in \mathcal{H}$.

We shall say that $\mathcal{F} = f * \mathcal{H}$ is a *parabolic enrichment* of f.

It is often more convenient to work with the system of returns $\mathcal{RF}$ of $\mathcal{H}$ to X_f^-. A more precise account of parabolic enrichments will appear in the revisions.

Lemma 35 *Let $\mathcal{F}$ be a parabolic enrichment of f. Then $\mathcal{F}$ cannot be expressed as a one-generator system.*

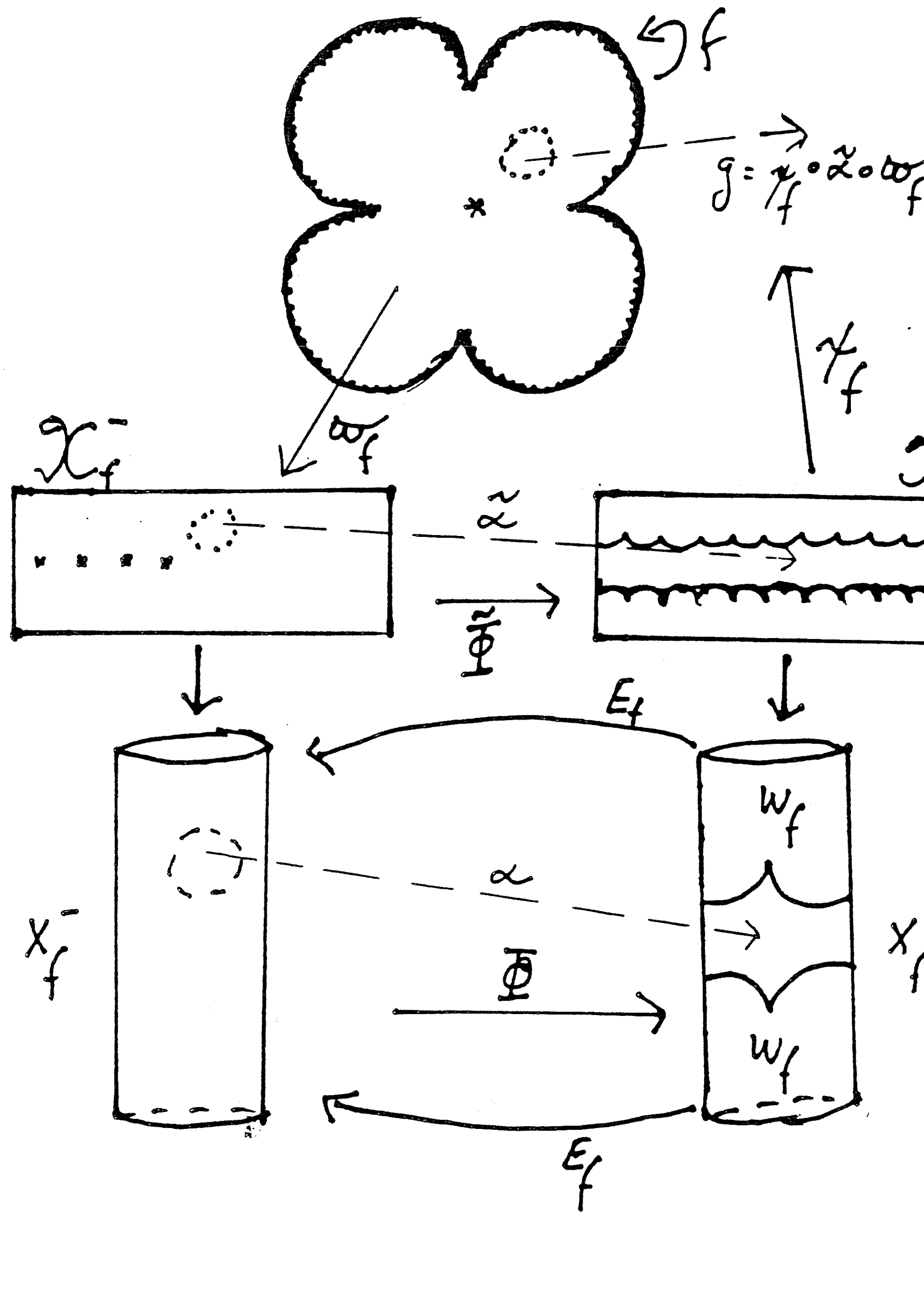

Figure 2.2: Parabolic Enrichment

Lemma 36 *Let $\mathcal{F}$ be directed with base f. Then $\mathcal{F} = f * \mathcal{H}$ for at most one $\mathcal{H}$; moreover, $\mathcal{H}$ is directed.*

Lemma 37 *Let $\mathcal{F} = f * \mathcal{H}$ be directed with base f, $U \subseteq \mathcal{B}_f$ open and connected.*

- *If U is simple for $\mathcal{F}$, then $\pi(U)$ is simple for $\mathcal{H}$.*
- *If $\pi(U)$ is simple for $\mathcal{H}$ but U is not simple for $\mathcal{F}$, then there exist $x \in U$ and $g \in \mathcal{F}_x$ with $g(x) \in \partial(f)$.*

Proof: Assume U is simple, and fix $x \in U$, $\gamma \in \mathcal{H}_x$. If $\gamma(\pi(x)) \in X_f^+$ then $\tilde{\gamma}(\varpi(x)) \in V_f$ for a suitable lift $\tilde{\gamma}$. By assumption, $\chi \circ \tilde{\gamma} \circ \varpi \in \mathcal{F}_x$ extends to some $g \in \mathcal{F}[U]$. In view of ??, $g = \chi \circ \tilde{\alpha} \circ \varpi$ for a unique lift *tilde*α of a unique $\alpha \in \mathcal{H}[\pi(U)]$; clearly, α extends γ. If $\gamma(\pi(x)) \in X_f^-$, we apply this argument to extend $\Phi \circ \gamma$ to $\pi(U)$. As $\gamma \preceq \Phi \circ \gamma$ and $\mathcal{H}$ is directed, γ again extends to $\pi(U)$. Thus, U is simple.

Suppose on the other hand that $\pi(U)$ is simple. If U is not simple, there exist connected open $V \subseteq U$ and $h \in \mathcal{F}[V]$ with no extension in $\mathcal{F}[U]$. Clearly, $h \notin \langle f \rangle$; we may assume that h is maximally continued. As $\pi(U)$ is simple, $h = \xi \circ \tilde{\alpha} \circ \varpi$ where $\tilde{\alpha}$ is a lift of some $\alpha \in \mathcal{F}[\pi(U)]$. Then $\tilde{\alpha}(V) \subseteq V_f$ but $\tilde{\alpha}(U) \nsubseteq V_f$. Consequently, $\tilde{\alpha}(\pi(x)) \in \partial V_f = \partial(\chi)$ for some $x \in U$, and thus $g(x) \in \partial(f)$ for $g = \chi \circ \sigma^{-1} \circ \varpi \in \mathcal{F}_x$. □

Thus, $J(\mathcal{F}) \subseteq J(f) \cup \pi^{-1}(J(f))$.

2.4 Towers

Fix n, a positive integer or infinity, and consider a sequence f_k on complex 1-manifolds X_k, where $1 \leq k < n+1$. Suppose that for $k > 1$, $X_{k+1} = \hat{X}^-_{f_k} \cup \hat{X}^+_{f_k}$ and $f_{k+1} = \Phi_k \cup E_{f_k}$ for some choice of transit map Φ_k between $\hat{X}^-_{f_k}$ and $\hat{X}^+_{f_k}$. By induction on finite n, we obtain systems

$$\langle f_1 \mid \cdots \mid f_n \rangle = f_1 * \langle f_2 \mid \cdots \mid f_n \rangle$$

on X_1; if n is infinite, we define

$$\langle f_1 \mid \cdots \rangle = \bigvee_k \langle f_1 \mid \cdots \mid f_k \rangle.$$

Then $\langle f_1 \mid \cdots \rangle = f_1 * \langle f_2 \mid \cdots \rangle$ by Lemma ??. We call such a conformal dynamical system an n *stage construct*. Constructs are abelian systems.

Definition. A *tower of height* n, where n is a positive integer or infinity, is a directed n stage construct.

Let $\mathcal{F} = \langle f_1 \mid \cdots \mid f_n \rangle$ or $\langle f_1 \mid \cdots \rangle$ be a tower of height $n \leq \infty$. As $\mathcal{F}$ is directed, f_1 is the unique base. By Lemma 36, if $n > 1$ and $\mathcal{F} = f * \mathcal{H}$ then $\mathcal{H} = \langle f_2 \mid \cdots \mid f_n \rangle$, respectively $\langle f_2 \mid \cdots \rangle$; moreover, $\mathcal{H}$ is a tower of height $n-1$. By Lemma 35 and induction, n and the sequence f_k are uniquely determined; we write $height(\mathcal{F}) = n$. For $1 \leq m < n+1$, $\mathcal{F}^m = \langle f_1 \mid \cdots \mid f_m \rangle$ is a tower of height m. In view of Lemma ?? these *subtowers* form an ascending sequence.

We assign heights to the elements $g \in \mathcal{F}$ with connected domain. By convention, $ht_{\mathcal{F}}(Id_U) = 0$, while for non-trivial $g \in \mathcal{F}[U]$ we set

$$ht_{\mathcal{F}}(g) = \min_{g \in \mathcal{F}^m} m < \infty.$$

If $ht_{\mathcal{F}}(g) \geq 2$ then $U \subseteq \mathcal{B}_f$, $g = \chi\tilde{\alpha}\varpi$ for a unique lift $\tilde{\alpha}$ of a unique $\alpha : \pi(U) \to X_f^+$ in $\mathcal{H}$, and $ht_{\mathcal{F}}(g) = ht_{\mathcal{H}}(\alpha) + 1$. Note that as the towers $\mathcal{F}^m$ are ascending, $ht(g) = ht(g_{|V})$ for any $g \in \mathcal{F}[U]$ and connected $V \subseteq U$; we set $ht_{\mathcal{F}}(\xi) = ht_{\mathcal{F}}(g)$ for $\xi = [g] \in \mathcal{F}_x$. Moreover, by an easy induction on heights, if $\mathcal{F}$ and $\mathcal{G}$ are towers and $\mathcal{F} \trianglelefteq \mathcal{G}$ then $ht_{\mathcal{F}}(g) = ht_{\mathcal{G}}(g)$ for every $g \in \mathcal{F}$. It follows that for any ascending sequence of towers $\mathcal{F}_k$, the system $\mathcal{F} = \bigvee_k \mathcal{F}_k$ is a tower and $height(\mathcal{F}) = \sup height(\mathcal{F}_k)$.

Lemma 38 *Let $\mathcal{F}$ be a tower on X, $U \subseteq X$ open and connected, $g \in \mathcal{F}[U]$, $h \in \mathcal{F}[g(U)]$. Then $ht_{\mathcal{F}}(h \circ g) = \max(ht_{\mathcal{F}}(h), ht_{\mathcal{F}}(g))$.*

Proof: We proceed by induction on $n = \min(ht_{\mathcal{F}}(h), ht_{\mathcal{F}}(g))$. The claim is trivial for $n = 0$. Suppose first that $n = 1$. If $ht_{\mathcal{F}}(h) = 1 = ht_{\mathcal{F}}(g)$ then h and g are restrictions of some f^m and f^ℓ, where f is the base of $\mathcal{F}$ and $m, \ell > 0$, hence $ht_{\mathcal{F}}(h \circ g) = ht_{\mathcal{F}}(f^{m+\ell}) = 1$. Otherwise, $height(\mathcal{F}) \geq 2$, and we write $\mathcal{F} = f * \mathcal{H}$. If $ht_{\mathcal{F}}(h) = 1 < ht_{\mathcal{F}}(g)$, then $h = f^m_{|g(U)}$ as above and $g = \chi \circ \tilde{\alpha} \circ \varpi$, where $\tilde{\alpha} : \varpi(U) \to V_f$ is a lift of some $\alpha \in \mathcal{H}[\pi(U)]$. Thus,

$$h \circ g = f^m \circ \chi \circ \tilde{\alpha} \circ \varpi = \chi \circ \sigma^m \circ \tilde{\alpha} \circ \varpi$$

where $\sigma^m \circ \tilde{\alpha}$ is another lift of α, so $ht_{\mathcal{F}}(h \circ g) = ht_{\mathcal{F}}(g)$. On the other hand, if $ht_{\mathcal{F}}(h) = 1 > ht_{\mathcal{F}}(g)$, then $g = f^\ell_{|U}$ and $h = \chi \circ \tilde{\beta} \circ \varpi$ for a lift

$\tilde{\beta} : \varpi(g(U)) \to V_f$ of some $\beta \in \mathcal{H}[\pi(g(U))]$. Again,

$$h \circ g = \chi \circ \tilde{\beta} \circ \varpi \circ f^{\ell} = \chi \circ \tilde{\beta} \circ \sigma^{\ell} \circ \varpi$$

where $\tilde{\beta} \circ \sigma^{\ell}$ is another lift of β, so $ht_{\mathcal{F}}(h \circ g) = ht_{\mathcal{F}}(h)$.

Now suppose $n > 1$. Then $g = \chi \circ \tilde{\alpha} \circ \varpi$ and $h = \chi \circ \tilde{\beta} \circ \varpi$ as above; moreover, U and $g(U)$ lie in $\mathcal{B}_f$, so $\tilde{\alpha}(\varpi(U)) \subseteq \tilde{W}_f$ and therefore $\alpha(\pi(U)) \subseteq W_f$. Consequently,

$$h \circ g = \chi \circ \tilde{\beta} \circ \varpi \circ \chi \circ \tilde{\alpha} \circ \varpi = \chi \circ \tilde{\gamma} \circ \varpi$$

where $\tilde{\gamma}$ is a lift of $\gamma = \beta \circ E_f \circ \alpha$. As $ht_{\mathcal{H}}(E_f) = 1 \leq \min(ht_{\mathcal{H}}(\beta), ht_{\mathcal{F}}(\alpha))$, it follows by induction that $ht_{\mathcal{H}}(\gamma) = \max(ht_{\mathcal{H}}(\beta), ht_{\mathcal{H}}(\alpha))$, hence

$$ht_{\mathcal{F}}(h \circ g) = \max(ht_{\mathcal{H}}(\beta), ht_{\mathcal{H}}(\alpha)) + 1 = \max(ht_{\mathcal{F}}(h), ht_{\mathcal{F}}(g)). \quad \square$$

In particular, if $g, h \in \mathcal{F}$ have overlapping connected domains and $g \preceq h$, then $ht_{\mathcal{F}}(g) \leq ht_{\mathcal{F}}(h)$. These height relations apply equally to germs.

Given a tower $\mathcal{F}$ on X, we may also assign heights to points and simple open subsets of X:

$$ht_{\mathcal{F}}(x) = \sup_{\xi \in \mathcal{F}_x} ht_{\mathcal{F}}(\xi), \qquad ht_{\mathcal{F}}(U) = \sup_{g \in \mathcal{F}[U]} ht_{\mathcal{F}}(g).$$

Thus, $ht_{\mathcal{F}}(U) = ht_{\mathcal{F}}(x)$ if U is simple and $x \in U$. Clearly, $ht_{\mathcal{F}}(x) = 0$ if and only if $x \in X - W$, and $ht_{\mathcal{F}}(x) \leq 1$ for $x \notin \mathcal{B}_f$, where $f : W \to X$ is the base of $\mathcal{F}$. Moreover, $ht_{\mathcal{F}}(x) = ht_{\mathcal{H}}(\pi(x)) + 1$ for $\mathcal{F} = f * \mathcal{H}$ and $x \in \mathcal{B}_f$.

Lemma 39 *Let $\mathcal{F}$ be a tower on X, $x \in X$, and $g \in \mathcal{F}_x$. Then*

$$ht_{\mathcal{F}}(x) = \max(ht_{\mathcal{F}}(g), ht_{\mathcal{F}}(g(x)).$$

Proof: By definition, $ht_{\mathcal{F}}(x) \geq ht_{\mathcal{F}}(g)$. If equality holds and $h \in \mathcal{F}_{g(x)}$, then $ht_{\mathcal{F}}(h \circ g) = ht_{\mathcal{F}}(x)$, hence $ht_{\mathcal{F}}(h) \leq ht_{\mathcal{F}}(x)$ by Lemma 38. Thus,

$$ht_{\mathcal{F}}(g(x)) = \sup_{h \in \mathcal{F}_{g(x)}} ht_{\mathcal{F}}(h) = ht_{\mathcal{F}}(x).$$

Otherwise, $ht_{\mathcal{F}}(x) > ht_{\mathcal{F}}(g)$, and as $ht_{\mathcal{F}}(g)$ is finite, $ht_{\mathcal{F}}(h) > ht_{\mathcal{F}}(g)$ for some $h \in \mathcal{F}_{g(x)}$. By Lemma 38, $ht_{\mathcal{F}}(h \circ g) = ht_{\mathcal{F}}(h)$, and thus

$$ht_{\mathcal{F}}(g(x)) = \sup_{h \in \mathcal{F}_{g(x)}} ht_{\mathcal{F}}(h) = \sup_{h \in \mathcal{F}_{g(x)}} ht_{\mathcal{F}}(h \circ g) = ht_{\mathcal{F}}(x). \quad \square$$

Suppose $\alpha, \beta \in \mathcal{F}_x$ with $\alpha \preceq \beta$. Then $ht_{\mathcal{F}}(\alpha(x)) \leq ht_{\mathcal{F}}(\beta(x))$. Consequently, when $h \in \mathcal{F}_{g(x)}$ is large, $ht_{\mathcal{F}}(h(x))$ is equal to the *eventual height*

$$\overline{ht}_{\mathcal{F}}(x) = \min_{g \in \mathcal{F}_x} ht_{\mathcal{F}}(g(x)).$$

Note that $\overline{ht}_{\mathcal{F}}(x) = 0$ if and only if some necessarily unique $g \in \mathcal{F}_x$ sends x outside the domain of the base. If $\overline{ht}_{\mathcal{F}}(x) \geq 1$ then $x \in \mathcal{B}_f$ and

$$\overline{ht}_{\mathcal{RF}}(\pi(x)) = \overline{ht}_{\mathcal{F}}(x) - 1. \tag{2.1}$$

Points of infinite height are of special interest. Certainly, $ht_{\mathcal{F}}(x) = \infty$ if $\overline{ht}_{\mathcal{F}}(x) = \infty$. Conversely, suppose $ht_{\mathcal{F}}(x) = \infty$, and let $g \in \mathcal{F}_x$. By Lemma 39, $ht_{\mathcal{F}}(g(x)) = \infty$ as $ht_{\mathcal{F}}(g) < \infty$. Thus,

$$ht_{\mathcal{F}}(x) = \infty \Leftrightarrow \overline{ht}_{\mathcal{F}}(x) = \infty.$$

Recall that $g(U)$ is simple for $g \in \mathcal{F}[U]$ and simple U, hence

$$ht_{\mathcal{F}}(U) = \max(ht_{\mathcal{F}}(g), ht_{\mathcal{F}}(g(U)).$$

Consequently, if Ω is open and forward invariant with simple components then $ht_{\mathcal{F}}(\alpha_* U) \leq ht_{\mathcal{F}}(\beta_* U)$ for any component U and $\alpha \preceq \beta \in \mathcal{F}[U]$. As above, for large $h \in \mathcal{F}[U]$, $ht_{\mathcal{F}}(h_* U) = \overline{ht}_{\mathcal{F}}(U)$ where

$$\overline{ht}_{\mathcal{F}}(U) = \min_{g \in \mathcal{F}[U]} ht_{\mathcal{F}}(g_* U),$$

equal to $\overline{ht}_{\mathcal{F}}(x)$ for any $x \in U$. Observe that $\overline{ht}_{\mathcal{F}}(U) = 0$ if and only if U escapes, and again $\overline{ht}_{\mathcal{F}}(U) = \infty$ if and only if $ht_{\mathcal{F}}(U) = \infty$.

Let $\mathcal{F}$ be a tower on X, $U \subseteq X$ open and connected, $g \in \mathcal{F}[U]$ with $ht_{\mathcal{F}}(g) = n \geq 1$. We say g is *primitive* of height n if for any *alpha* $\in \mathcal{F}[U]$ and $\beta \in \mathcal{F}[\alpha(U)]$ with $g = \beta \circ \alpha$, either $ht_{\mathcal{F}}(\alpha) < n$ or $ht_{\mathcal{F}}(\beta) < n$.

The primitive elements of height 1 are the restrictions of the base f. Clearly, if $g \in \mathcal{F}[U]$ is primitive and $h \in \mathcal{F}[U]$ with $ht_{\mathcal{F}}(h) = ht_{\mathcal{F}}(g)$ then h is also primitive. It further follows from Lemma 38 that $\alpha \circ g$, for $\alpha \in \mathcal{F}[g(U)]$ with $ht_{\mathcal{F}}(\alpha) < n$, is primitive. Again, if $V \subseteq X$ is connected and $\beta \in \mathcal{F}[V]$ with $ht_{\mathcal{F}}(\beta) < n$, then $g \circ \beta$ is primitive.

Lemma 40 *Let $\mathcal{F}$ be a tower on X, $x \in X$.*

- *For $0 < n \leq ht_{\mathcal{F}}(x)$, there is a primitive element of height n in $\mathcal{F}_x$.*
- *If $g \in \mathcal{F}_x$ with $ht_{\mathcal{F}}(g) = n$, then g is a composition of primitive elements of height n; the length of the composition depends only on g.*

- *If $n < ht_{\mathcal{F}}(x)$ then $\mathcal{F}_x$ contains arbitrarily long compositions of primitive elements of height n.*

Proof: See revisions, where we will further comment on the natural order on orbits of $\mathcal{F}$.

Let $\mathcal{F}$ be a tower with base f. As $\mathcal{F}$ is a parabolic, hence proper enrichment of f, $J(\mathcal{F})$ and $\Omega(\mathcal{F})$ are invariant. For $0 < n < height(\mathcal{F}) + 1$, $\mathcal{F}^n \trianglelefteq \mathcal{F}$, so $J(\mathcal{F}^n) \subseteq J(\mathcal{F})$ for $0 < n < height(\mathcal{F})$. It is easily checked that

$$J(\mathcal{F}) = \overline{\bigcup_{n=1}^{\infty} J(\mathcal{F}^n)} \tag{2:2}$$

for infinite height $\mathcal{F}$.

Lemma 41 *Let $\mathcal{F}$ be a tower, $0 < n < height(\mathcal{F})$.*

- *If $\mathcal{F}_x = \mathcal{F}^n_x$ then $ht_{\mathcal{F}}(x) = ht_{\mathcal{F}^n}(x) \leq n$ and $\overline{ht}_{\mathcal{F}^n}(x) = \overline{ht}_{\mathcal{F}}(x)$.*
- *Otherwise, $x \in \Omega(\mathcal{F}^n)$ and $\overline{ht}_{\mathcal{F}^n}(x) = ht_{\mathcal{F}^n}(x) = n < \overline{ht}_{\mathcal{F}}(x)$.*

Proof: As $ht_{\mathcal{F}}(g) = ht^n_{\mathcal{F}}(g)$ for every $g \in \mathcal{F}^n$, it follows that $ht_{\mathcal{F}}(x) = ht^n_{\mathcal{F}}(x) \leq n$ if and only if $\mathcal{F}_x = \mathcal{F}^n_x$. Moreover, if $\mathcal{F}_x \neq \mathcal{F}^n_x$ then $ht^n_{\mathcal{F}}(x) = n < ht_{\mathcal{F}}(x)$ by Lemma 40, and $x \in \Omega(\mathcal{F}^n)$ by Lemma 25.

As $ht_{\mathcal{F}}(g(x)) \leq ht_{\mathcal{F}}(x)$ for $g \in \mathcal{F}_x$, it follows that $\overline{ht}_{\mathcal{F}^n}(x) = \overline{ht}_{\mathcal{F}}(x)$ when $ht_{\mathcal{F}}(x) \leq n$. On the other hand, if $ht_{\mathcal{F}}(x) > n$ then $\overline{ht}_{\mathcal{F}^n}(x) = ht_{\mathcal{F}^n}(g(x))$ for some g with $ht_{\mathcal{F}}(g) = n$. By Lemma 39, $\overline{ht}_{\mathcal{F}^n}(x) = ht_{\mathcal{F}^n}(x) = n$. □

Consider, for $0 \leq n \leq \infty$, the invariant set

$$X_n(\mathcal{F}) = \{x \in X : \overline{ht}_{\mathcal{F}}(x) = n\}.$$

Then $J_n(\mathcal{F}) = J(\mathcal{F}) \cap X_n(\mathcal{F})$ and $\Omega_n(\mathcal{F}) = \Omega(\mathcal{F}) \cap X_n(\mathcal{F})$ are also invariant. Clearly,

$$J_0(\mathcal{F}) = \bigcup_{g \in \mathcal{F}} g^{-1}(\partial(f)) \tag{2.3}$$

is the smallest backward invariant set containing $\partial(f)$, and $\Omega_0(\mathcal{F})$ consists of the escaping components of $\Omega(\mathcal{F})$. We further denote

$$X_+(\mathcal{F}) = \{x \in X : \overline{ht}_{\mathcal{F}}(x) > 0\}$$

and define $J_+(\mathcal{F})$ and $\Omega_+(\mathcal{F})$ accordingly; this convention is consistent with our earlier usage of $J_+(f)$ and $\Omega_+(f)$. By ?? and 2.1,

$$\pi^{-1}(J_{n-1}(\mathcal{R}\mathcal{F})) \subseteq J_n(\mathcal{F})$$

for $n \geq 2$, while

$$\pi^{-1}(J_0(\mathcal{R}\mathcal{F})) \subseteq J_0(\mathcal{F}) \cup J_1(\mathcal{F}). \tag{2.4}$$

Similarly, $\Omega_n(\mathcal{F}) \subseteq \mathcal{B}_f$ for $n \geq 2$ and

$$\pi(\Omega_n(\mathcal{F})) \subseteq \Omega_{n-1}(\mathcal{R}\mathcal{F}),$$

while

$$\pi(\mathcal{B}_f \cap (\Omega_0(\mathcal{F}) \cup \Omega_1(\mathcal{F})),) \subseteq \Omega_0(\mathcal{R}\mathcal{F}).$$

Note that the function $ht_{\mathcal{F}} : X \to [0, \infty]$ is lower semi-continuous; that is

$$O_n(\mathcal{F}) = \{x \in X : ht_{\mathcal{F}}(x) > n\}$$

is open for every n. The infinite height points thus constitute an invariant G_δ in X.

Lemma 42 *Let $\mathcal{F}$ be a tower, $0 < n < height(\mathcal{F})$. Then $\Omega(\mathcal{F}^n) = O_n \cup \Omega(\mathcal{F})$. Each component of O_n is a component of $\Omega_n(\mathcal{F}^n)$. Every other component of $\Omega(\mathcal{F}^n)$ is a component of $\Omega(\mathcal{F})$ with $ht_{\mathcal{F}}(U) = ht_{\mathcal{F}^n}(U)$.*

Proof: Without loss of generality, $height(\mathcal{F}) \geq 2$, and we write $\mathcal{F} = f * \mathcal{H}$ with transit map Φ. By Lemma 41, $O_n \cup \Omega(\mathcal{F}) \subseteq \Omega(\mathcal{F}^n)$ and it suffices to show that any component U of $\Omega(\mathcal{F}^n)$ intersecting $O_n(\mathcal{F})$ is contained in $O_n(\mathcal{F})$. Clearly, $U \subseteq \mathcal{B}_f$, and Φ is defined on the cylinder containing $\pi(U)$. Suppose $n = 1$. For each $x \in U$ there is a lift $\tilde{\Phi}$ with $\tilde{\Phi}(\varpi(x)) \in V_f$, and $g = \chi \circ \tilde{\Phi} \circ \varpi \in \mathcal{F}_x$ has $ht_{\mathcal{F}}(g) = 2$; consequently, $U \subseteq O_2(\mathcal{F})$. On the other hand, if $n > 1$ then $\pi(U) \subseteq \mathcal{R}U$ intersects $O_{n-1}(\mathcal{H})$, and by induction $\mathcal{R}U \subseteq O_{n-1}(\mathcal{H})$; consequently, $U \subseteq O_n(\mathcal{F})$. □

Thus, $J(\mathcal{F}^n)$ consists of the points of $J(\mathcal{F})$ with $ht_{\mathcal{F}}(x) \leq n$. Lemma 41 implies more generally that

$$J_m(\mathcal{F}^n) = \{x \in J(\mathcal{F} : \ ht_{\mathcal{F}}(x) = m, \ \overline{ht}_{\mathcal{F}}(x) = n\}$$

for every m. Consequently,

$$J_m(\mathcal{F}) = \bigcup_{g \in \mathcal{F}} g^{-1}(J_m(\mathcal{F}^m)) \tag{2.5}$$

for $0 < m < height(\mathcal{F}) + 1$, and thus for infinite height $\mathcal{F}$,

$$J_m(\mathcal{F}) = \bigcup_{n=1}^{\infty} J_m(\mathcal{F}^n) \tag{2.6}$$

for $0 < m < \infty$. Similarly,

$$\Omega_m(\mathcal{F}) = \bigcup_{g \in \mathcal{F}} g^{-1}(\Omega_m(\mathcal{F}^{m+1}))$$

for $0 \leq m < height(\mathcal{F})$. As $J_1(\mathcal{F}^1) = J_+(f)$, it follows from 2.3 and 2.5 that $J_0(\mathcal{F}) \cup J_1(\mathcal{F})$ is the smallest backward invariant set containing $J(f)$.

Proposition 5 *Let $\mathcal{F}$ be a tower with base f. Then $J_0(\mathcal{F}) \cup J_1(\mathcal{F})$ is dense in $J(\mathcal{F})$; for typical f, $J_0(\mathcal{F})$ is dense.*

Proof: As the second statement is an immediate consequence of the first, we may prove them simultaneously by induction on $n = height(\mathcal{F})$. In view of 2.2 and 2.6 we may assume $n < \infty$, and for $n = 1$ there is nothing to prove. Assume $n \geq 2$, and let $J = J_0(\mathcal{F}) \cup J_1(\mathcal{F})$. As $J(f) \subseteq J(\mathcal{F})$, it suffices to show $J(\mathcal{F}) \cap \mathcal{B}_f \subseteq \overline{J}$.

Fix $x \in J(\mathcal{F}) \cap \mathcal{B}_f$, and connected open $U \subseteq \mathcal{B}_f$ containing x. Suppose $\pi(U)$ is simple for $\mathcal{RF}$. By the second part of Lemma 37, either $U \cap J \neq \emptyset$ or U is simple for $\mathcal{F}$. In the latter case, $\mathcal{F}[U]$ cannot be a normal family, so $U \cap J \neq \emptyset$ by Montel's Theorem. As any smaller neighborhood of $\pi(x)$ is simple, it follows that $x \in \overline{J}$.

On the other hand, if no such $\pi(U)$ is simple then $\pi(x)$ has no simple neighborhood. Consequently, $\pi(x) \in J(\mathcal{RF})$. By induction, $\pi(x) \in \overline{J_0(\mathcal{RF})}$ as $\mathcal{RF}$ has typical base. In view of 2.4, $x \in \pi^{-1}(\overline{J_0(\mathcal{RF})}) \subseteq \overline{J}$. □

It immediately follows that $\Omega_m(\mathcal{F})$ is the interior of $X_m(\mathcal{F})$ for $1 < m <\leq \infty$. In particular, $J_m(\mathcal{F})$ has empty interior for $1 < m \leq \infty$.

Let $\mathcal{F}$ be a tower, U a component of $\Omega(\mathcal{F})$. *A priori*, the semigroup $Fix_{\mathcal{F}}(U)$ fixing U might have elements of many different heights. In fact, the actual possibilities are quite restricted. Recall that the renormalized towers $\mathcal{R}^m\mathcal{F}$ for $1 \leq m < height(\mathcal{F})$ have positive base elements.

Lemma 43 *Let $\mathcal{F}$ be a tower with positive base, U an equatorial component of $\Omega(\mathcal{F})$. Then $ht_{\mathcal{F}}(U) \leq 1$.*

Proof: Let f be the base of $\mathcal{F}$. If $ht_{\mathcal{F}}(U) > 0$ then U lies in a periodic component of $\Omega(f)$, and the latter component V is also equatorial. In view of Lemma 34, V is not a parabolic domain; thus, $U = V$ and $ht_{\mathcal{F}}(U) = 1$. □

Proposition 6 *Let $\mathcal{F}$ be a tower, U a component of $\Omega(\mathcal{F})$, $g \in Fix_{\mathcal{F}}(U)$ nontrivial. Then $ht_{\mathcal{F}}(g) \leq ht_{\mathcal{F}}(U) \leq ht_{\mathcal{F}}(g) + 1$.*

Proof: By definition, $ht_{\mathcal{F}}(g) \leq ht_{\mathcal{F}}(U)$; we establish the other inequality by induction on $n = ht_{\mathcal{F}}(g)$. Let f be the base of $\mathcal{F}$. If $n = 1$ then g is the restriction of an iterate of f and U lies in a periodic component of $\Omega(f)$. Moreover, if $ht_{\mathcal{F}}(U) > 1$ the latter component is a parabolic domain. Let γ be a path in U with endpoints x and $g(x)$. As $\varpi(g(x)) = \varpi(x) + m$ for some positive integer m, the curve $\pi \circ \gamma$ in $\mathcal{R}U$ is closed and homotopically non-trivial on X_f, hence $\mathcal{R}U$ is equatorial. As $\mathcal{R}\mathcal{F}$ has positive base, we conclude from Lemma 43 that $ht_{\mathcal{F}}(U) = ht_{\mathcal{R}\mathcal{F}}(\mathcal{R}U) + 1 \leq 2$. On the other hand, if $n > 1$ then $\mathcal{R}g \in Fix_{\mathcal{R}\mathcal{F}}(\mathcal{R}U)$ is nontrivial. By induction,

$$ht_{\mathcal{F}}(U) = ht_{\mathcal{R}\mathcal{F}}(\mathcal{R}U) + 1 \leq ht_{\mathcal{R}\mathcal{F}}(\mathcal{R}g) + 1 = ht_{\mathcal{F}}(g) + 1. \quad \square$$

Corollary 3 *Let $\mathcal{F}$ be a tower, U a component of $\Omega(\mathcal{F})$ with $ht_{\mathcal{F}}(U) = \infty$. Then U wanders.*

Proof: Suppose $g, h \in \mathcal{F}[U]$ with $g_*U = h_*U$. Without loss of generality, $h = \alpha \circ g$ for some $\alpha \in Fix_{\mathcal{F}}(g_*U)$. If $g \neq h$ then α is nontrivial, and $ht_{\mathcal{F}}(U) \leq ht_{\mathcal{F}}(g) + 1 < \infty$ by the Proposition; thus $g = h$. As $\overline{ht}_{\mathcal{F}}(U) = \infty$, U cannot escape. Consequently, U wanders. □.

Let $\mathcal{F}$ be a tower, U a component of $\Omega(\mathcal{F})$, and consider the quantities

$$ht^\flat_{\mathcal{F}}(U) = \min\{ht_{\mathcal{F}}(g) : g \in Fix_{\mathcal{F}}(U) \text{ nontrivial }\},$$

$$ht^\#_{\mathcal{F}}(U) = \max\{ht_{\mathcal{F}}(g) : g \in Fix_{\mathcal{F}}(U) \text{ nontrivial }\}.$$

By convention, $ht^\flat_{\mathcal{F}}(U) = 0 = ht^\#_{\mathcal{F}}(U)$ for wandering U, and both are otherwise positive. In view of Proposition 6,

$$ht^\flat_{\mathcal{F}}(U) \leq ht^\#_{\mathcal{F}}(U) \leq ht_{\mathcal{F}}(U) \leq ht^\flat_{\mathcal{F}}(U) + 1.$$

We say that U is a *type I* component when $ht^\flat_{\mathcal{F}}(U) = ht^\#_{\mathcal{F}}(U)$, a type II component otherwise.

Lemma 44 *Let $\mathcal{F}$ be a tower, U a non-wandering component of $\Omega(\mathcal{F})$, $m = ht^\flat_{\mathcal{F}}(U) \geq 1$. The elements $h \in Fix_{\mathcal{F}}(U)$ with $ht_{\mathcal{F}}(h) = m$ are the iterates of a unique $g \in Fix_{\mathcal{F}}(U)$ with $ht_{\mathcal{F}}(g) = m$.*

Proof: Fix $g \in Fix_{\mathcal{F}}(U)$ with $ht_{\mathcal{F}}(g) = m$ of minimal rank ℓ. If γ is another such element then without loss of generality $g \preceq \gamma$, hence $\gamma = \alpha \circ g$ for some $\alpha \in Fix_{\mathcal{F}}(U)$. As g and γ have the same rank, $ht_{\mathcal{F}}(\alpha) < m$; thus, $\alpha = Id_U$, so g is unique. By the minimality of ℓ, each $h \in Fix_{\mathcal{F}}(U)$ with $ht_{\mathcal{F}}(h) = m$ has rank $n\ell$ for some $n \geq 1$. As $g^n \in Fix_{\mathcal{F}}(U)$, it follows that $h = g^n$. □

We say g is the *generator* of height $ht^{\flat}_{\mathcal{F}}(U)$. In the type II case there are height $ht^{\#}_{\mathcal{F}}(U)$ elements of minimal rank, and for any two such $h_1 \preceq h_2$ there exists ℓ with $h_2 = g^{\ell} \circ h_1$. However, there is no canonical generator of height $ht^{\#}_{\mathcal{R}}(U)$.

Lemma 45 *Let $\mathcal{F}$ be a tower, U a component of $\Omega(\mathcal{F})$. Then $ht^{\#}_{\mathcal{F}}(U) = ht_{\mathcal{F}}(U)$ if and only if U is final. In particular, every type II component is final.*

Proof: Suppose first that U is final. For each $\alpha \in \mathcal{F}[U]$ there exists $\beta \in Fix_{\mathcal{F}}(U)$ with $\alpha \preceq \beta$, hence $ht_{\mathcal{F}}(\alpha) \leq ht_{\mathcal{F}}(\beta)$. Consequently, $ht^{\#}_{\mathcal{F}}(U) = ht_{\mathcal{F}}(U)$.

Assume conversely that $ht^{\#}_{\mathcal{F}}(U) = ht_{\mathcal{F}}(U)$. Fix $g \in Fix_{\mathcal{F}}(U)$ of height $ht^{\#}_{\mathcal{F}}(U)$ and minimal rank ℓ, and suppose $h \in \mathcal{F}[U]$. If $ht_{\mathcal{F}}(h) < ht^{\#}_{\mathcal{F}}(U)$ then $h \preceq g$. Otherwise, $ht_{\mathcal{F}}(h) = ht^{\#}_{\mathcal{F}}(U)$ of rank m, and $g^n \preceq h$ when $n\ell > m$. Thus, U is final. □

Recall that as $\mathcal{F}$ is hierarchical, each $\alpha \in \mathcal{F}[U]$ determines a homomorphism

$$\alpha_* : Fix_{\mathcal{F}}(U) \longrightarrow Fix_{\mathcal{F}}(\alpha_* U)$$

where $\alpha_*(g) \circ \alpha = \alpha \circ g$.

Lemma 46 *Let $\mathcal{F}$ be a tower, U a component of $\Omega(\mathcal{F})$. Then α_* is an isomorphism.*

Proof: See revisions. □

In particular, $ht_{\mathcal{F}}(\alpha_* g) = ht_{\mathcal{F}}(g)$ for every $g \in Fix_{\mathcal{F}}(U)$. Thus, $ht^{\flat}_{\mathcal{F}}(\alpha_* U) = ht^{\flat}_{\mathcal{F}}(\alpha_* U)$, and $ht^{\#}_{\mathcal{F}}(\alpha_* U) \leq ht^{\#}_{\mathcal{F}}(\alpha_* U)$

In addition, if $U \subseteq \mathcal{B}_f$ then $g \rightsquigarrow \mathcal{R}g$ gives a surjective homomorphism

$$\mathcal{R} : Fix_{\mathcal{F}}(U) \rightarrow Fix_{\mathcal{RF}}(\mathcal{R}U). \tag{2.7}$$

For $\alpha \in \mathcal{F}[U]$ with $\alpha_* U \subseteq \mathcal{B}_f$ the diagram

$$\begin{array}{ccc} Fix_{\mathcal{F}}(U) & \xrightarrow{\alpha_*} & Fix_{\mathcal{F}}(\alpha_* U) \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ Fix_{\mathcal{RF}}(\mathcal{R}U) & \xrightarrow{(\mathcal{R}\alpha)_*} & Fix_{\mathcal{RF}}(\mathcal{R}\alpha_* U) \end{array}$$

commutes. Recall that if $\mathcal{R}g = \mathcal{R}h$ with $g \leq h$ then $h = f^m \circ g$ for some $m \geq 0$. Thus, 2.7 is an isomorphism if and only if $ht^{\flat}_{\mathcal{F}}(U) > 1$.

In the revisions, we will discuss the various dynamical possibilities for final components of $\Omega(\mathcal{F})$.

2.5 Complete Towers

Definition. Let X and Y be compact Riemann surfaces, $W \subseteq X$ open and connected. We shall say $y \in Y$ is a *neglected value* of $f : W \rightarrow Y$ if there exist a connected open set U intersecting ∂W and a component V of $U - \partial W$ such that $V \cap f^{-1}(y) = \emptyset$. An analytic map $f : W \rightarrow Y$ is *complete* if the set $\mathcal{N}(f)$ of neglected values is finite.

Observe that we are including the possibility that $\partial(f) = \emptyset$, in which case f is a finite degree surjection and $\mathcal{N}(f) = \emptyset$; we shall say that $\deg f = \infty$ when $\partial(f) \neq \emptyset$. Thus, $f(W) \supseteq Y - \mathcal{N}(f)$ for any complete map $f : W \rightarrow Y$.

By definition, a complete map has no removable singularities. Conversely, let X and Y be compact Riemann surfaces, $W \subseteq X$ open, and assume that $f : W \to Y$ has no removable singularities. Suppose further that $\partial(f)$ is countable. Then $W = X - \partial(f)$ is connected, and as the isolated points of $\partial(f)$ are dense, it follows from Picard's Theorem that f is complete with $\#\mathcal{N}(f) \leq 2$.

Let X and Y be compact Riemann surfaces, $W \subseteq X$ open and connected, $f : W \to Y$ analytic. Given $A \subseteq X$, consider the subset

$$f_\circ A = \{y \in Y : \ f^{-1}(y) \subseteq A\}$$

of $f(A) \cup \mathcal{N}(f)$. Clearly, $f_\circ A \subseteq f_\circ B$ for $A \subseteq B$. If f is a complete map of infinite degree and A is finite, then $f_\circ A \subseteq \mathcal{N}(f)$.

More generally, let X and Y be complex 1-manifolds with compact components, and $W \subseteq X$ open. We shall say that $f : W \to X$ is complete if $f_{|V}$, for each component V of W, is complete as a map to the component of Y containing $f(V)$. Then $\mathcal{N}(f) = \bigcup_V \mathcal{N}(f_{|V})$ is countable. Similarly, set $f_\circ A = \bigcup_V f_{|V\circ}(V \cap A)$ for $A \subseteq X$.

Lemma 47 *Let X, Y and Z be complex 1-manifolds with compact components, $V \subseteq X$ and $W \subseteq Y$ connected open sets, $f : V \to Y$ and $g : W \to Z$ complete analytic maps. Then $g \circ f$ is complete, and*

$$\mathcal{N}(g \circ f) \subseteq \mathcal{N}(g) \cup g_\circ\mathcal{N}(f).$$

Proof: See revisions.

In particular, $\mathcal{N}(g \circ f_{|U}) = \mathcal{N}(g)$ when $\deg f = \infty$.

More generally, let X and Y be complex 1-manifolds with compact components, and $W \subseteq X$ open. We shall say that $f : W \to X$ is complete if $f_{|V}$, for each component V of W, is complete as a map to the component of Y containing $f(V)$. Then $\mathcal{N}(f) = \bigcup_V \mathcal{N}(f_{|V})$ is countable. Similarly, set $f_\diamond A = \bigcup_V f_{|V\diamond}(V \cap A)$ for $A \subseteq X$.

Under these conventions, if f and g are complete and composable, then $g \circ f$ is complete,

$$\mathcal{N}(g \circ f) \subseteq \mathcal{N}(g) \cup g_\diamond \mathcal{N}(f).$$

It follows from Lemma 47 that the iterates of a complete analytic map f are complete. Moreover, if $\mathcal{N}(f)$ is finite, for example when $W(f)$ is connected, then $\mathcal{N}(f^n) \subseteq \mathcal{N}(f)$ for every n. In general, $\bigcup_{n=1}^{\infty} \mathcal{N}(f)$ is the smallest set A containing $\mathcal{N}(f)$ such that $f_\diamond A \subseteq A$. Let

$$\mathcal{N}\mathcal{E}(f) = \mathcal{E}(f) \cup \bigcup_{n=1}^{\infty} \mathcal{N}(f),$$

where $\mathcal{E}(f)$ is the set of exceptional values; then $\mathcal{N}\mathcal{E}(f^n) \subseteq \mathcal{N}\mathcal{E}(f)$ for every n. Furthermore, any return map f^Z is complete, and $\mathcal{N}(f^Z) \subseteq \mathcal{N}\mathcal{E}(f)$.

Consider the special case in which X is a Riemann surface and $\partial(f)$ is countable. As $\partial(f^n)$ is countable, it follows that $\mathcal{N}(f^n) \subseteq \mathcal{N}(f)$ for any n. If f is typical, then $\mathcal{E}(f) = \emptyset$, and it follows that $\mathcal{N}\mathcal{E}(f) = \mathcal{N}(f)$; by the discussion above, $\#\mathcal{N}\mathcal{E}(f) \leq 2$. We may still conclude $\#\mathcal{N}\mathcal{E}(f) \leq 2$ when f is exceptional: if $\deg f < \infty$ then $\mathcal{N}(f) = \emptyset$, while the backward invariance of $\mathcal{E}(f)$ forces $\mathcal{E}(f) \subseteq \mathcal{N}(f)$ when $\deg f = \infty$. Moreover, $\{\infty\} \subseteq \mathcal{N}(f)$ for entire f, equality holding unless f is conjugate to $z \rightsquigarrow p(z)e^{g(z)} + c$, where g is holomorphic on $\mathbf{C}$, p is a polynomial, and $c \in \mathbf{C}$. If f is Radström, then $\mathcal{N}(f) = \mathcal{E}(f) = \{0, \infty\}$.

Let f is a complete analytic map with empty elementary part, on a complex 1-manifold X. Then $Z \cap J(f)$ is infinite for every component Z of X with an essential successor. Recall that a set R in a topological space T is *residual* if R contains the intersection of countably many open dense subsets of T. By the Baire Category Theorem [4], a residual subset of a complete metric space is dense.

Lemma 48 *Let f a complete analytic map on X. Assume that f has empty elementary part, and that every component of X has an essential successor. Then $W(f^n) \cap J(f)$, for each $n \geq 1$, is open and dense in $J(f)$. Consequently, $J_+(f)$ is residual in $J(f)$.*

Proof: We proceed by induction on n. Suppose $n = 1$. Fix a component V of $W(f)$, and let Z be the component of X containing $f(V)$. By assumption, $Z \cap J(f) - \mathcal{N}(f_{|V}) \neq \emptyset$, hence $\partial V \subseteq \overline{f^{-1}(J(f))} \subseteq \overline{W(f) \cap J(f)}$. Consequently,

$$\overline{W(f) \cap J(f)} = \partial(f) \cup (W(f) \cap J(f)) = J(f).$$

Now suppose $n > 1$. By induction, $W(f^{n-1}) \cap J(f)$ is open and dense in $J(f)$. As f is an open map and

$$f^{-1}(W(f^{n-1}) \cap J(f)) = W(f^n) \cap J(f),$$

we conclude that $W(f^n) \cap J(f)$ is open and dense in $J(f)$. Therefore,

$$J_+(f) = \bigcap_{n=1}^{\infty} W(f^n) \cap J(f)$$

is residual in $J(f)$. $\Box$.

Under the above hypotheses, it follows immediately that

$$\partial(f^m) \subseteq \partial(f^n) = \overline{f^{-(n-m)}(\partial(f^m))} \tag{2.8}$$

for $m < n$. Furthermore, if f is typical and U is a component of $\Omega_+(f)$, then the component of $W(f^n)$ containing U is a proper subset of the corresponding component of $W(f^m)$.

Let f a non-elementary complete analytic map on a Riemann surface X. Suppose $y \in X - \mathcal{NE}(f)$, and let $U \subseteq X$ be an open set intersecting $J(f)$. For exceptional f, it follows from the definition of $\mathcal{E}(f)$ that $y \in f^n(U)$ for some $n \geq 0$. This observation underlies the classical arguments that prove $J(f)$ is perfect. Given $x \in J(f)$ and $y \notin \mathcal{E}(f)$, there exists a backward orbit string $\cdots \xrightarrow{f} y_{-1} \xrightarrow{f} y$ converging to x; we may arrange that $y_{-1} \neq y$, and thus $y_{-k} \neq x$ for infinitely many k.

On the other hand, if f is typical then U intersects $\partial(f^n)$ for some $n \geq 1$, and we again conclude that $y \in f^n(U)$. In general, any closed backward invariant set either contains $J(f)$ or lies in $\mathcal{NE}(f)$.

Lemma 49 *Let f be a complete analytic map on a Riemann surface. Assume that f is typical. Then there exists $\ell \leq 3$ such that for any $n \geq 1$, every point in $\partial(f^n)$ is an accumulation point of $\partial(f^{\ell+n})$.*

Proof: Let $\ell = \sup_{x \in \partial(f)} \ell_x$, where ℓ_x is the least $m \geq 0$ such that x is an accumulation point of $\partial(f^{m+1})$. Clearly, $\ell_x = 0$ if and only if x is isolated in $\partial(f)$, so $\ell = 0$ if and only if $\partial(f)$ is perfect. Suppose $\ell > 0$. It follows from Picard's Theorem that X is a torus or sphere; moreover, $\ell = 1$ on the torus, while

$$\ell \;=\; \min\{m \geq 1 : \#\partial(f^m) \geq 3\} \;\leq\; 3$$

on the sphere. The conclusion follows by 2.8. □

In particular, $J(f)$ is perfect, hence uncountable. Consequently, any open set U intersecting $J(f)$ contains points in $J(f) - \mathcal{NE}(f)$, and thus

$$J(f) \subseteq \overline{\bigcup_{n=0}^{\infty} f^{-n}(U)}$$

by the remarks above.

Proposition 7 *Let $\mathcal{F}$ be a tower on X with complete base f. Assume that every component of X has an essential successor. Then $J(\mathcal{F})$ is perfect.*

Proof: In view of Proposition 5 and Lemma 20, it suffices to treat the height 1 case. Suppose that $x \in J(f)$ is isolated. If $x \in J_+(f)$ then $W_\infty(f)$ contains a neighborhood U of x. Moreover, some sequence $f^{n_k}_{|U}$ must fail to be normal, and we may assume that the images $f^{n_k}(U)$ all lie in the same component Z of X. On the other hand, $Z \cap J(f)$ is infinite, and it follows by Montel's Theorem that x cannot be isolated. Thus, $x \in \partial(f^n)$ for some n, so $U - \{x\} \subseteq W(f^n)$ for some neighborhood U. Let Z be the component of X containing $f^n(U - \{x\})$. Again, $Z \cap J(f)$ is infinite, and it now follows from Picard's Theorem that x cannot be isolated. □

We digress slightly to present some simple observations crucial to the construction of towers in the next Chapter. Let f be a complete analytic map on a complex 1-manifold X, and suppose U is a finite order component of $\Omega(f)$, that is $f^m_{|U} = Id_U$ for $m \geq 1$. Further, let V be the component of $W(f^m)$ containig U, and Z the component of X containing V. Then $Id_V = f^m_{|V}$, and therefore $V = Z$; thus, Z lies in the elementary part of f. Consequently, if f is an analytic map on X and $\hat{f}$ is complete with empty elementary part, then $\Omega(f)$ has no finite order components. Similar considerations prove the following:

Lemma 50 *Let f be an analytic map on a complex 1-manifold X. Assume that $\hat{f}$ is complete with empty elementary part, and that every component of X has an essential successor. Further, let $\mathcal{F}$ be a tight system with base f. Then f is the unique base of $\mathcal{F}$.*

Proof: Assume without loss of generality that f is complete, and let g be another base element. Then f and g have the same domain, and $f_{|U} \neq g_{|U}$ on some component U; let $Y = f(U)$, $Z = g(U)$. As f and g are base elements, there exist $\alpha \in \mathcal{F}[Y]$ and $\beta \in \mathcal{F}[Z]$ with $g_{|U} = \alpha \circ f_{|U}$ and $f_{|U} = \beta \circ g_{|U}$. Clearly, $\beta \circ \alpha = Id_Y$ and $\alpha \circ \beta = Id_Z$.

As f is a base element, either $\alpha = f^m_{|Y}$ for some $m > 0$ or else $f^n_{|Y} \preceq \alpha$ for every n; similarly, $\beta = f^m_{|Z}$ for some $m > 0$, or else $f^n_{|Z} \preceq \beta$ for every n.

Consequently, $Y \subseteq W_\infty(f)$, and $f^n_{|Y}$ is injective for every n. By completeness, Y is a component of X and therefore compact. It follows that $Y_k = f^k(Y)$ is a component of X, and $f_{|Y_k} : Y_k \to Y_{k+1}$ a bijection, for every $k \geq 0$. By assumption, $Y_k = Y_\ell$ for some $k \neq \ell$; but then Y_k lies in the elementary part of f. □

We record the obvious relation between neglected and singular values.

Lemma 51 *Let $f : W \to Y$ be a complete analytic map. Then $\mathcal{N}(f) \subseteq S(f)$.*

Proof: We may assume without loss of generality that W and Y are connected, and that $\deg f = \infty$. If $y \notin S(f)$ then any simply connected neighborhood $D \subseteq Y - S(f)$ is evenly covered. Fixing $z \in D - \mathcal{N}(f)$ and $x \in \partial(f)$, choose $z_k \in f^{-1}(z)$ with $z_k \to x$; let $g_k : D \to W$ be the local inverse to f with $g_k(z) = z_k$. As the regions $g_k(D)$ are disjoint, the sequence g_k is a normal, hence $g_k \to x$. In particular, $y_k \to x$, where $y_k = g_k(y) \in f^{-1}(y)$. Consequently, $y \notin \mathcal{N}(f)$. □

Definition. Let f be an analytic map on X. The *post-singular set* $PS(f)$ is the smallest forward invariant set containing $S(f)$; we denote its closure $\mathcal{PS}(f)$.

Clearly, $\mathcal{E}(f) \subseteq S(f)$ when f is complete, and thus $\mathcal{NE}(f) \subseteq PS(f)$.

Lemma 52 *Let f be a complete analytic map on X with empty elementary part, $\chi : \mathcal{V} \to X$ a global linearizing or Fatou parameter. Then χ is complete, and $\mathcal{N}(\chi) \subseteq \mathcal{NE}(f)$.*

Proof: Consider first the main component V of $\mathcal{V}$, and let Z be the component of X containing $\chi(V)$, and $h = f^{\{Z\}}$. If $\partial(f) = \{\infty\}$ then $\#\mathcal{N}(\chi) \leq 2$ by Picard's Theorem and the first part of Lemma 31; it follows that $\mathcal{N}(\chi) = \mathcal{E}(h)$ as both sets are countable and backward invariant under h. Suppose, on the other hand, that $\partial(f) \neq \{\infty\}$. By the second part of the Lemma, any open U intersecting $\partial(V)$ must intersect $\chi^{-1}(\partial(h^n))$ for some $n \geq 1$. As h is complete, $\chi = h^n \circ \chi_{|V} \circ \tau^{-n}$ assumes every value $x \notin \mathcal{NE}(h)$ in U, that is, $\mathcal{N}(\chi_{|V}) \subseteq \mathcal{NE}(h)$.

Final details will be supplied in the revisions.

Note that

$$\tau(\partial(\chi)) \subseteq \partial(\chi)$$

by 2.8

Recall that an element $g \in \mathcal{F}[U]$ of a conformal dynamical system $\mathcal{F}$ is maximally continued when U is connected and g extends in $\mathcal{F}$ to no larger connected open set. We write $g \in \mathcal{F}_{MAX}[U]$, and denote $\mathcal{F}_{MAX}$ the collection of all maximally continued elements of $\mathcal{F}$. Let us say that a tower $\mathcal{F}$ is complete if every $g \in \mathcal{F}_{MAX}$ is a complete map. Then

$$\mathcal{NE}(\mathcal{F}) = \mathcal{E}(f) \cup \bigcup_{g \in \mathcal{F}_{MAX}} \mathcal{N}(g)$$

is a countable set.

Lemma 53 *Let $\mathcal{F}$ be a complete mixing tower on X. Assume $height(\mathcal{F}) \geq 2$, and let $\mathcal{Y}$ be the set of components of $\hat{X}_f^-$ where the transit map is defined. Then the tower $(\mathcal{RF})^{\mathcal{Y}}$ is mixing.*

Proof: Suppose Y_1, $Y_2 \in \mathcal{Y}$, and fix components Z_1 and Z_2 of X such that $Z_i \cap \pi^{-1}(Y_i) \neq \emptyset$. By assumption, there exist a connected open $V \subseteq Z_1 \cap \pi^{-1}(Y_1)$ with $y \in \pi(V)$ and $g \in \mathcal{F}_{MAX}[V]$ with $ht_{\mathcal{F}}(g) = 2$. Let Z be the component of X containing $g(V)$. As $\mathcal{F}$ is mixing, there exist a connected open $W \subseteq Z$ and $h \in \mathcal{F}_{MAX}[W]$ with $h(W) \subseteq Z_2$. Fix $z \in Z_2 \cap \pi^{-1}(Y_1) - \mathcal{NE}(\mathcal{F})$, and $x \in g^{-1}(W)$ with $h \circ g(x) = z$. Then $\pi(x) \in Y_1$ and $\alpha(\pi(x)) = \pi(z) \in Y_2$, where $\alpha = \mathcal{R}(h \circ g) \in \mathcal{RF}_{\pi(x)}$ is nontrivial as $ht_{\mathcal{RF}}(\alpha) \geq ht_{\mathcal{F}}(g) - 1 \geq 1$. □

If the base of $\mathcal{F}$ is itself mixing, we may choose h with $ht_{\mathcal{F}}(h) = 1$ to obtain α with $ht_{\mathcal{RF}}(\alpha) = 1$. It follows then that $\mathcal{RF}$ has mixing base.

Lemma 54 *Let $\mathcal{F}$ be a complete mixing tower on X. Suppose $y \in X - \mathcal{NE}(\mathcal{F})$, and let $U \subseteq X$ be an open set intersecting $J(\mathcal{F})$. Then $y = g(x)$ for some $x \in U$ and $g \in \mathcal{F}_x$.*

Proof: By Proposition 5, we may choose open $V \subseteq U$ and $\alpha \in \mathcal{F}[U]$ with $\alpha(V)$ intersecting $J(f)$. Let Z_1 and Z_2 be the components of X containing $\alpha(V)$ and y, and set $\beta = f^{\{Z_2\}}$. By assumption, we may choose a maximally continued $\gamma \in \mathcal{F}[W]$ such that $W \subsetneq Z_1$ and $\gamma(W) \subseteq Z_2$. By assumption,

$y = \gamma(w)$ for some $w \in W$, and there exist $z \in \alpha(V)$ and $m \geq 0$ with $\beta^m(z) = w$. Fix $x \in V$ with $\alpha(x) = z$, and take $g = \gamma \circ \beta^m \circ \alpha$. Then $g \in \mathcal{F}$ and $g(x) = y$. $\square$

Lemma 55 *Let $\mathcal{F}$ be a complete mixing tower on X, and let $\mathcal{Z}$ be a set of components of X. Then $J(\mathcal{F}^{\mathcal{Z}}) = J(\mathcal{F}) \cap \bigcup \mathcal{Z}$.*

Proof: It is enough to show $J(\mathcal{F}^{\{Z\}}) = J(\mathcal{F}) \cap Z$ for every component Z. Clearly, $J(\mathcal{F}^{\{Z\}}) \subseteq J(\mathcal{F}) \cap Z$, so it suffices to show that $J(\mathcal{F}^{\{Z\}})$ is a dense subset. Let U be an open set intersecting $J(\mathcal{F})$, and fix $y \in J(\mathcal{F}^{\{Z\}}) - \mathcal{NE}(f)$. By Lemma 54, there exist $x \in U$ and $g \in \mathcal{F}_x$ with $g(x) = y$. As $y \in Z$, $g \in \mathcal{F}_x^{\{Z\}}$, hence $x \in J(\mathcal{F}^{\{Z\}})$. $\square$

Proposition 8 *Let $\mathcal{F}$ be a complete mixing finite type tower. Assume that $height(\mathcal{F}) \geq 2$. Then $J(\mathcal{F}) = \overline{\pi^{-1}(J(\mathcal{RF}))}$.*

Proof: Clearly, $\overline{\pi^{-1}(J(\mathcal{RF}))} \subseteq J(f) \cup \pi^{-1}(J(\mathcal{RF})) \subseteq J(\mathcal{F})$, where f is the base of $\mathcal{F}$; we show that $\pi^{-1}(J(\mathcal{RF}))$ is dense. Let U be an open set intersecting $J(\mathcal{F})$, and fix $y \in \mathcal{B}_f - \mathcal{NE}(\mathcal{F})$ with $\pi(y) \in J(\mathcal{RF}))$. In view of Lemma 54, $y = g(x)$ for some $x \in U$ and $g \in \mathcal{F}_x$, and $x \in \mathcal{B}_f$ by Lemma ??. Consequently, $\pi(y) = \pi \circ g(x) = (\mathcal{R}g) \circ \pi(x)$, where $\mathcal{R}g \in \mathcal{RF}_{\pi(x)}$, and it follows that $\pi(x) \in J(\mathcal{RF})$. $\square$

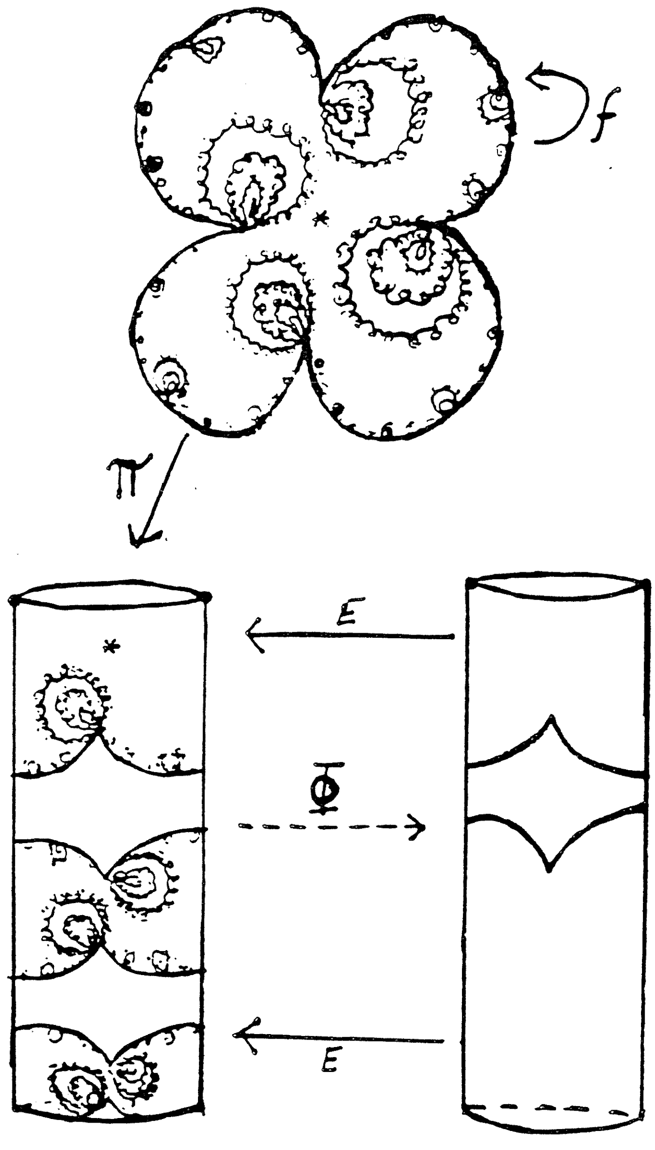

Figure 2.3: $J(\mathcal{F})$ and $J(\mathcal{RF}) = J(\Phi \circ E)$

Chapter 3

Finite Type Maps and Towers

3.1 Finite Type Maps

Definition: Let W and X be complex 1-manifolds, $f : W \to X$ analytic. We say that f is a map of *finite type* if X is compact and $S(f)$ is a finite set.

Given an open set $Y \subseteq X$, we will often write Y^* for $Y - S(f)$ and $Y^\times$ for $Y - \overline{f^{-1}(S(f))}$. When W and X are connected, we take $\deg f$ to be the degree of the covering space $f_{|W^\times} : W^\times \to X^*$. In view of 1.1 we observe:

Lemma 56 *Let f and g be composable analytic maps of finite type. Then $g \circ f$ is a finite type analytic map, and $S(g \circ f) \subseteq S(g) \cup g(S(f))$.*

Recall that the singular set of an analytic map $f : W \to X$ contains every critical or asymptotic value. Conversely, let $x \in S(f)$ be an isolated point, and fix a path $\gamma : [0,1] \to X$ with $\gamma(0) \notin S(f)$ and $\gamma(1) = x$. Each choice of $\tilde{y} \in f^{-1}(y)$ determines a unique lift $\tilde{\gamma} : [0,1) \to W$ with $\tilde{\gamma}(0) = \tilde{y}$.

If $\tilde{\gamma}(t_k) \to w \in W$ for some $t_k \to 1$, then $f(w) = \lim_{k\to\infty} \gamma(t_k) = x$. Thus, $\tilde{\gamma}$ either extends to a closed path in W or tends to infinity. If every lift is bounded in W, then some lift limits at a critical point of f; it follows that x is a critical or asymptotic value. In particular:

Lemma 57 *Let f be a finite type analytic map. Then $S(f) = C(f) \cup A(f)$.*

Every positive degree analytic map of compact Riemann surfaces is a map of finite type. Such a map has no asymptotic values; the possibilities are restricted by the Riemann-Hurwitz formula ??. Our treatment of the infinite degree case begins with a discussion of the covering properties of a finite type map near its domain boundary.

Let $f : W \to X$ be an analytic map of complex 1-manifolds, $B \subseteq X$ a Jordan domain, D a component of $f^{-1}(B)$; we say that D is a *proper preimage* of B if $f_{|D} : D \to B$ is a branched cover, necessarily of finite degree. We will refer to a Jordan domain B containing a single point of $S(f)$ with boundary disjoint from $S(f)$ as an *isolating neighborhood.* If the cyclic cover $f_{|D^\times} : D^\times \to B^*$ has finite degree, then $D^\times$ is conformally equivalent to a punctured disc. The puncture is a removable singularity, so D is a disc if $D \subseteq W$; in this case D is a proper preimage. On the other hand, if $\deg f_{|D^\times} = \infty$ then D is conformally equivalent to the disc, and we refer to D as a *tract.* Its boundary in W is the open arc $f^{-1}(\partial B)$; its *accumulation* is the closed set $\partial D \cap \partial W$.

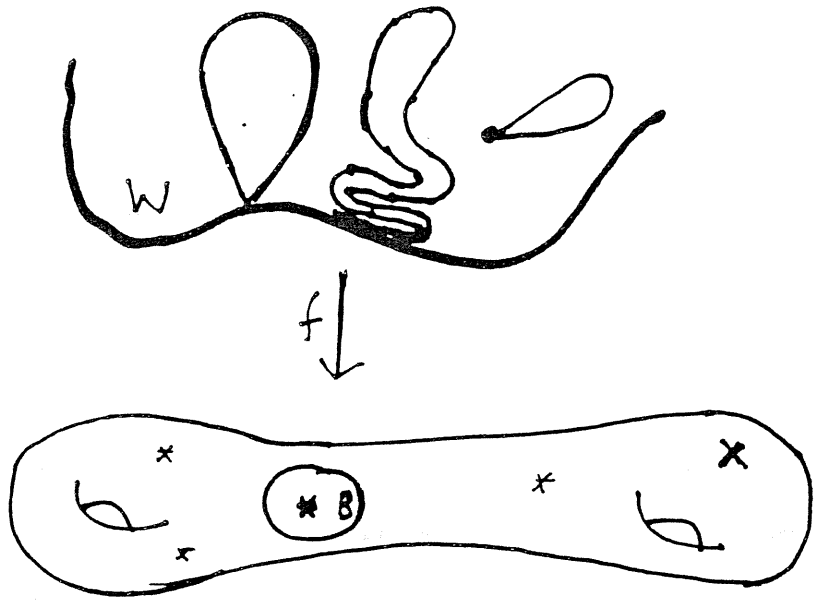

Figure 3.1: Tracts

Lemma 58 *Let $f : W \to X$ be a finite type analytic map, where W lies in a complex 1-manifold Y. Further, let B be an isolating neighborhood of $x \in A(f)$, and T a tract covering B^*. Then $f^{-1}(\partial B)$ is dense in ∂D.*

The proof will show that the ends of $f^{-1}(\partial B)$ have the same accumulation in ∂W. We will require some basic facts concerning the boundary behavior of conformal maps.

Definition. Let U be an open subset of a topological space X. A boundary point $x \in \partial U$ is *accessible* if $x = \lim_{t \to 1} \gamma(t)$ for some path $\gamma : [0, 1) \to U$.

As arc and path connectedness are equivalent for metric spaces, we are free to upgrade paths to arcs or even geodesics when discussing accessible boundary points on Riemann surfaces. It is really no more difficult to work in the more general setting of path-metric spaces. Recall that the length of a path $\gamma : [0,1] \to X$ in a metric space (X, d) is

$$\ell(\gamma) = \sup_{n,\ 0=t_0\leq\cdots\leq t_n=1} \sum_{i=0}^{n} d(\gamma(t_i), \gamma(t_{i+1})) \leq \infty,$$

and γ is *rectifiable* when $\ell(\gamma)$ is finite. The distance function d is a *path-metric* if $d(x, y) = \inf_\gamma \ell(\gamma)$, where γ ranges over all paths in X between x and y.

Lemma 59 *Let U be an open subset of a path-metric space X. Then the accessible boundary points of $U \subseteq X$ are dense in ∂U.*

Proof: Fixing $x \in \partial U$, choose $x_k \in U$ with $d(x, x_k) < \frac{1}{2^k}$. By assumption, we may construct a path $\gamma : [0, \infty) \to X$ with $\gamma(k) = x_k$ and $\ell(\gamma_{|[k,k+1]}) < \frac{3}{2^k}$ for every integer $k \geq 0$; as γ has finite length, we obtain a closed path on setting $\gamma(\infty) = x$. Clearly, x is accessible if $\gamma(t) \in U$ for sufficiently large t; otherwise,

$$t_k = \sup\{t > k : \gamma([k, t]) \subseteq U\} < \infty,$$

$\gamma(t_k) \in \partial U$ and $\gamma(t_k) \to x$. Each $\gamma(t_k)$ is an endpoint of the path $\gamma_{|[k,t_k]}$, hence accessible. □

Lemma 60 *Let X be a Riemann surface, $\phi : \mathbf{\Delta} \to D \subseteq X$ a conformal isomorphism, and α an open arc in D. If α tends to a point of ∂D, then the arc $\phi^{-1} \circ \alpha$ tends to a point of $\partial \mathbf{\Delta}$. Moreover, if two such arcs have distinct endpoints in ∂D, then their inverse images have distinct endpoints in $\partial \mathbf{\Delta}$.*

The above is shown in [30] for plane domains, via the Fatou-Riesz theorems on radial limits of functions holomorphic in the disc; the general case follows on passage to the universal cover. We now prove Lemma 58.

Proof: A conformal isomorphism $\beta : (\mathbf{\Delta}, 0) \to (B, x)$ extends to a homeomorphism of the closures [30]. Choosing a universal cover $\pi : \mathbf{\Delta} \to \mathbf{\Delta}^*$ and a lift $\tilde{\beta} : \mathbf{\Delta} \to T$ of $\beta_{|\mathbf{\Delta}^*}$, we may therefore extend $\tilde{\beta}$ to a homeomorphism $\mathbf{\Delta} \cup \alpha \to D \cup f^{-1}(\partial B)$, where α is the arc complementary to some point of $\partial\mathbf{\Delta}$. Every point in $f^{-1}(\partial B)$ is accessible, while by Lemma 60, at most one accessible boundary point of T belongs to ∂W. In view of Lemma 59, $f^{-1}(\partial B)$ is dense in ∂T. □

Lemma 61 *Let $f : W \to X$ be an analytic covering map of Riemann surfaces, where W lies in a Riemann surface Y. Further, let $U \subseteq Y$ be a connected open set, and assume that $f(w_k)$ is bounded in X for some sequence $w_k \in W$ tending to $U \cap \partial W$. Then any Jordan domain $B \subseteq X$ with $\overline{B} \cap S(f) = \emptyset$ has infinitely many proper preimages compactly contained in U.*

Proof: We may assume without loss of generality that $f(w_k) \to x \in X$. Let r be the diameter of B in the path-metric induced from d_{X^*}. By assumption, $diam_{W^*} D < r < \infty$ for every preimage D of B. Fix $\eta > r + d_{W^*}(x, B)$. In view of Corollary ??, for large enough k the neighborhood

$$R_k = \{w \in W^* : d_{W^*}(w, w_k) < \eta\}$$

lies inside U and is isometric to the hyperbolic disc of radius η. Consequently, R_k contains a preimage of B; as w_k tends to ∂W there are infinitely many such preimages in U. □

Note that if the w_k belong to the same component V of $U \cap W$, then these preimages lie compactly within V.

Proposition 9 *Let $f : W \to X$ be a finite type analytic map, where X is a Riemann surface and W lies in a complex 1-manifold Y, and let $U \subseteq Y$ be a connected open set intersecting ∂W and containing no removable singularities. Then any Jordan domain $B \subseteq X$ with $\overline{B} \cap S(f) = \emptyset$ has infinitely many proper preimages compactly contained in U.*

Proof: By compactness, there exist $v_k \in W$ with $v_k \to y \in U \cap \partial W$ and $f(v_k) \to x \in X$. Suppose $x \in S(f)$ and fix an isolating neighborhood N. If k is sufficiently large then w_k lies in a preimage of N. By assumption, the boundary of this preimage intersects U; by Lemma 58, the corresponding component of $f^{-1}(\partial N)$ intersects U. Shrinking U, we obtain $w_k \to y$ with $f(w_k) \in \partial B$. We may therefore assume at the outset that $x \notin S(f)$, and obtain the desired preimages by Lemma 61. □

It should be possible to adapt this argument to the case where B is an isolating neighborhood of a singular value x of *bounded ramification*:

$$x \notin A(f) \text{ and } \sup_{f(w)=x} \deg_w f < \infty.$$

With unbounded ramification, the proper preimages, if they exist at all, might remain large as they approach ∂W so there is no guarantee that any lie completely inside U.

Suppose further that $\partial U \cap \partial W = \emptyset$, and let B be an isolating neighborhood of $x \in S(f)$. Then U is disjoint from the closure of all but finitely many preimages of B. By Proposition 9 there exist $w_k \in W$ with $w_k \to y \in U \cap \partial W$ and $f(w_k) \to x \in X$, and we may assume that each w_k lies in a preimage of B. A preimage T containing infinitely many w_k must be a tract, as $y \in \partial T$. If there is no such T, then U intersects infinitely many preimages and compactly contains all but finitely many; each of the latter is either a proper preimage or a tract accumulating in $U \cap \partial W$. We have shown:

Corollary 4 *Let $f : W \to X$ be a finite type analytic map of Riemann surfaces, where W lies in a Riemann surface Y, and let $U \subseteq Y$ be an open set intersecting ∂W and containing no removable singularities. Assume that $\partial U \cap \partial W = \emptyset$, and let $B \subseteq X$ be an isolating neighborhood of $x \in S(f)$. Then either U compactly contains infinitely many proper preimages of B, or else some preimage is a tract with accumulation in $U \cap \partial W$.*

The deployment of preimages near an isolated boundary point y is especially simple: some tract or sequence of proper preimages accumulates at y, and the diameters of the proper preimages tend to 0 even if x has unbounded ramification.

It is worth comparing Proposition 9 to a classical theorem involving no assumption of finite type.

Ahlfors' Islands Theorem *Let $B_1, \ldots, B_5$ be Jordan domains in $\hat{\mathbf{C}}$ with disjoint closures. Then:*

- *For any transcendental analytic map $f : \mathbf{C} \to \hat{\mathbf{C}}$, some B_i has infinitely many unramified proper preimages;*
- *The analytic maps $f : \Delta \to \hat{\mathbf{C}}$ under which no B_i has infinitely many unramified proper preimages form a normal family.*

In Ahlfors' terminology, a proper preimage is referred to as an *island.* The number 5 may be reduced slightly if ramified islands are allowed, or if there is an omitted point. The Islands Theorem is a deep result in Ahlfors' theory of covering surfaces. Its proof [16, 33] entails an analysis of the mean covering properties of the analytic maps under consideration; like that of Proposition 9, it is metric in character. We shall not make use of Ahlfors' Theorem in this work.

Let us return to the setting of Proposition 9. If $U \cap \partial W$ is totally disconnected, the use of Lemma 58 may be replaced by a purely topological argument. Otherwise, $U \cap W$ may have more that one component, and we wish to show that each contains proper preimages. The argument below involves considerations of Brownian motion. Through the work of Sullivan [?], the proof may be recast in the language of geodesic flows, and thereby related to that of Lemma 58.

Lemma 62 *Let $f : W \to X$ be an analytic map, where X is a compact Riemann surface, $S(f)$ has capacity 0, and W lies in a complex 1-manifold Y. Further, let $U \subseteq Y$ be a connected open set intersecting ∂W in a set of positive capacity, and let V be a component of $U \cap W$. Then any Jordan domain $B \subseteq X$ with $\overline{B} \cap S(f) = \emptyset$ has infinitely many proper preimages compactly contained in V.*

Proof: By assumption, a continuous path in U starting from a fixed $v \in V - f^{-1}(S(f))$ hits ∂W with positive probability, and avoids $f^{-1}(S(f))$. On the other hand, a path in X starting from $f(v)$ almost surely passes through B infinitely often, while avoiding $S(f)$. It follows that the set of paths $\gamma : [0, \infty) \to U$ such that $\gamma(0) = v$, $\gamma(T) \in U \cap \partial W$ for some finite T, $\gamma(t) \in V$ for $t \in [0, T)$, and $\gamma(t_k) \in f^{-1}(B)$ for some $t_k \nearrow T$, has positive measure. In particular, there exist at least one such path; the conclusion follows by Lemma 62. □

Proposition 10 *Let $f : W \to X$ be a finite type analytic map with no removable singularities, where W lies in a complex 1-manifold Y. Further, let $U \subseteq Y$ be a connected open set intersecting ∂W, and let V be a component of $U \cap W$. Then any Jordan domain $B \subseteq X$ with $\overline{B} \cap S(f) = \emptyset$ has infinitely many proper preimages compactly contained in U.*

In particular, a finite type analytic map $f : W \to X$, where W lies in a complex 1-manifold Y with compact components, is complete if and only if there are no removable singularities.

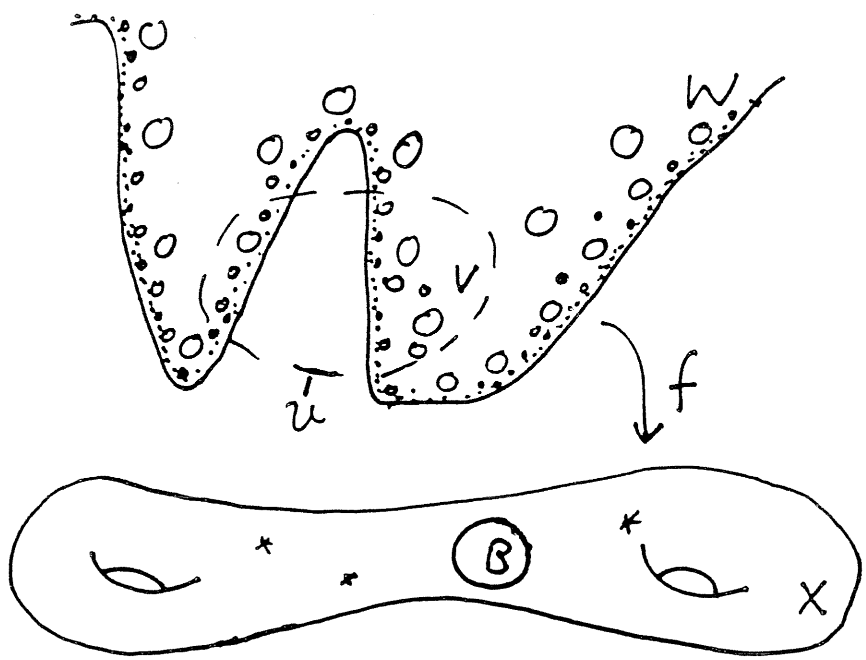

Figure 3.2: Proper preimages near $\partial(f)$

Observe that by Lemma 56, the iterates of a finite type analytic map are maps of finite type.

Lemma 63 *Let f be a finite type analytic map on X, and let $\mathcal{Z}$ be an essential set of components. Then $f^{\mathcal{Z}}$ is map of finite type.*

Proof: As $S(f)$ is finite, there exists m such that for each $y \in S(f)$, the least if any $k \geq 0$ with $f^k(y) \in \bigcup \mathcal{Z}$ is less than or equal to m. Fix a component V of $W(f^{\mathcal{Z}})$; then $f^{\mathcal{Z}}_{|V} = f^n_{|V}$ for some n. For $0 \leq \ell \leq n$, denote V_ℓ the component of $W(f^{n-\ell})$ containing $f^\ell(V)$; observe that $V_0 = V$, V_n is a component in $\mathcal{Z}$, and V_ℓ for $0 < \ell < n$ is disjoint from $\bigcup \mathcal{Z}$. By Lemma 56 and induction, any singular value of $f^{\mathcal{Z}}_{|V}$ can be expressed as $f^k(x)$, where

$0 \leq k < n$ and $x \in S(f) \cap V_{n-k}$; necessarily, $k \leq m$, and it follows that

$$S(f^Z) = \bigcup_V S(f^Z_{|V}) \subseteq \bigcup_{k=0}^{m} f^k(S(f))$$

is finite. □

Similarly, if $x \in X$ has infinite forward orbit, $x_n = f^{n-1}(x)$, then for large enough n the backward orbit of any point in $f^{-1}(x_{n+1}) - \{x_n\}$ is disjoint from $PS(f)$. Furthermore:

Lemma 64 *Let f be a complete finite type analytic map on X. Assume that f has empty elementary part, and let $x \in X$ be a point with infinite forward orbit. Then $f^{-1}(x_{n+1}) - \{x_n\} \neq \emptyset$ for large n.*

Let f be a finite type analytic map on a complex 1-manifold. Setting $\mathcal{NS}_1(f) = S(f)$, let $\mathcal{NS}_{n+1}(f) = S(f) \cup f_\diamond \mathcal{NS}_n(f)$ for $n \geq 1$, and consider

$$\mathcal{NS}(f) = \bigcup_{n=1}^{\infty} \mathcal{NS}_n(f).$$

Lemma 65 *Let f be a complete finite type analytic map on X with empty elementary part. Then $\mathcal{NS}(f)$ is finite.*

Proof: If $\mathcal{NS}(f)$ is infinite, then there exists $x \in S(f)$ whose forward orbit is infinite and lies in $\mathcal{NS}(f)$. Let W_n be the component of $W(f)$ containing $x_n = f^{n-1}(x)$, and Z_n the corresponding component of X. If n is sufficiently large then $x_{n+1} \notin S(f)$, and thus $V \cap f^{-1}(x_{n+1}) \neq \emptyset$ for every component V of $W(f)$ with $f(V) \subseteq Z_{n+1}$; by assumption, $V_n \cap f^{-1}(x_{n+1}) \subseteq \mathcal{NS}(f)$ for some such component V_n. As above, $\mathcal{NS}(f) \cap f^{-1}(x_{n+1}) = \{x_n\}$ for large enough n, so $V_n \cap f^{-1}(x_{n+1}) = \{x_n\}$, and therefore $V_n = W_n$; that is, $f_{|W_n}$ is injective for sufficiently large n. But then $f^{\{Z\}}$ is elementary for some component Z of X. Consequently, $\mathcal{NS}(f)$ is finite. □

As $\mathcal{E}(f) \cup \mathcal{N}(f) \subseteq \mathcal{S}(f)$, it follows that $\mathcal{NE}(f) \subseteq \mathcal{NS}(f)$ is finite.

3.2 Finite Type Towers

Definition. Let $\mathcal{F}$ be a tower on X. We say that $\mathcal{F}$ is a tower of *finite type* if the base f is a finite type map on X.

In view of ??, we observe:

Lemma 66 *Let $\mathcal{F}$ be a finite type tower. Then $\mathcal{RF}$ is a finite type tower.*

Recall that the definition of towers includes the infinite condition that no element be locally invertible. This condition is satisfied automatically in the case of finite type.

Proposition 11 *Let $\mathcal{F}$ be a construct admitting a finite type base element with empty elementary part. Then $\mathcal{F}$ is a finite type tower.*

Proof: We must show that $\mathcal{F}$ is directed. It suffices to prove this for n stage constructs with n finite, and we proceed by induction on n. By Lemma 50, the unique base of $\mathcal{F}$ is a finite type map f with empty elementary part, so

$\mathcal{F}$ is directed for $n = 1$. For $n > 1$ write $\mathcal{F} = f * \mathcal{H}$, where $\mathcal{H}$ is an $n-1$ stage construct with finite type base $\Phi \cup E$. In view of Lemma 32, $\Phi \cup E$ has empty elementary part; by induction, $\mathcal{H}$ is directed.

Suppose $g \circ f_{|U} = Id_U$ where U is connected and $g \in \mathcal{F}[f(U)]$. As seen above, $g \notin \langle f \rangle$, so $g = \chi \circ \tilde{\alpha} \circ \varpi$ for a lift $\tilde{\alpha}$ of some $\alpha \in \mathcal{H}[\pi(U)]$. Then

$$\chi \circ \tilde{\alpha} \circ \sigma \circ \varpi = \chi \circ \tilde{\alpha} \circ \varpi \circ f = Id_U.$$

As $\varpi \circ \chi \circ \tilde{\alpha} \circ \sigma$ is a lift of $E \circ \alpha$, it follows that $E \circ \alpha = Id_{\pi(U)}$, contrary to what we have just shown. Therefore, $\mathcal{F}$ is directed. □.

Recall that $O_n(\mathcal{F}) = \{x \in X : ht_{\mathcal{F}}(x) > n\}$ is an open set for any tower $\mathcal{F}$ and any n.

Lemma 67 *Let $\mathcal{F}$ be a complete mixing finite type tower, and suppose $n < height(\mathcal{F})$. Then $O_n(\mathcal{F}) \cap J(\mathcal{F})$ is open and dense in $J(\mathcal{F})$.*

Proof: We argue by induction on n. Clearly, $O_0(\mathcal{F}) = W(f)$ where f is the base of $\mathcal{F}$, and thus $O_0(\mathcal{F}) \cap J(\mathcal{F})$ is dense in $J(\mathcal{F})$. Suppose $n \geq 1$. By induction, in view of Lemma 53, $O_{n-1}(\mathcal{RF}) \cap J(\mathcal{RF})$ is dense in $J(\mathcal{RF})$. As π is an open map and

$$\pi^{-1}(O_{n-1}(\mathcal{RF}) \cap J(\mathcal{RF})) \subseteq O_n(\mathcal{F}) \cap J(\mathcal{F}),$$

it follows from Proposition 8 that $O_n(\mathcal{F}) \cap J(\mathcal{F})$ is dense in $J(\mathcal{F})$. □

In view of Lemma 42, we observe

$$\partial O_n(\mathcal{F}) = J(\mathcal{F}^n) = \{x \in J(\mathcal{F}) : ht_{\mathcal{F}}(x) \leq n\}.$$

Recall that $J_0(\mathcal{F}) \cup J_1(\mathcal{F})$ is dense in $J(\mathcal{F})$ for any tower $\mathcal{F}$, and that $J_0(\mathcal{F})$ is dense when $\mathcal{F}$ has typical base. By comparison:

Proposition 12 *Let $\mathcal{F}$ be a complete mixing finite type tower, $1 \leq m \leq n$, where $n = height(\mathcal{F})$. Then $J_m(\mathcal{F})$ is dense in $J(\mathcal{F})$. Furthermore, $J_n(\mathcal{F})$ is residual.*

Proof: We proceed by induction on m. Suppose $m = 1$. By Lemma 48, $J_+(f)$ is residual, and thus dense in $J(f)$. Let U be an open set intersecting $J(\mathcal{F})$. In view of Proposition 5, there exist open $V \subseteq U$ and $g \in \mathcal{F}[V]$ with $g(V) \cap J(f) \neq \emptyset$. By Lemma 48, there exists $x \in V$ with $g(x) \in J_+(f)$; thus, $y \in U \cap J_1(\mathcal{F})$. It follows that $J_1(\mathcal{F})$ is dense in $J(\mathcal{F})$. Moreover, for $n = 1$, $\mathcal{F} = \langle f \rangle$ and thus $J_1(\mathcal{F}) = J_+(f)$ is residual.

Suppose now that $1 < m < \infty$. By induction, $J_{m-1}(\mathcal{RF})$ is dense in $J(\mathcal{RF})$ and residual if $m - 1 = height(\mathcal{RF}) = n - 1$. As π is an open map, $J_m(\mathcal{F}) = \pi^{-1}(J_{m-1}(\mathcal{RF}))$ is dense in $\pi^{-1}(J(\mathcal{RF}))$ and residual if $m = n$; the conclusion follows by Proposition 8.

Finally, if $m = \infty = n$ then $J_\infty(\mathcal{F}) = \bigcap_{\ell=0}^{\infty} O_\ell(\mathcal{F}) \cap J(\mathcal{F})$. By Lemma 67, each set in the intersection is open and dense in $J(\mathcal{F})$. Consequently, $J_\infty(\mathcal{F})$ is residual. □

Proposition 13 *Let f be a complete finite type map with empty elementary part, $\varpi : \mathcal{B} \to \hat{\mathbf{C}}$ a global Fatou or linearizing coordinate. Further, let U be a connected open set intersecting $\partial\mathcal{B}$, and V a component of $U \cap \mathcal{B}$. Then any Jordan domain $B \subseteq \mathbf{C}$, where $0 \notin \overline{B}$ in the case of a linearizing coordinate,* *has infinitely many unramified proper ϖ-preimages compactly contained in V.*

Proof: Let Z be the component of X containing the associated fixed point, $B_m = \tau^m(B)$ for $m \geq 0$. Then $f^n(V) \subseteq Z$ for some n, and by Lemma ?? there exist $x \in \partial V \cap U \cap W(f^n)$ and a disc $D \subseteq U \cap W(f^n)$ containing x such that $f^n_{|D} : D \to f^n(D)$ is either a homeomorphism or a branched cover with unique critical value $f^n(x) \in \partial\mathcal{B}$. Fix a component P of $D \cap V$, and let Q be the component of $f^n(D) \cap \mathcal{B}$ containing P; then f^n maps P homeomorphically onto Q. If Q contains infinitely many unramified proper ϖ-preimages of $B_n = \tau^n(B)$, then P contains infinitely many such preimages of B. We may therefore assume without loss of generality that $U \subseteq Z$. Then $\partial\mathcal{B}$ is infinite, closed, and backward invariant, and consequently equal to $J(f)$.

By Lemma ?? there exists n such that $\overline{B}_m \cap S(\varpi) = \emptyset$ for $m \geq n$. Suppose first that $U \cap \partial\mathcal{B}$ contains a nondegenerate continuum, and let $x \in \partial V \cap U \cap W(f^n)$, D, P, and Q be as above. By Lemma ??, $f^n(D) \cap \partial\mathcal{B}$ contains a nondegenerate continuum, and it follows by Lemma 62 that Q contains infinitely many unramified proper ϖ-preimages of B_n; as above, P contains infinitely many such preimages of B. Otherwise, $\partial\mathcal{B}$ is a Cantor set, and f is consequently rational or typical. Fix $z \in \mathcal{B}$ with infinite forward orbit, and let $z_m = f^m(z)$; in view of Lemma 64, for large enough m there exist $y_m \in \mathcal{B}$ such that $f^m(y_m) = z_m$ but $f^{m-1}(y_{m-1}) \neq z_{m-1}$. By construction, the y_m are distinct points in $\varpi^{-1}(z)$; as X is compact, there exist $m_k \to \infty$ and $y \in \partial\mathcal{B}$ such that $y_{m_k} \to y$. It follows by Lemma 61 that any neighborhood of y contains infinitely many unramified proper ϖ-preimages of each B_m with $m \geq n$. If f is rational then $y \notin \mathcal{E}(f)$, and thus $y = f^m(x)$ for some $x \in U \cap W(f^m)$; as above, $V = U - \partial\mathcal{B}$ contains infinitely many proper ϖ-preimages of B. In the typical case, Lemma 48 implies $U \cap \partial(f^m) \neq \emptyset$ for some $m \geq n$, and the existence of the desired preimages follows from Proposition 9. □

In particular, ϖ is complete, with $\mathcal{N}(\varpi) = \{\infty\}$ for a Fatou coordinate, and $\{\infty\} \subseteq \mathcal{N}(\varpi) \subseteq \{0, \infty\}$ for a linearizing coordinate.

Theorem 1 *Let $\mathcal{F}$ be a tower with complete base f. Then $\mathcal{F}$ is complete, and $\mathcal{NE}(\mathcal{F}) = \mathcal{NE}(f)$.*

Proof: By definition,

$$\mathcal{NE}(f) = \mathcal{E}(f) \cup \bigcup_{n=1}^{\infty} \mathcal{N}(f^n),$$

so it suffices to show $\mathcal{N}(g) \subseteq \mathcal{NE}(f)$ for every $g \in \mathcal{F}_{MAX}$ with $ht_{\mathcal{F}}(g) \geq 2$. Writing $\mathcal{F} = f * \mathcal{H}$, we have $g = \chi \circ \tilde{\alpha} \circ \varpi_{|V}$, where V is the domain of g and $\tilde{\alpha}$ is a lift of some $\alpha \in \mathcal{H}_{MAX}$. By Lemma 47,

$$\mathcal{N}(g) \subseteq \mathcal{N}(\chi) \cup \chi_\circ \mathcal{N}(\tilde{\alpha} \circ \varpi),$$

where

$$\mathcal{N}(\tilde{\alpha} \circ \varpi) \subseteq \mathcal{N}(\tilde{\alpha}) \cup \tilde{\alpha}_\circ \mathcal{N}(\varpi).$$

In view of Lemma 13, $\tilde{\alpha}_\circ\mathcal{N}(\varpi) \subseteq \tilde{\alpha}_\circ\underline{\infty} = \emptyset$, while $\mathcal{N}(\tilde{\alpha}) \subseteq \underline{\infty}$ by Lemmas ?? and ??. Therefore,

$$\chi_\circ\mathcal{N}(\tilde{\alpha} \circ \varpi) \subseteq \chi_\circ\underline{\infty} = \mathcal{N}(\chi);$$

by Lemma 52,

$$\mathcal{N}(g) \subseteq \mathcal{N}(\chi) \subseteq \mathcal{N}\mathcal{E}(f). \quad \square$$

3.3 Repelling Fixed Points

It was established classically by means of Montel's Theorem that the Julia set of a non-elementary rational map is the closure of the set of repelling periodic points. Later on, this principle was extended to entire [1] and Radström maps [3], and more recently to meromorphic maps [2], through the application of Ahlfors' Islands Theorem. Using instead Proposition 9 and the stratification of the Julia set, we prove the density of repelling points for maps, and then towers, of finite type.

Schwarz' Lemma gives rise to repelling fixed points in many settings. We begin with the simplest configuration.

Lemma 68 *Let $f : D \twoheadrightarrow B$ be analytic, where B and D are simply connected and $D \subset\subset B$. Suppose that f is either a homeomorphism or a branched cover whose only critical value lies outside $\overline{D}$. Then D contains a repelling fixed point of f.*

Proof: Without loss of generality, we may assume that B and D are Jordan domains. If $f_{|D}$ is a homeomorphism, there is an inverse $g : B \to D$. By Schwarz' Lemma, D contains a fixed point, attracting for g hence repelling for f. In the branched case, we may slit B along a closed arc leading from the critical value to ∂B and avoiding $\overline{D}$. The slit region B^- is simply connected, as are its preimages. Fix such a preimage D^- lying in D. By construction, $f_{|D^-} : D^- \to B^-$ is a homeomorphism and $D^- \subset\subset B^-$; as above, D^- contains a repelling fixed point. □

Note that as $J(f)$ is perfect, the density of repelling periodic points for a typical finite type map on a Riemann surface follows directly from Proposition 9 and the stratification of $J(f)$. To treat exceptional finite type maps without invoking Ahlfors' Islands Theorem, we must produce repelling fixed points near the essential singularities.

Lemma 69 *Let f be an analytic map on X, and let B and T be Jordan regions in X, and $x \in B \cap \partial T$. Suppose that $f_{|T}$ is a universal cover of $B - \{x\}$. Then x is a limit of repelling fixed points of f.*

Proof: Observe that we are free to replace B and T by any smaller Jordan neighborhood of x and the corresponding preimage component. The portion of T outside any given disc about x is compact and maps into the exterior of some other such disc. Consequently, we may assume that $\overline{B \cup T}$ lies in a contractible neighborhood of x, and that $f_{|\overline{T}-\{x\}}$ is a universal cover of $\overline{B} - \{x\}$. As the curve ∂B already bounds B, and thus cannot bound a disc

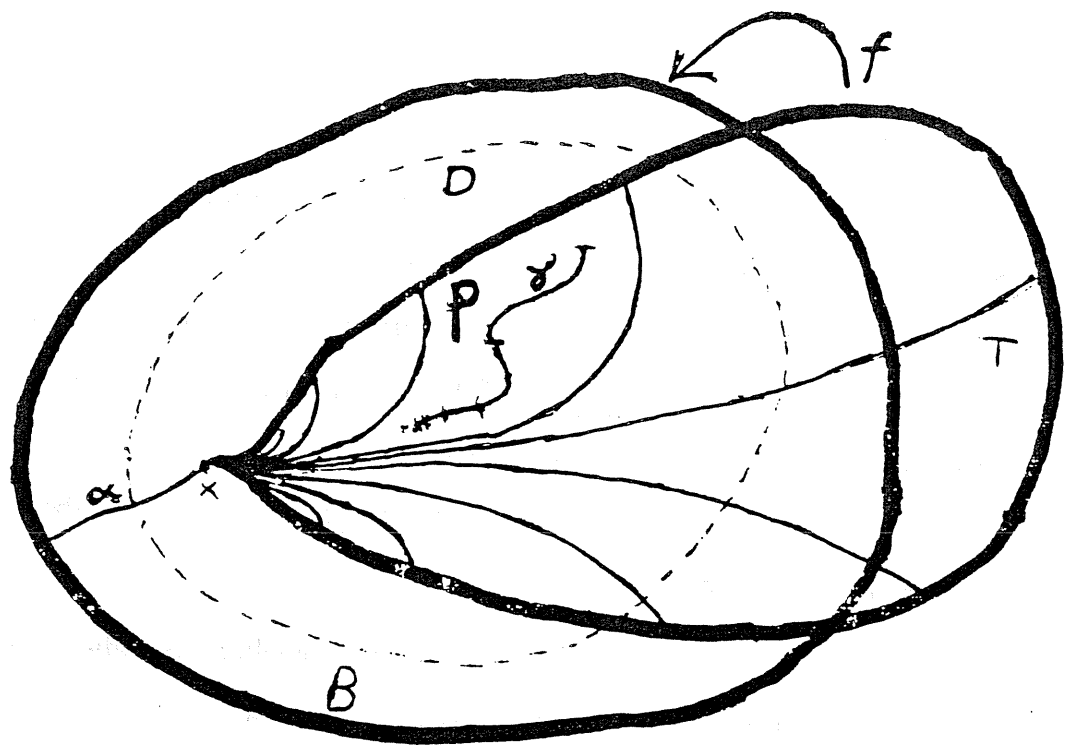

Figure 3.3: Fixed points in a tract

in T, it follows that $\partial B \not\subseteq T$. Hence, it is possible to connect x to ∂B by an arc α in B which intersects $\overline{T}$ only at x, and slit B along α to obtain a simply connected region B^-. The preimage of B^- in T consists of infinitely many regions, each bounded by two consecutive components of $f^{-1}(\alpha)$ meeting at x and an arc of ∂T.

Let $B^* = B - \{x\}$; in view of Corollary 1, there exists a smaller Jordan neighborhood D with $\eta^T_{B^*}(w) \geq 2$ for every $w \in D \cap T$. If ∂D is disjoint from $\overline{T}$ then $T \subset\subset D$; otherwise, ∂T has first and last intersections with ∂D, the ends of $\partial T - \{x\}$ lie inside D, and the subtended arc of ∂D intersects at most finitely many components of $f^{-1}(\alpha)$. Consequently, D contains all but finitely many of the preimages of B^- in T. Fixing a preimage $P \subseteq D$, let $\gamma : [0,1] \to P$ be a rectifiable path with $f(\gamma(1)) = \gamma(0)$. As $P \subseteq B^-$, we may extend to a path $\check{\gamma} : [0, \infty) \to P$ with $f(\check{\gamma}(t+1)) = \check{\gamma}(t)$. Let $\ell_{B^*}(n)$ and $\ell_T(n)$ be the Poincaré lengths in B^* and T of the segment $\check{\gamma}_{|[n,n+1]}$; then $\ell_{B^*}(n+1) \leq \frac{1}{2}\ell_T(n+1) = \ell_{B^*}(n)$ for $n \geq 0$. Consequently, $\check{\gamma}$ has finite length in B^*, and $\lim_{t\to\infty} \check{\gamma}(t) \in P$ is a repelling fixed point of f. □

Eremenko and Lyubich [12] investigated the more general class of entire maps for which $S(f) - \{\infty\}$ is bounded. In conjunction with Lemma 70 to follow, the argument above establishes the density of repelling periodic points for this class.

Proposition 14 *Let f be a complete finite type analytic map on a Riemann surface X. Then every point of $\partial(f)$ is a limit of repelling fixed points of f.*

Proof: Fix $x \in \partial(f)$ and an isolating neighborhood B of x. By Lemma 68, any proper preimage $D \subset\subset B$ contains a repelling fixed point of f. If there is no such preimage then it follows from Proposition 9 that x is isolated in $\partial(f)$. In view of Corollary 4, some preimage of B is a tract accumulating at x, and the existence of the desired fixed points follows from Lemma 69. □

It follows that $\partial(f^n)$, for $n \geq 1$, is in the accumulation of the set of repelling fixed points of f^n; conversely, any limit of such fixed points lies in $\partial(f^n)$. In view of Lemma 48, $\partial(f^n)$, for $n \geq 1$, is the accumulation of the set of period n repelling points of f.

Lemma 70 *Let $g : W \to X$ and $h : V \to X$ be analytic maps on X, and let $x \in V$ such that $\zeta = h(x)$ is a repelling or parabolic fixed point of g. Suppose*

$x = g^m(w)$ for some $m \geq 0$ and some $w \neq \zeta$ in a linearizing neighborhood or repelling petal N for ζ. Then for large k there exist $x_k \to x$, each x_k a repelling fixed point of $h \circ g^{m+k}$.

Proof: Let $\gamma : N \to N$ be the distinguished local inverse of g. Fix a Jordan neighborhood A of ζ, and let B be the component of $h^{-1}(A)$ containing x, and C the component of $g^{-m}(B)$ containing w. If A is sufficiently small then B is a Jordan domain, and $h_{|B} : B \to A$ is either a homeomorphism or a branched cover with critical value ζ. Shrinking A and hence B, we may assume as well that $C \subset\subset N - \{\zeta\}$ is a Jordan domain, and that $g_{|C} : C \to B$ is either a homeomorphism or a branched cover with critical value x. For $k \geq 0$, $C_k = \gamma^k(C)$ is a Jordan domain and $g^k_{|C_k} : C_k \to C$ is a homeomorphism. Moreover, $\zeta \notin \overline{C_k}$, and $C_k \subset\subset A$ when k is sufficiently large; for such k, let $D_k \subset\subset B$ be a component of $h^{-1}(C_k)$. Then D_k is a Jordan domain and x lies outside $\overline{D_k}$. By Lemma 68, D_k contains a repelling fixed point x_k of $h \circ g^{m+k}$, and $x_k \to x$ by construction. □

Theorem 2 *Let $\mathcal{F}$ be a complete mixing finite type tower on X. Then $J(\mathcal{F})$ is the closure of the set of repelling points of $\mathcal{F}$.*

Proof: Every repelling fixed point belongs to $J(\mathcal{F})$. Conversely, we show by induction on finite $n \leq height(\mathcal{F})$ that every open set intersecting $J(\mathcal{F}^n)$ contains repelling fixed points of height n. In view of Lemma 55, we may assume without loss of generality that X is connected.

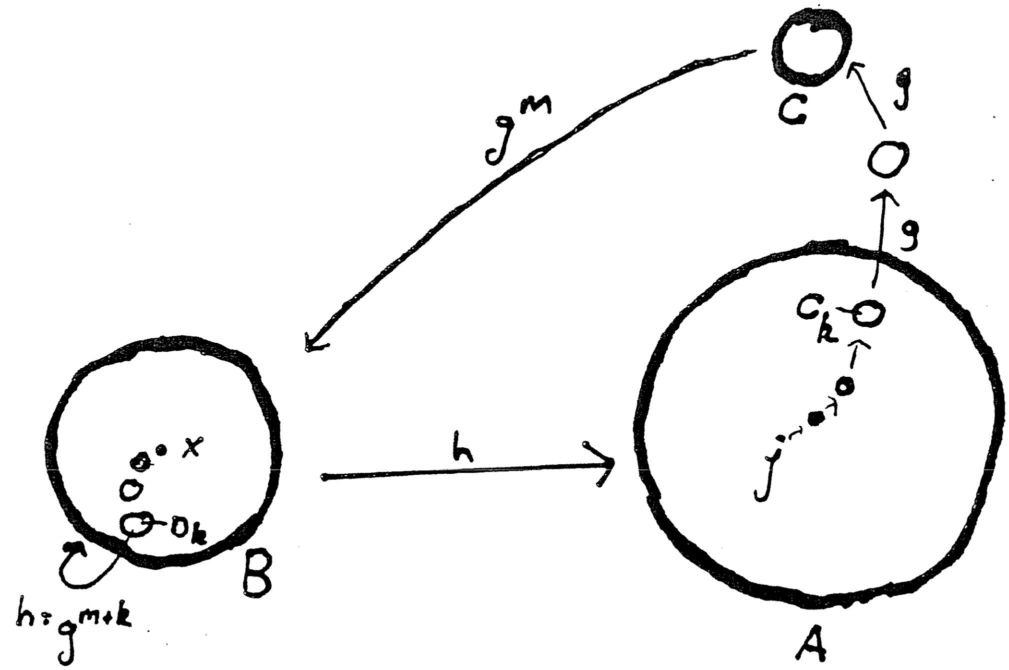

Figure 3.4: Creating more fixed points

We show first that the base f has a repelling or parabolic fixed point $\zeta \notin \mathcal{NE}(f)$. By Proposition 14, if $\deg f = \infty$ then f has infinitely many repelling fixed points; only finitely many can lie in $\mathcal{NE}(f)$. For rational f, a standard global index formula [30] guarantees the existence of a repelling or parabolic fixed point ζ, and $\zeta \notin \mathcal{NE}(f)$ as $\mathcal{NE}(f) = \mathcal{E}(f)$ consists of superattracting points. If f is a toral endomorphism then $\mathcal{NE}(f) = \emptyset$ and the existence of repelling fixed points can be verified by inspection. Fix an associated linearizing neighborhood or repelling petal N, and suppose U intersects $J(\mathcal{F}^1) = J(f)$. By the remarks after Theorem 1, $\zeta = f^{\ell}(x)$ for some $x \in U - \mathcal{NE}(f)$ and $\ell \geq 0$. Again, $x = f^m(w)$ for some $w \in N$ and $m \geq 0$. It follows by Lemma 70 that x is a limit of repelling fixed points x_k of $h_k = f^{\ell+m+k} \in \mathcal{F}$, and $ht_{\mathcal{F}}(x_k) = ht_{\mathcal{F}}(h_k) = 1$.

Now suppose $U \cap J(\mathcal{F}^n) \neq \emptyset$ where $n \geq 2$. By Proposition 8 and induction, there exists $x \in U$ such that $\pi(x)$ is a repelling fixed point of $\mathcal{RF}$ with $ht_{\mathcal{RF}}(\pi(x)) = n - 1$. As $J(\mathcal{F}^n)$ is perfect, we may assume without loss of generality that $x \notin \mathcal{NE}(f)$. It follows from Lemma ?? that there exist $\ell \geq 0$ and $g \in \mathcal{F}_{MAX}$ with $ht_{\mathcal{F}}(g) = n$ for which $\zeta = f^{\ell}(x)$ is a repelling fixed point. In view of Theorem 1, there exists $m \geq 0$ such that $x = g^m(w)$ for some w in a linearizing neighborhood. We conclude by Lemma 70 that x is a limit of repelling fixed points x_k of $h_k = f^{\ell} \circ g^{m+k} \in \mathcal{F}$, and $ht_{\mathcal{F}}(x_k) = ht_{\mathcal{F}}(h_k) = n$. □

3.4 Final Fatou Components

We now complete the classification of final Fatou components for towers of finite type. In the case of maps, it remains to show that $\Omega(f)$ has no Baker or exotic domains. The nonexistence of Baker domains for finite type entire maps was by Eremenko and Lyubich [12] with an argument involving a logarithmic change of variable and the Koebe-$\frac{1}{4}$ Theorem. The result was extended to finite type Radström maps [20] and certain finite type meromorphic maps [21] by Kotus; see also [6]. Although this argument adapts to the general case, it does not address the issue of exotic domains. We rule out both possibilities in the following:

Proposition 15 *Let $f : W \to X$ be a complete finite type analytic map on X. A fixed component U of $\Omega(f)$ on which $f^n_{|U}$ tends to infinity is a parabolic*

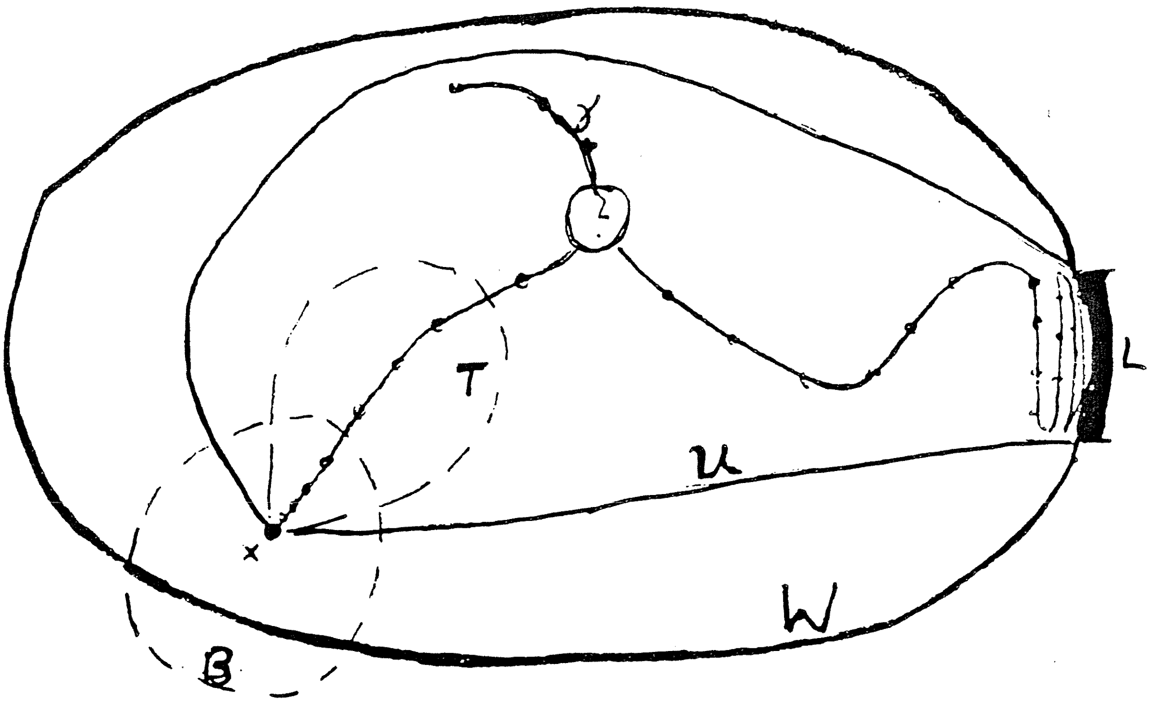

Figure 3.5: The Snake Argument

domain.

Proof: Let $\gamma : [0, \infty) \to U$ be a locally rectifiable forward invariant path. By assumption, γ tends to infinity in U, and the tail avoids $S(f)$. As in Lemma 28, it suffices to show that γ does not tend to infinity in W. We argue by comparing the lengths $\ell_Y(n)$ of the segments $\gamma_{|[n,n+1]}$ in the Poincaré metrics on various regions Y in X; as previously, we write Y^* and $Y^\times$ for $Y - S(f)$ and $Y - \overline{f^{-1}(S(f))}$. Without loss of generality, f has empty elementary part, so U is hyperbolic; in view of Lemma 29 we may further assume that X^* is hyperbolic. By Schwarz' Lemma, $0 < \ell_{X^*}(n) \leq \ell_{U^*}(n)$ and $\ell_U(n) \geq \ell_U(n+1)$ for $n \geq 0$. In view of Proposition 1 and the finiteness of the singular set, $\eta_U^{U^*}(\gamma(t)) \to 1$ as $t \to \infty$. Consequently, $\ell_{X^*}(n)$ is bounded.

If γ tends to infinity in W, then the set L of accumulation points is a continuum in $\partial(f)$. As above, $\eta_{X^\bullet}^{W^\times}(\gamma(t)) \to 1$ as $t \to \infty$. By Corollary 1, $\eta_{X^\bullet}^{W^{\times\bullet}}(\gamma(t)) \to \infty$ if L consists of more than one point; then $\eta_{X^\bullet}^{W^\times}(\gamma(t)) \to \infty$, so $\ell_{X^\bullet}(n+1) = \ell_{W^\times}(n) \geq 2\ell_{X^\bullet}(n)$ for large n, and thus $\ell_{X^\bullet}(n) \to \infty$. On the other hand, if $L = \{x\}$ then x is an asymptotic value. Fix an isolating neighborhood B of x; the tail of γ lies in a tract T, and $\eta_{B^\bullet}^{T}(\gamma(t)) \to \infty$ by Corollary 1. Again, $\ell_{B^\bullet}(n+1) = \ell_T(n) \geq 2\ell_{B^\bullet}(n)$ for large n, so $\ell_{B^\bullet}(n) \to \infty$; by Proposition 1, $\eta_{X^\bullet}^{B^\bullet}(\gamma(t)) \to 1$, and thus $\ell_{X^\bullet}(n) \to \infty$. Therefore, γ has a limit point in W, so U must be a parabolic domain. □

In view of the remarks preceding Lemma 50, every periodic component of $\Omega(f)$ is of one of the five standard types. By induction, we conclude:

Theorem 3 *Let $\mathcal{F}$ be a complete finite type tower. A type I final component U of $\Omega(\mathcal{F})$ is a superattracting, attracting, parabolic, Siegel, Hermann, or Baker domain; in the last case, the associated limit point is a parabolic fixed point of height m, where $0 < m < ht_{\mathcal{F}}(U)$. A Type II final component is an attracting, Siegel, or Hermann domain.*

Recall that an attracting or parabolic basin for a finite type map with empty elementary part contains a singular value. Consequently, there are finitely many fields of attracting, superattracting, parabolic, or Baker domains in the Fatou set of a finite type tower. There is a more subtle association of singular values to rotation domains and Cremer points. Let us first note the following consequence of Theorem 2.

Lemma 71 *Let f be a complete finite type analytic map on X. Further, let U be a connected open set intersecting $J(f)$, and suppose $g_k \to g$, where each $g_k : U \to X$ is a local inverse of f^{n_k} and $n_k \to \infty$. Then g is constant.*

Proof: If g is non-constant, then f^{n_k} converges locally uniformly on $g(U)$ to g^{-1}. Consequently, the derivatives $f^{n_k}{}'$, computed in fixed local coordinates, are locally uniformly convergent. On the other hand, U contains a repelling periodic point x, and $f^n{}'(x) \to \infty$. Therefore, g is constant. □

Proposition 16 *Let $f : W \to X$ be a finite type analytic map with empty elementary part. The boundary of any Siegel disc or Hermann ring lies in $\mathcal{PS}(f)$. Similarly, any Cremer point is an accumulation point of $\mathcal{PS}(f)$.*

Proof: In view of Lemma 29, we may assume that $X - \mathcal{PS}(f)$ is hyperbolic. Let U be a Siegel disc or Hermann ring, and fix a sequence $g_k : U \to U$ of inverse branches of f^{n_k} with $g_k \to Id_U$. Suppose $\partial U \not\subseteq \mathcal{PS}(f)$, and fix an open set V disjoint from $\mathcal{PS}(f)$ and intersecting $\partial U \subseteq J(f)$; without loss of generality, $U \cup V$ is simply connected. Each g_k extends to an analytic map $\breve{g}_k : U \cup V \to X$. By assumption, the sequence of extensions is normal, hence $\breve{g}_k \to Id_{U \cup V}$ by Vitali's Theorem. In view of Lemma 71, this is impossible.

Similarly, let x be a Cremer point, and suppose that x is disjoint from the accumulation of $\mathcal{PS}(f)$. Then there exist a simply connected neighborhood V and inverse branches $g_n : V \to X$ of f^n with $g_n(x) = x$. Observe that $|g_n'(x)| = 1$ in any local coordinate. The sequence of injective maps g_n is

therefore normal with non-constant limits; again, this contradicts Lemma 71.
□

3.5 Geometric Finiteness I

In Section 3.3 we established a topological alternative for the Julia set of a finite type analytic map on a complex 1-manifold X: either $J(f)$ is nowhere dense or else $J(f) = X$. The corresponding question in measure theory has engendered much speculation. The best general result is due to Lyubich [12]: if f is a rational map for which every infinite forward critical orbit tends to a periodic cycle, then $J(f)$, if nowhere dense, has Lebesgue measure 0. This condition on the critical orbits is the next simplest after outright finiteness; such rational maps correspond in Sullivan's dictionary to the geometrically finite Kleinian groups.

Definition. A finite type analytic map f on a complex 1-manifold is *geometrically finite* if every infinite forward singular orbit tends to a periodic cycle.

In view of Proposition 16 we observe:

Lemma 72 *A geometrically finite map with empty elementary part has no irrationally indifferent periodic points and no Hermann rings.*

To extend Lyubich's result to infinite degree maps of finite type, we must distinguish between preperiodic and terminating finite orbits. Indeed, Mc-

Mullen has shown that the entire maps $f_\lambda(z) = \lambda \sin z$ always have positive measure Julia set [26]; on the other hand, it is easy to find parameter values for which $J(f_\lambda)$ is nowhere dense and both finite singular values are preperiodic. Let f be an analytic map on a complex 1-manifold. We denote $TS(f)$ the union of all terminal singular orbits in $J(f)$.

Definition. A geometrically finite map f is *strongly geometrically finite* if $TS(f) = \emptyset$, that is, if every singular value in $J(f)$ is preperiodic.

Note that entire and Radström maps are excluded, as their essential singularities are singular values. If f is typical then $\partial(f)$ may itself have positive measure; the correct generalization of Lyubich's theorem concerns the measure of $J_+(f)$ for strongly geometrically finite maps.

We recall the standard considerations of density and distortion in the plane. Let E and B be Lebesgue measurable subsets of $\mathbf{C}$. The density of E in B is the quantity

$$den(E : B) = \frac{meas(B \cap E)}{meas(B)}.$$

The density of E at $x \in \mathbf{C}$ is defined to be

$$den_x(E) = \lim_{r \to 0} den(E : B_r(x)),$$

where $B_r(x)$ is the Euclidean r-ball about x, whenever the limit exists; if $den_x(E) = 1$ then x is a *density point* of E. By the Lebesgue Density Theorem [?], almost every point of E is a density point.

For open $B \subseteq \mathbf{C}$ containing x, the scale-invariant quantity

$$M_x(B) = \frac{\sup_{y \in \partial B} ||x - y||}{\inf_{y \in \partial B} ||x - y||}$$

measures the *eccentricity* of B relative to x. Note that B occupies a definite proportion, depending only on the eccentricity, of the area of the circumscribed ball about x. A family of neighborhoods of x whose eccentricity relative to x is uniformly bounded is said to be of *bounded shape* relative to x. If x is a point of density of E and B_k is a *fundamental system* of bounded shape neighborhoods of x, that is, if $diam\ B_k \to 0$, then $den(E : B_k) \to 1$.

The *distortion* of a map $g : D \to \mathbf{C}$ defined on an open set $D \subseteq \mathbf{C}$ is the ratio

$$dist(f) = \frac{\sup_{x,y \in D} dist(f; x, y)}{\inf_{x,y \in D} dist(f; x, y)},$$

where

$$dist(f; x, y) = \frac{||f(x) - f(y)||}{||x - y||}.$$

Clearly,

$$den(E : D) \leq dist(f)^2 \cdot den(f(E) : f(D)) \tag{3.1}$$

for measurable $E \subseteq \mathbf{C}$. The distortion of a smooth map is easily bounded in terms of the partial derivatives. The notions of density point and bounded shape neighborhood therefore make sense on a Riemann surface.

Koebe Distortion Theorem *Let $g : \Delta_R \to \mathbf{C}$ be an injective analytic map. Then $dist(f_{|\Delta_r}) \leq \kappa(\frac{r}{R})$ for $0 < r \leq R$, where $\kappa(t) \to 1$ as $t \to 0$.*

See [33] for one of many proofs.

Lemma 73 *Let f be a finite type analytic map on a complex 1-manifold X. Assume that $J(f)$ is nowhere dense. Then $\omega(x) \subseteq \mathcal{PS}(f) \cap J(f)$ for every density point $x \in J_+(f)$.*

Proof: Suppose $f^{n_k}(x) \to y \notin \mathcal{PS}(f)$, where $n_k \to \infty$. Choose a conformal disc $U \subseteq X - \mathcal{PS}(f)$ centered at y, and let $D \subset\subset B \subset\subset U$ be concentric subdiscs. For large k there are inverse branches $g_k : U \to X$ of f^{n_k} with $g_k(f^{n_k}(x)) = x$, and $g_k \to x$ by Lemma 71. The open sets $B_k = g_k(B)$ with $f^{n_k}(x) \in D$ form a fundamental system of bounded shape neighborhoods of x. Consequently, $den(J_+(f) : B_k) \to 1$, so $den(J_+(f) : B) = 1$ by 3.1 and the Koebe Distortion Theorem. But then $B \subseteq J(f)$, contrary to assumption. □

If f is geometrically finite, then $\mathcal{PS}(f) \cap J(f)$ is a finite set consisting of parabolic cycles, repelling cycles, and $TS(f)$. Assume further that $J(f)$ is nowhere dense, and let $x \in J_+(f)$ be a Lebesgue density point. In view of Lemmas ?? and 73, if $\omega(x) \not\subseteq TS(f)$ then $\omega(x)$ is a parabolic or repelling cycle, and x is consequently preperiodic. As the set of preperiodic points is countable, the Lebesgue Density Theorem yields:

Lemma 74 *Let f be geometrically finite with nowhere dense Julia set. Then $\omega(x) \subseteq TS(f)$ for almost every $x \in J_+(f)$.*

We may immediately conclude:

Theorem 4 *Let f be strongly geometrically finite with nowhere dense Julia set. Then $J_+(f)$ has measure 0.*

Furthermore, if $\partial(f)$ has measure 0 then $f^{-n}(\partial(f))$ has measure 0 for $n \geq 0$, proving:

Corollary 5 *Let f be strongly geometrically finite with nowhere dense Julia set, and assume $\partial(f)$ has measure 0. Then $J(f)$ has measure 0.*

In certain situations, the requirement of strong geometrically finiteness may be relaxed. Lyubich proved Theorem 4 for geometrically finite entire maps satisfying a growth condition which, while dynamically unnatural, may be verified in concrete cases. Further arguments of McMullen establish measure 0 when the Julia set is sufficiently thin at infinity.

Let us say that a singular value y of a finite type map is *nearly unramified* if y is not an asymptotic value, and if the local degree is 1 at all but finitely many preimages.

Proposition 17 *Let f be geometrically finite with nowhere dense Julia set. Assume that $deg_x f = 1$ for $x \in W(f) \cap TS(f)$, and that every singular value in $TS(f)$ is nearly unramified. Then $J_+(f)$ has measure 0.*

Proof: It suffices to show that no density point $x \in J_+(f)$ has ω-limit set in $TS(f)$. Fix an isolating neighborhood D_ζ about each $\zeta \in \partial(f) \cap TS(f)$; for $w \in f^{-n}(\zeta) \cap TS(f)$, let D_w be the component of $f^{-n}(D_\zeta)$ containing w. If the original neighborhoods are sufficiently small, the discs D_w for $w \in TS(f)$ are pairwise disjoint, and $f_{|D_w}$ is a homeomorphism when $w \in W(f)$. We may further assume that the preimages of D_w intersecting D_ζ have degree 1 for every $w, \zeta \in TS(f)$ with $\zeta \in \partial(f)$.

Suppose $\omega(x) \subseteq TS(f)$. Without loss of generality, the orbit of x never leaves $\bigcup_{w \in TS(f)} D_w$. Moreover, there exist $n_k \to \infty$ and $\zeta \in \partial(f) \cap TS(f)$ such that $f^{n_k}(x) \to \zeta$. For large enough k, x lies in a component B_k of $f^{-n_k}(D_\zeta)$. By the Koebe Distortion Theorem and Lemma 71, the B_k form a fundamental system of bounded shape neighborhoods of x. As above, $den(J_+(f) : B_\zeta) = 1$, hence $B \subseteq J_+(f)$ contrary to hypothesis. □

Geometrically finite maps arise quite naturally from finite type towers of infinite height. The most basic measure of the dynamical complexity of a finite type analytic map f on a complex 1-manifold X is the number $\delta(f)$ of distinct forward infinite grand singular orbits. Let $\mathcal{F}$ be a finite type tower with base f. Assume that $height(\mathcal{F}) \geq 2$, and let r be the base of $\mathcal{RF}$. Then

$$\delta(r) \leq \#(S_f - P_f) = \#\pi(\mathcal{B}_f \cap S(f)) \leq \delta(f). \tag{3.2}$$

If equality holds on the left-hand side then every point of $S_f - P_f$ has infinite forward orbit under r, and any two points lie in distinct grand orbits. Consequently, every component of $\hat{X}_f^-$ is essential for r, and there are no *singular orbit relations*: that is, $r^p(x) = r^q(y)$ for positive p, q and $x, y \in S(r)$ implies $p = q$ and $x = y$. Of course, there may still be orbit relations among the critical points. If equality holds on the right-hand side then every infinite forward singular orbit of f lies in $\mathcal{B}_f$, so f is geometrically finite.

Suppose now that $height(\mathcal{F}) = \infty$, and for $m \geq 0$ let r_m be the base of

$\mathcal{R}^{m-1}\mathcal{F}$. By the above, if $\delta(r_{m-1}) = \delta(r_m) = \delta(r_{m+1})$ then r_m is strongly geometrically finite with no singular orbit relations outside the pole-set; in particular, $J(r_m)$ has measure 0. In view of 3.2, the sequence $\delta(r_m)$ is non-increasing, and therefore stabilizes; let $\delta(\mathcal{F})$ be the eventual value, and $M_{\mathcal{F}}$ the least m for which $\delta(r_m) = \delta(\mathcal{F})$. In view of Theorem 4 and the analyticity of the projections π_{r_m}, we conclude:

Corollary 6 *Let $\mathcal{F}$ be a finite type tower of infinite height, and suppose $M_{\mathcal{F}} < m < \infty$. Then r_m is strongly geometrically finite with no singular orbit relations outide the pole-set, and $J_m(\mathcal{F})$ has measure* 0.

We assert nothing about the measure of $J_\infty(\mathcal{F})$. Indeed, if $\mathcal{F}$ has infinite height and the sequence $\delta(f_m)$ is constant, then $\Omega(\mathcal{F})$ has no final components. It will follow from Theorem 7 in the next chapter that $J(\mathcal{F}) = X$ if the domain of the base f is dense in X; thus $J_\infty(\mathcal{F})$ may have full measure. The use of different techniques will enable us to comment on the ergodic theory of $J_\infty(\mathcal{F})$.

Chapter 4

Rigidity and Finiteness Theorems

4.1 Marked Points

Definition Let $\mathcal{F}$ be a tower.

$$\mathcal{M}(\mathcal{F}) = \{x \in X : \phi(x) = x \text{ for every } \phi \in Homeo_0(\mathcal{F})\}.$$

We refer to the elements of $\mathcal{M}(\mathcal{F})$ as *marked points.*

Lemma 75 *Let $\mathcal{F}$ be a finite type tower, $E \subseteq X$ a totally disconnected set preserved by every $\phi \in Homeo(\mathcal{F})$. Then $E \subseteq \mathcal{M}(\mathcal{F})$.*

Proof: Evaluation at a fixed $x \in E$ gives a continuous map $Homeo(\mathcal{F}) \to E$. Consequently, $\phi(x) = x$ for $\phi \in Homeo_0(\mathcal{F})$. □

Proposition 18 *Let $\mathcal{F}$ be a finite type tower with least element f. Then $\mathcal{M}(\mathcal{F})$ is the smallest closed invariant set containing $S(f)$ and the fixed points of $\mathcal{F}$.*

Proof: For $n \geq 1$, $S(f^n)$ is a finite set preserved by $Homeo(\mathcal{F})$, hence $S(f^n) \subseteq \mathcal{M}(\mathcal{F})$. Similarly, all fixed points are marked. And it follows from Lemma 75 that $\mathcal{M}(\mathcal{F})$ is invariant. □

It follows that components of $X - \mathcal{M}(\mathcal{F})$ are $\mathcal{F}$-simple, and $g : U \to g_* U$ is a covering space for $g \in \mathcal{F}[U]$.

Corollary 7 *Let $\mathcal{F}$ be a finite type tower. The canonical map $\pi_0(X - \mathcal{M}(\mathcal{F})) \to \pi_0(\Omega(\mathcal{F})$ is surjective. Moreover,*

- *If U is an attracting, parabolic, or wandering domain, then $U - \mathcal{M}(\mathcal{F})$ is connected.*
- *If U is a Siegel disc, Hermann ring, or superattracting domain. then $U - \mathcal{M}(\mathcal{F})$ "splits into annuli".*

Lemma 76 *Let $\mathcal{F}$ be a finite type tower, U a component of $\Omega(\mathcal{F})$. Suppose some $\alpha \in \mathcal{F}[U]$ is not a covering space. Then there is a least such α.*

Lemma 77 *Let $\mathcal{F}$ be a finite type tower, U a wandering component of $\Omega(\mathcal{F})$. Then the maps in $\vec{\mathcal{F}}_U$ are eventually covers.*

Proof: By hypothesis, there exists $g \in \mathcal{F}[U]$ such that $h_* U \cap S(f) = \emptyset$ for all $h \in \mathcal{F}[g_* U]$; moreover, $\mathcal{F}[g_* U]$ is non-trivial. We claim that every map in $\mathcal{F}[g_* U]$ is a covering space. Otherwise, by Lemma 76 there is a least $\alpha \in \mathcal{F}[g_* U]$ which is not a cover. Now $\alpha = f \circ \beta$ for some $\beta \in \mathcal{F}[U]$, and

$f_{|\beta_* U}$ is a cover as $\alpha_* U \cap S(f) = \emptyset$. But $\beta \preceq \alpha$ implies β is a cover, hence α is a cover, contrary to assumption. $\square$

For a tower $\mathcal{F}$ with least element f, let $\mathcal{PS}(\mathcal{F})$ be the smallest closed forward $\mathcal{F}$-invariant set containing $S(f)$, $\mathcal{GS}(\mathcal{F})$ the smallest closed $\mathcal{F}$-invariant set containing $S(f)$.

$$\hat{\mathcal{C}}(\mathcal{F}) = \mathcal{F} - \text{ invariant conformal structures}$$

$$\mathcal{C}(\mathcal{F}) = \mathcal{F} - \text{invariant structures} = \text{fiducial on } X - W$$

Lemma 78 *Let $\mathcal{F}$ be a finite type tower, C a closed $\mathcal{F}$-invariant set containing $S(f) \cup J(f)$.*

A. $\hat{QC}_0(\mathcal{F}, C) = \hat{QC}(\mathcal{F}) \cap QC_0(X, C)$,

B. $\hat{QC}_0(\mathcal{F}, C) = \bigcap_{n=1}^{\infty} \hat{QC}_0(\mathcal{F}^n, C)$,

C. $QC_0\mathcal{F}, C) = QC(\mathcal{F}) \cap \hat{QC}_0(\mathcal{F}, C)$.

Proof: For all three assertions, the righthand side contains the lefthand side by definition.

A. Suppose $\phi \in \hat{QC}(\mathcal{F}) \cap QC_0(X, C)$. Fixing a bounded functorial isotopy Ξ, define bijections $\Phi_t : X \to X$ for $t \in [0, 1]$ by

$$\Phi_t(x) = \begin{cases} \Xi(\phi_{|U})_t & \text{for } x \text{ in a component } U \text{ of } X - C \\ x & \text{for } x \in C \end{cases}$$

As Ξ is bounded, each Φ_t is continuous, hence $\Phi_t \in QC(X, C)$ by Bers' Lemma. By functoriality, $\Phi_t \in \hat{QC}(\mathcal{F})$ for every t, hence $\Phi_t \in \hat{QC}_0(\mathcal{F}, C)$.

Proposition 19 *Let $\mathcal{F}$ be a finite type tower with base $f : W \to X$, A a closed forward $\mathcal{F}$-invariant set containing $S(f)$, $c_1, c_2 \in \mathcal{C}(\mathcal{F})$, $\phi \in QC_0(X, A)$ with $\phi^* c_2 = c_1$. If c_1 and c_2 agree on $X - W$, or if $\phi \in QC_0(X, A \cup \partial(f)$, then $\phi \in QC_0(\mathcal{F}, A^{\mathcal{F}})$.*

Proof: By Lemma ??, it suffices to show $\phi \in QC_0(X, C)$, for some $\mathcal{F}$-invariant closed $C \supseteq S(f) \cup J(\mathcal{F})$, under the inductive assumption that the Proposition holds for all towers of lesser height.

Suppose $height(\mathcal{F}) = 1$. By hypothesis, $f^! \phi \in QC_0(X, f^{-1}(A) \cup \partial(f))$, and $(f^!\phi)^* c_2$ and c_1 agree on W. In the first case, it follows that $(f^!\phi)^* c_2 = c_1$, hence $f^!\phi \sim \phi$ rel A, as $A \subseteq f^{-1}(A) \cup (X - W)$. Consequently, $f^!\phi = \phi$, that is, $\phi \in QC(f)$. In the secod case, consider $\psi : X \to X$ given by

$$\psi(x) = \begin{cases} f^!\phi(x) & \text{for } x \in W \\ \phi(x) & \text{for } x \in X - W \end{cases}$$

In view of the additional assumption on ϕ, Bers' Lemma implies

$$\psi \in QC_0(X, f^{-1}(A) \cup \partial(f)),$$

and $\phi^* c_2 = c_1$. As before, $\psi = \phi$, so $\phi \in QC(f)$. Inductive application of the homotopy lifting property yields

$$\phi \in QC_0(X, \bigcup_{j=0}^{m} f^{-j}(A \cup \partial(f))$$

for every $m \geq 0$. By Corollary 2, $\phi \in QC_0(X, C)$ where $C = \overline{\bigcup_{j=0}^{\infty} f^{-j}(A \cup \partial(f))}$.

Assume $1 < N < \infty$. Let $\Phi : X_f \to X_f^\dagger$ be the transit map.

By induction, $\phi \in QC_0(f, A^f)$. Consequently, ϕ induces

$$\varphi \cup \varphi^\dagger \in QC_0(x_f \cup X_f^\dagger, E \cup E^\dagger)$$

where E is $\pi(A^f)$ union the poles of the cylinders in X_f, $E^\dagger$ the union of $Q(\chi^{-1}(A^f))$ and the poles of $X_f^\dagger$. Then $\Phi^{-1}(E^\dagger) \supseteq E$ is forward $\mathcal{RF}$-invariant. As $\partial(\epsilon_f) \subseteq E^\dagger$,

$$\varphi \in QC_0(\mathcal{RF}, \Phi^{-1}(E^\dagger)^{\mathcal{RF}})$$

by induction. Note that $A^{\mathcal{F}} = \pi^{-1}(\Phi^{-1}(E^\dagger)^{\mathcal{RF}}) \cup \mathcal{M}(f)$. By Bers'Lemma,

$$\psi(x) = \begin{cases} \pi^! \varphi(x) & \text{for } x \in B_f \\ \phi(x) & \text{for } x \in X - B_f \end{cases}$$

defines a quasiconformal homeomorphism in $QC_0(X, A^{\mathcal{F}})$, and $\psi = \phi$ as $\psi^* c_2 = c_1$.

Finally, suppose $N = \infty$. By induction, $\phi \in QC_0(\mathcal{F}^n, A^{\mathcal{F}^n})$ for finite n. As $A^{\mathcal{F}} = \overline{\bigcup_{n=1}^{\infty} A^{\mathcal{F}^n})}$, $\phi \in QC_0(X, A^{\mathcal{F}})$ by Corollary 2. □

4.2 Geometric Finiteness II

The central conjecture concerning iterated rational maps, the density in parameter space of structurally stable maps, reduces to establishing *conformal rigidity* on the Julia set: that invariant complex structures agreeing on $\Omega(f)$ be equal [?]. While there have been encouraging partial results, this question is still very much open. In our setting, we must require such complex structures to agree on $\partial(f)$. Such rigidity is automatic when $J(f)$ has measure

0, as was shown in section ?? for strongly geometrically finite maps. In this section, we present an independent route to rigidity through considerations of Teichmüller Theory. While we will not recover the measure dichotomy, we may weaken the dynamical hypothesis to geometric finiteness, thereby readmitting certain entire and Radström maps.

The *affine* toral endomorphisms $z \rightsquigarrow mz$, $m \geq 2$ are exceptions to this conformal rigidity. These maps occur in one complex-dimensional families parametrized by the Teichmüller space of the underlying torus; any two of the same degree m^2 are conjugate by an affine stretch. As observed by Lattès, these torus maps commute with the involution $z \rightsquigarrow -z$ and descend to *affine* rational maps of $\hat{\mathbf{C}}$ with similar properties; he also exhibited related rigid examples arising from tori admitting complex multiplication. Details may be found in [24], where rigidity is proved for post-critically finite rational maps using Teichmüller's Existence and Uniqueness Theorems (see also [41], [9]). Our adaptation of this argument employs Strebel's more general Frame Mapping Theorem, which McMullen first used to prove rigidity in a different setting [25].

The ultimate mechanism behind conformal rigidity for geometrically finite maps is a weak contraction principle concerning the infinitessimal Teichmüller metric: the non-existence of invariant quadratic differentials. We must consider quadratic differentials with countably many singularities. For the purposes of this section, invariance refers to f^*; invariance under f_* will be the relevant notion when we apply the principle again in section ??.

Recall the relation

$$ord_z f^* q = d\ ord_{f(z)} q + 2n - 2 \text{ where } d = \deg_z f. \tag{4.1}$$

Proposition 20 *Let X be a Riemann surface, $f : W \to X$ a finite type iterable analytic map, $q \in Q(X, C)$ where C is a countable closed set. For $q \neq 0$, the following are equivalent:*

1. $f_* q = q$;

2. $f^* q = (\deg f) q$, *hence* $\deg f < \infty$;

3. $f^* q$ *and* q *determine the same measurable linefield on* W.

Furthermore, if $\deg f > 1$ *and any of the above conditions holds for some* $q \neq 0$, *then* $\hat{f}$ *is an affine endomorphism of a sphere or torus.*

Proof: We may assume without loss of generality that f is complete. Then by Lemma ??, either $\deg f < \infty$ and $W = X$ or $\deg f_{|V} = \infty$ for every component V of W. In view of Lemma 16 and the inequality

$$||f_* q|| \leq \sum_V ||f_{|V*} q|| < \sum_V ||q_{|V}|| \leq ||q||,$$

$f_* q = q$ implies $\deg f < \infty$ and $f^* q = f^* f_* q = (\deg f) q$. In particular, $f^* q$ and q determine the same linefield on X. Conversely, suppose $f^* q$ and q determine the same linefield on a component V. Both quadratic differentials are holomorphic on the complement of the countable set $(W \cap C) \cup f^{-1}(C)$. By Lemma 15, $(f^* q)_{|V}$ and $q_{|V}$ are positive scalar multiples, so $||(f^* q)_{|V}|| < \infty$.

Consequently, $V = X$, $\deg f < \infty$, and $f^*q = (\deg f)q$; by Lemma 16, $f_*q = \frac{1}{\deg f} f_* f^* q = q$.

For the second part, assume $f^*q = (\deg f)q$ with $\deg f$ finite. Without loss of generality, f is a rational map with $\deg f \geq 2$, and q is not holomorphic at any point of C. By Lemma 17, $C = P \cup E$, where P is the set of poles and E is the set of accumulation points; moreover, every pole is simple. Let Z be the set of zeros of q. By the relation

$$ord_z f^*q = d\ ord_{f(z)}q + 2n - 2 \text{ where } d = \deg_z f, \tag{4.2}$$

P is forward invariant, Z is backward invariant, and E is invariant. By the remarks after Lemma ??, the closed countable backward invariant set $Z \cup E$ must lie in $\mathcal{E}(f)$. We further deduce from 4.2 that every superattracting point lies in E. As the points of $\mathcal{E}(f)$ are superattracting, we conclude that $Z = \emptyset$. It further follows from 4.2 that every critical point outside E is simple, and that every critical value outside E is a pole.

Any backward orbit string of poles must accumulate in E. As E consists of superattracting points, there can be no such backward orbit string; thus, every pole lies in $PS(f)$. On the other hand, it follows from Lemma 64 that any pole with infinite forward orbit gives rise to poles not belonging to $PS(f)$. Consequently, P is finite, $E = \emptyset$, and q is meromorphic on $\hat{\mathbf{C}}$. By the index relation ??, there are precisely four poles and, as demonstrated in [24], f is affine. □

Corollary 8 *Let $f : W \to X$ be a finite type iterable analytic map with empty affine and elementary parts, $q \in Q(X, C)$ where C is a countable closed set. If $f_* q = q$ then $q = 0$.*

Proof: We may assume without loss of generality that f is complete. Suppose $f_* q = q$, and let Z_i be the components of X. By ??, $q_{|Z} = 0$ for inessential Z_i. Moreover,

$$||f_* q|| \leq \sum_{W_{ij} \neq \emptyset} ||f_{W_{ij}*} q|| \leq \sum_{W_{ij} \neq \emptyset} ||q_{|W_{ij}}|| \leq ||q||,$$

where $W_{ij} = Z_i \cap f^{-1}(Z_j)$. As above, it follows that $||f_{W_{ij}*} q|| = ||q_{|W_{ij}}||$, hence $W_{ij} = Z_j$, whenever $W_{ij} \neq \emptyset$. In particular, $f^{\{Z_i\}}$ is exceptional for essential Z_i. Thus, $f^{\{Z_i\}} = f^m$ for some $m \geq 1$, and $f^{m-1}(Z_i)$ is the unique essential component intersecting $f^{-1}(Z_i)$. Consequently, $f_*^{\{Z_i\}} q_{|Z_i} = (f_*^m q)_{|Z_i} = q_{|Z_i}$. By Proposition 20, $q = 0$ in view of the hypothesis. □

As $\mathcal{PS}(f)$ may be infinite, we must verify Strebel's relaxation condition. The key is the local geometry near parabolic cycles.

Lemma 79 *Let $\phi : X \to X_1$ be a quasiconformal conjugacy between geometrically finite maps $f : W \to X$ and $f_1 : W_1 \to X_1$. Suppose ϕ is conformal on $\Omega_+(f)$. Then ϕ relaxes rel $\mathcal{PS}(f)$.*

Proof: Fixing $\epsilon > 0$, we construct a new map ψ in the same isotopy class rel $\mathcal{PS}(f)$, agreeing with ϕ outside a specified neighborhood of $\mathcal{PS}(f)$, and with dilatation less than $1+\epsilon$ inside an even smaller neighborhood. By hypothesis,

ϕ is already conformal near the attracting and superattracting points. We present a construction to relax ϕ near a parabolic point, leaving the map unchanged outside a small neighborhood. After performing the construction in disjoint neighborhoods of the parabolic points, it will remain, as is always possible, to relax ϕ near a finite set of isolated points.

Observe that the induced homeomorphism $\phi^\dagger : X^\dagger \to X_1^\dagger$ is conformal at both ends of each quotient cylinder. More generally, let $\varphi : \mathbf{C}^* \to \mathbf{C}^*$ be a quasiconformal homeomorphism, conformal near 0 and ∞, and $\tilde{\varphi} : \mathbf{C} \to \mathbf{C}$ a lift. For some $T_0 > 0$,

$$\tilde{\varphi}(z) = \begin{cases} z + \alpha + \eta(z) & \text{for } \Im z > T_0 \\ z + \beta + \eta(z) & \text{for } \Im z < -T_0 \end{cases}$$

where $e^{2\pi i \alpha}$ and $e^{2\pi i \beta}$ are the eigenvalues of φ at ∞ and 0, η is a bounded analytic function on $\{z : |\Im z| > T_0\}$, and $\eta(z) = O(e^{-|\Im z|})$. Schwarz' Lemma yields the further estimate $\eta'(z) = O(e^{-|\Im z|})$. By ??, we obtain a uniform $|\tilde{\varphi}(z) - z| < M_0$.

Recalling the construction of ??, consider (writing $z = x + iy$)

$$\gamma_T(z) = \begin{cases} (1 + \frac{y}{T}(\tilde{\varphi}(x + iT) - x) & \text{for } 0 \le \Im z \le T \\ (1 - \frac{y}{T}(\tilde{\varphi}(x - iT) - x) & \text{for } -T \le \Im z \le 0 \\ \tilde{\varphi}(z) & \text{for } |\Im z| > T. \end{cases}$$

It is easily verified that $|\gamma_T(z) - z| < M_0$ for $T \ge T_0$; by ??, γ_T is $\kappa(T)$-quasiconformal, where $\kappa(T) \to 1$ as $T \to \infty$. Fixing T with $\kappa(T) < 1 + \epsilon$ and $M > M_0$, consider

$$\begin{aligned} A &= \{z : \Re z \le -M,\ |\Im z| \le T\}, \\ B &= \{z : -M \le \Re z \le 0,\ |\Im z| \le T\}, \\ C &= \{z : |\Im z| \ge T\}, \\ D &= A \cup B \cup C. \end{aligned}$$

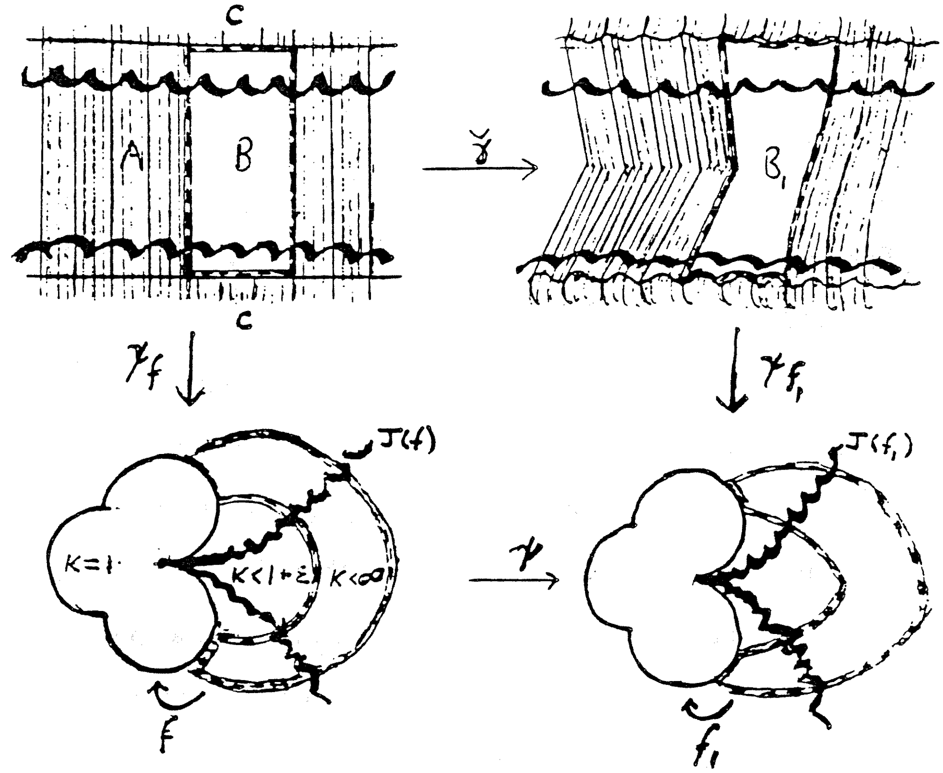

Figure 4.1[illegible] Relaxing conjugacy at parabolic point

Write ∂B as the union of closed segments $\delta_1, \ldots, \delta_4$ as labelled in Figure 4.2. Then $\gamma_T(\delta_1)$ and $\gamma(\delta_3)$ are disjoint, and $\gamma_T(\delta_1) \cup \gamma(\delta_2 \cup \delta_3 \cup \delta_4)$ is a quasicircle bounding a quasidisc B_1. Fixing quasiconformal $\sigma : B \to B_1$ agreeing with γ_T on δ_1 and γ on $\delta_2 \cup \delta_3 \cup \delta_4$, set

$$\check{\gamma}(z) = \begin{cases} \gamma_T(z) & \text{for } z \in A \cup C \\ \sigma(z) & \text{for } z \in B \\ \gamma(z) & \text{for } z \in \mathbf{C} - D. \end{cases}$$

Then $\check{\gamma} : \mathbf{C} \to \mathbf{C}$ is quasiconformal, with $K(\check{\gamma}_{|A \cup C}) < 1 + \epsilon$. Moreover, $\check{\gamma} \sim \gamma$ rel $A \cup B$, as $A \cup B \cup \{\infty\}$ is homeomorphic to the closed disc.

Let ζ be a multiplicity d parabolic fixed point of f, $\zeta_1 = \phi(\zeta)$. The choice of origins in the associated planes of $\tilde{X}_f^\dagger$ and $\tilde{X}_{f_1}^\dagger$, determines lifts $\gamma^1, \ldots, \gamma^{d-1} : \mathbf{C} \to \mathbf{C}$, of $\phi^\dagger$. Denote by χ^i the restriction of χ_f to the i-th plane in the cluster. We may arrange that the χ^i be defined and injective on D, with pairwise disjoint images $\chi^i(D)$ lying in a specified neighborhood of ζ; let $\xi^i : \chi^i(D) \to D$ be the corresponding inverse branches. Define quasiconformal $\psi : X \to X$ by

$$\psi(x) = \begin{cases} \chi_{f_1} \circ \check{\gamma}^i \circ \xi^i(x) & \text{for } z \in \chi^i(D), \\ \phi(x) & \text{elsewhere,} \end{cases}$$

with $\check{\gamma}^i$ obtained from γ^i as above, using large enough M so that

$$\mathcal{PS}(f) \cap \bigcup_{i=1}^{d-1} \chi^i(A \cup C) = \emptyset.$$

Near ζ, ψ has dilatation less than $1 + \epsilon$, and $\psi \sim \phi$ rel $\mathcal{PS}(f)$. □

We are now in a position to recapitulate McMullen's argument.

Theorem 5 *Let $f : W \to X$ be geometrically finite with empty affine part and suppose $c_1, c_2 \in \mathcal{C}(f)$ agree on $X - J_+(f)$. Then $c_1 = c_2$.*

Proof: We may assume without loss of generality that X is connected, $(X, \mathcal{PS}(f))$ is hyperbolic, and $\deg f \geq 2$. By Lemma 79, the quasiconformal map $Id_X : (X, c_1) \to (X, c_2)$ relaxes rel $\mathcal{PS}(f)$. In view of the Strebel-Teichmüller Theorems, there is a unique $\psi \in QC_0(X, \mathcal{PS}(f))$ with

$$K_{c_1,c_2}(\psi) = K = \inf_{\phi \in QC_0(f)} K_{c_1,c_2}(Id_X).$$

But $f^!\psi \in QC_0(X, \mathcal{PS}(f))$ and $K_{c_1,c_2}(f^!\phi) \leq K$ as c_1 and c_2 agree on $X - W$. Consequently, $f^!\psi = \psi$. By Proposition 19, $\psi \in QC_0(f)$; in particular, ψ is the identity on $J(f)$.

Moreover, $\psi : (X, c_1) \to (X, c_2)$ is either conformal or the Teichmüller map associated to some non-zero $q \in Q(X, \mathcal{PS}(f))$. In the latter case, as ψ is a conjugacy, q and f^*q determine the same measurable line field on W; but then f is affine by Proposition 20. Consequently, ψ is conformal, so $c_1 = \psi^* c_2$. As c_1 and c_2 agree on $\Omega(f)$, $\psi : X \to X$ is conformal on $\Omega(f)$. By Bers' Lemma, ψ is 1-quasiconformal, hence conformal. Thus, $\psi = Id_X$ and $c_1 = c_2$. □

See [11] for an application, involving Radström maps, to real dynamics.

4.3 Teichmüller Spaces

We pursue a discussion parallel to that of Chapter 1, in the category of finite type towers. Define

$$Teich(\mathcal{F}) = \mathcal{C}(\mathcal{F})/QC_0(\mathcal{F}).$$

There is a canonical map $Teich(\mathcal{F}) \to Teich(X, \mathcal{M}(\mathcal{F}))$. Additionally, if $\mathcal{G}$ is a subtower, there is a canonical map $Teich(\mathcal{F}) \to Teich(\mathcal{G})$.

The forgetful map $Teich(X, \mathcal{M}(\mathcal{F})) \to Teich(\mathcal{F})$ is natural in the sense that

$$\begin{array}{ccc} Teich(X, \mathcal{M}(\mathcal{F})) & \overset{\phi^\#}{\longleftarrow} & Teich(X_1, \mathcal{M}(\mathcal{F}_1)) \\ \downarrow & & \downarrow \\ Teich(\mathcal{F}) & \overset{\phi^\#}{\longleftarrow} & Teich(\mathcal{F}_1) \end{array}$$

commutes for every quasiconformal conjugacy $\phi : X \to X_1$ from $\mathcal{F}$ to $\mathcal{F}_1$.

Similarly for $Teich(\mathcal{F}) \to Teich(\mathcal{G}) \ldots$.

As in Chapter 1, we may split $Teich(\mathcal{F})$ according to support. For $\mathcal{F}$-invariant measurable $A \subseteq X$ with $\partial A \subset \mathcal{M}(\mathcal{F})$, let

$$Teich^A(\mathcal{F}) = \mathcal{C}^A(X)/QC_0(\mathcal{F})$$

where $\mathcal{C}^A(\mathcal{F}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{C}(X)$. As $\mathcal{C}^A(\mathcal{F})$ is always connected, $Teich^A(\mathcal{F})$ is either trivial or of positive topological dimension.

Lemma 80 *Let $\mathcal{F}$ be a finite type tower on a complex 1-manifold X, $\mathcal{A}$ a partition of X into $\mathcal{F}$-invariant measurable sets with boundary in $\mathcal{M}(\mathcal{F})$. Then the canonical map*

$$Teich(\mathcal{F}) \to \prod_{A \in \mathcal{A}} Teich^A(\mathcal{F})$$

is injective. If $Teich^A(\mathcal{F})$ *is trivial for all but finitely many* A*, then* $Teich(\mathcal{F})$ *and* $\prod_{A\in\mathcal{A}} Teich^A(\mathcal{F})$ *are canonically homeomorphic.*

Proof:

We may immediately dispose of some special cases.

Lemma 81 *Let* $\mathcal{F}$ *be a finite type tower on a complex 1-manifold* X.

A. *Let* $\mathcal{A}$ *be the set of essential components* Z *where* f^Z *is conjugate to* $z \rightsquigarrow z^n$ *for some* $n \in \mathbf{Z}$*, snd let* $\mathcal{B}$ *be the complementary set. Then* $Teich(\mathcal{F})$ *is canonically isomorphic to* $Teich(\mathcal{F}^{\mathcal{B}})$.

B. *Let* $\mathcal{A}$ *be the affine part of* $\mathcal{F}$*,* $\mathcal{B}$ *the complementary part. Then* $Teich^+(\mathcal{F})$ *is homeomorphic to* $Teich^+(\mathcal{F}^{\mathcal{A}}) \times Teich^+(\mathcal{F}^{\mathcal{B}})$.

Proof:

We will find it useful to split according to eventual height and membership in the Fatou or Julia sets. For $n \in \mathbf{N} \cup \{\infty, +\}$, we will employ the abbreviations

$$\begin{aligned} Teich^n(X) &= Teich^{X_n(\mathcal{F})}(\mathcal{F}) \\ Teich^{\Omega_n}(X) &= Teich^{\Omega_n(\mathcal{F})}(\mathcal{F}) \\ Teich^{J_n}(X) &= Teich^{J_n(\mathcal{F})}(\mathcal{F}). \end{aligned}$$

Lemma 82 *Let* $\mathcal{F}$ *be a finite type tower,* $0 \le m < \infty$, $0 \le n \le \infty$. *Under the canonical map* $Teich(\mathcal{F}) \to Teich(\mathcal{F}^n)$:

- *For* $m < n$, $Teich^m(\mathcal{F}) \cong Teich^m(\mathcal{F}^n)$*, respecting the further splitting* Ω *vs.* J.

- $Teich^n(\mathcal{F}) \hookrightarrow Teich^n(\mathcal{F}^n)$, *again respecting the splitting* Ω *vs.* J.

- *For* $m > n$, $Teich^m(\mathcal{F}) \to Teich^{\Omega_n}(\mathcal{F}^n)$.

Proof: It is easily seen that $\mathcal{C}^m(\mathcal{F}) = \mathcal{C}^m(\mathcal{F}^n)$ for $m < n$, and $\mathcal{C}^n(\mathcal{F}) \subseteq \mathcal{C}^n(\mathcal{F}^n)$, corresponding statements holding for $\mathcal{C}_\Omega, \mathcal{C}_J$. Furthermore, $\mathcal{C}^m(\mathcal{F}) \subseteq \mathcal{C}^n_\Omega(\mathcal{F})$ when $m > n$. □

Suppose $height(\mathcal{F}) \geq 2$. Recall that $c \in \mathcal{C}(\mathcal{F})$ descends to $R(c) \in \mathcal{C}(\mathcal{R}\mathcal{F})$.

Lemma 83 *There is a continuous induced map* $R : Teich(\mathcal{F}) \to Teich(\mathcal{R}\mathcal{F})$. *Moreover,*

$$R : Teich^+(\mathcal{F}) \xrightarrow{\cong} Teich^1(\mathcal{F}) \times Teich^+(\mathcal{R}\mathcal{F}).$$

Proof:

Lemma 84 *Let* $O \subseteq X$ *be the smallest* $\mathcal{F}$*-invariant set containing a component* U *of* $\mho(\mathcal{F})$.

A. For wandering U, $Teich^O(\mathcal{F}) \cong \lim_{g \in \mathcal{F}[U]} Teich(g_*U)$.

B. For fixed U, $\Sigma = Fix_{\mathcal{F}}(U)$, $Teich^O(\mathcal{F}) \cong Teich(U)^\Sigma$.

Consequently, $Teich^O(\mathcal{F})$ is a complex Banach manifold.

4.4 Central Finiteness Theorem

We now prove the Central Finiteness Theorem, that is, the finite dimensionality of $Teich^+(\mathcal{F})$ for finite type towers. We obtain a dimension bound on

$Teich^+(\mathcal{F})$ by induction on $height(\mathcal{F})$. As a first step, we bound the dimension for finite type maps by exhibiting a natural injection of $Teich^+(f)$ into an auxiliary finite dimensional space of deformations. Sullivan's original proof for rational maps made use of the obvious finite dimensional parameter spaces; however, it is necessary to make certain normalizations, and one only obtains a locally defined map with totally disconnected fibre. We present an abstract functorial functorial construction applying in full generality. The considerations of Section ?? lead to a continuous injection of $Teich^+(f)$ into a space $Def(f)$ whose finite dimensionality follows from the contraction principle of Section ??.

Lemma 85 *Let $\mathcal{F}$ be a finite type tower on X. Then the canonical map*

$$Teich^+(\mathcal{F}) \to Teich(X, \mathcal{PS}(\mathcal{F}))$$

is injective.

Proof: Without loss of generality, $(X, \mathcal{PS}(\mathcal{F}))$ is hyperbolic. By Proposition 19, if $c_1, c_2 \in \mathcal{C}^+(\mathcal{F})$, $\phi \in QC_0(X, \mathcal{PS}(\mathcal{F}))$, and $\phi^* c_2 = c_1$, then $\phi \in QC_0(\mathcal{F})$. □

Let f be a finite type iterable analytic map. As $\mathcal{PS}(f)$ may be infinite, the above injection does not directly lead to a dimension bound. Note however that as

$$\mathcal{PS}(f) \subseteq (X - W) \cup f^{-1}(\mathcal{PS}(f)),$$

the forgetful map

$$Teich(\ X, (X-W)\cup f^{-1}(\mathcal{PS}(f))\ ) \rightarrow Teich(X, \mathcal{PS}(f))$$

composed with

$$f^{\#} : Teich(X, \mathcal{PS}(f)) \rightarrow Teich(\ X, (X-W)\cup f^{-1}(\mathcal{PS}(f))\ )$$

gives a self-map $\sigma : Teich(X, \mathcal{PS}(f) \longrightarrow Teich(X, \mathcal{PS}(f)$ whose fixed point set contains the image of $Teich^{+}(f)$. Recall the forgetful maps for $n \geq 1$:

$$\begin{array}{ccc} & \nearrow & Teich(X, S(f^n)) \\ Teich(X, \mathcal{PS}(f)) & & \downarrow p_n \\ & \searrow & Teich(X, S(f^{n-1})) \end{array}$$

As $S(f^{n-1}) \subseteq (X-W)\cup f^{-1}(S(f^n))$, we similarly obtain

$$\sigma_n : Teich(X, S(f^n)) \longrightarrow Teich(X, S(f^{n-1}))$$

on composing the forgetful map

$$Teich(\ X, (X-W)\cup f^{-1}(S(f^n)) \rightarrow Teich(X, S(f^{n-1}))$$

with

$$f^{\#} : Teich(X, S(f^n)) \rightarrow Teich(\ X, (X-W)\cup f^{-1}(S(f^n)).$$

Consider the commutative diagrams of analytic maps:

$$\begin{array}{ccccc} Teich(X, S(f^n)) & & \xrightarrow{\sigma_n} & & Teich(X, S(f^{n-1})) \\ & \nwarrow & & \nearrow & \\ p_n \downarrow & & Teich^{+}(f) & & \downarrow p_{n-1} \\ & \swarrow & & \searrow & \\ Teich(X, S(f^{n-1})) & & \xrightarrow{\sigma_{n-1}} & & Teich(X, S(f^{n-2})) \end{array}$$

Let $Def_n(f)$ be the analytic subvariety of $Teich(X, S(f^n))$ where the maps p_n and σ_n agree; their common restriction sends $Def_n(f)$ to $Def_{n-1}(f)$. Taken together, these varieties and projections constitute an inverse system, and

$$Def(f) = \varprojlim Def_n(f)$$

canonically injects into $\lim_{\leftarrow} Teich(X, S(f^n))$. By commutativity of the diagrams, $Def_n(f)$ contains the image of $Teich(f) \to Teich(X, S(f^n))$; consequently, there is an induced map $Teich(f) \to Def(f)$.

Proposition 21 *Let $f : W \to X$ be a finite type iterable analytic map. Then the canonical map*

$$Teich(f) \longrightarrow Def(f)$$

is injective.

Proof: The square of canonical maps commutes:

$$\begin{array}{ccc} Teich(f) & \to & Def(f) \\ \downarrow & & \downarrow \\ Teich(X, \mathcal{PS}(f)) & \to & \lim_{\leftarrow} Teich(X, S(f^n)) \end{array}$$

where

$$Teich(f) \hookrightarrow Teich(X, \mathcal{PS}(f))$$

by Lemma 85, and

$$Teich(X, \mathcal{PS}(f)) \hookrightarrow \varprojlim Teich(X, S(f^n))$$

by Proposition 3. □

It is important to take note of the sense in which the injection $Teich(f) \hookrightarrow Def(f)$ is natural. As discussed above, the forgetful maps $Teich(f) \rightarrow Teich(X, S(f^n))$ commute with the allowable bijections associated to quasiconformal conjugacies, and the p_n commute with all allowable bijections. The maps $\sigma_n : Teich(X, S(f^n)) \rightarrow Teich(X, S(f^{n-1}))$ are natural in the sense that

$$\begin{array}{ccc} Teich(X, S(f^n)) & \xleftarrow{\phi^\#} & Teich(X_1, S(f_1^n)) \\ \sigma_n \downarrow & & \downarrow \sigma_n \\ Teich(X, S(f^{n-1})) & \xleftarrow{\phi^\#} & Teich(X_1, S(f_1^{n-1})) \end{array}$$

commutes for every quasiconformal conjugacy $\phi : X \rightarrow X_1$ conformal on $X - W$. Consequently, such a conjugacy induces allowable bijections

$$\phi^\# : Def_n(f_1) \rightarrow Def_n(f)$$

and

$$\phi^\# : Def(f_1) \rightarrow Def(f)$$

Thus, $Teich(f) \hookrightarrow Def(f)$ is natural in the sense that

$$\begin{array}{ccc} Teich(f_1) & \xleftarrow{\phi^\#} & Teich(f) \\ \downarrow & & \downarrow \\ Def(f_1) & \xleftarrow{\phi^\#} & Def(f) \end{array}$$

commutes for quasiconformal conjugacies conformal on $X - W$.

For $n \geq 1$, let $\delta_n(f) = \#D_n$ where $D_n = S(f^n) - S(f^{n-1})$; $D_{n+1} \subseteq f(D_n)$ by Lemma 57, so $\delta_n(f)$ is non-increasing. Consequently, $f_{|D_n} : D_n \rightarrow D_{n+1}$ is eventually bijective, and $\delta_n(f)$ stabilizes; the eventual value is the number of distinct forward infinite grand singular orbits. Some of these orbits may be absorbed by the affine part of f; let $\delta(f)$ be the number of the remaining orbits.

Proposition 22 *Let $f : W \to X$ be a finite type iterable analytic map with empty elementary and affine parts. Then $Def_n(f) \hookrightarrow Teich(X, S(f^n))$ is a complex submanifold of dimension $\delta_n(f)$.*

Proof: We may express $Def_n(f)$ as the inverse image of the diagonal Δ under the analytic map

$$\alpha = \sigma_n \times p_n : Teich(X, S(f^n)) \to Teich(X, S(f^{n-1}))^2.$$

We will prove that α is transverse to Δ; once we have established this, it will follow by the Implicit Function Theorem that $Def_n(f)$ is a complex submanifold of dimension

$$\dim_{\mathbf{C}} Teich(X, S(f^n)) - \dim_{\mathbf{C}} Teich(X, S(f^{n-1})) = \delta_n(f).$$

We claim that

$$D\sigma_n - Dp_n : T_\tau Teich(X, S(f^n)) \to T_\xi Teich(X, S(f^{n-1}))$$

is surjective for all $\tau \in Teich(X, S(f^n))$ and $\xi \in Teich(X, S(f^{n-1}))$ with $\sigma_n(\tau) = \xi = p_n(\tau)$. By duality, it is equivalent to show that

$$D^*\sigma_n - D^*p_n : T_\xi^* Teich(X, S(f^{n-1})) \to T_\tau^* Teich(X, S(f^n))$$

is injective. By naturality, it is enough to check this at the base point, where the cotangent map is given by

$$f_* - Id : Q(X, S(f^{n-1})) \to Q(X, S(f^n)),$$

and injectivity follows by Corollary 8.

Given $w_1, w_2 \in T_\xi(X, S(f^{n-1}))$, choose $v \in T_\tau Teich(X, S(f^n))$ with

$$(D\sigma_n - Dp_n)v = (w_1 - w_2).$$

Then $(D\alpha)v + (w, w) = (w_1, w_2)$ where $w = w_1 - (D\sigma_n)v$. Thus,

$$D\alpha(\,T_\tau Teich(X, S(f^n))\,) + T_\tau\Delta = T_{\alpha(\tau)}Teich(X, S(f^{n-1}))$$

at every $\tau \in Def_n(f)$; that is, f is transverse to Δ. □

Corollary 9 *Let $f : W \to X$ be a finite type iterable analytic map. Then $\dim_{\mathbf{C}} Teich^+(f) < \infty$. Furthermore,*

$$\dim_{\mathbf{C}} Teich(f) \leq \delta(f)$$

for maps wit h empty affine and elementary parts.

Proof: By Lemma 81,

$$\dim_{\mathbf{C}} Teich(f) = \dim_{\mathbf{C}} Teich(f^{\mathcal{A}}) + \dim_{\mathbf{C}} Teich(f^{\mathcal{B}}) \leq aff(f) + \delta(f)$$

where $\mathcal{A}$ and $\mathcal{B}$ are the affine and complementary parts, and moreover, $\dim_{\mathbf{C}} Teich(f^{\mathcal{A}}) = aff(f)$. By Lemma 96, $Def(f^{\mathcal{B}})$ has topological dimension less than or equal to $2\delta(f^{\mathcal{B}})$, and $\delta(f) = \delta(f^{\mathcal{B}})$ by definition. □

Consequently, there are continuous maps

$$Teich^1(\mathcal{F}) \hookrightarrow Teich(f)$$

and

$$Teich(\mathcal{F}) \cong Teich^1(\mathcal{F}) \times Teich(\mathcal{RF})$$

which together provide the inductive step in bounding the dimension of $Teich(\mathcal{F})$ for finite height towers. To obtain a height-independent bound, we appeal to:

Lemma 86 *Let $\mathcal{F}$ be a finite type tower with base f and $height(\mathcal{F}) \geq 2$, r the base of $\mathcal{RF}$. Then $\delta(r) \leq \delta(f)$; if equality holds, then*

$$R : Teich(\mathcal{F}) \xrightarrow{\cong} Teich(\mathcal{RF}).$$

Proof: By hypothesis, f is geometrically finite, and $\Omega(\mathcal{F})$ has no height 1 component; thus $Teich^1(\mathcal{F})$ is trivial by either Theorem 4 or 5. □

Finally, a height-independent bound extends to infinite height towers by virtue of:

Lemma 87 *Let $\mathcal{F}$ be a finite type tower. Then the canonical map*

$$Teich(\mathcal{F}) \to \varprojlim Teich(\mathcal{F}^n)$$

is injective.

Proof:

By Corollary ??, $QC_0(\mathcal{F}) = \bigcap_n QC_0(\mathcal{F}^n)$. As $\mathcal{C}(\mathcal{F}) = \bigcap_{n=1}^{\infty} \mathcal{C}(\mathcal{F})^n$, there is a commutative square of canonical maps:

$$\begin{array}{ccc} Teich(\mathcal{F}) & \hookrightarrow & \mathcal{C}(X)/QC_0(\mathcal{F}) \\ \iota\downarrow & & \downarrow j \\ \lim_{\leftarrow} Teich(\mathcal{F}^n) & \hookrightarrow & \lim_{\leftarrow} \mathcal{C}(X)/QC_0(\mathcal{F}^n) \end{array}$$

By [*], the action of $QC_0(f)$ on $\mathcal{C}(X)$ is fixed-point free, hence j is injective in view of Lemma 11. It follows that ι is injective. □

Theorem 6 *Let $\mathcal{F}$ be a finite type tower. Then* $\dim_{\mathbf{C}} Teich(\mathcal{F}) < \infty$.

Proof: Let $N = height(\mathcal{F})$. For $1 \leq m < N$, let f_m be the least element of $\mathcal{R}^{m-1}\mathcal{F}$. Then

$$Teich^m(\mathcal{F}) \cong Teich^1(\mathcal{R}^{m-1}\mathcal{F}) \hookrightarrow Teich(f_m),$$

hence $\dim_{\mathbf{C}} Teich^m(\mathcal{F}) \leq \delta(f_m)$ by Corollary 9. Moreover, if $m+1 < N$ and $\delta(f_m) = \delta(f_{m+1})$, then $\dim_{\mathbf{C}} Teich^m(\mathcal{F}) = 0$ be Lemma 86. Summing over m yields the rough bound

$$\dim_{\mathbf{C}} Teich(\mathcal{F}) \leq \frac{1}{2}[\delta(f) + \delta(f)^2]$$

for finite height towers. By Lemma 87, the same bound holds for infinite height towers. □

Corollary 10 *Let $\mathcal{F}$ be a finite type tower. Then $Teich(\mathcal{F})$ is a contractible complex Banach manifold.*

Proof: By Lemma 80, $Teich(\mathcal{F})$ canonically injects into the product

$$\prod_{0 \leq n \leq \infty} Teich^n(\mathcal{F})$$

of complex Banach manifolds. In view of Theorem 6 every factor $Teich^n(\mathcal{F})$ with $n > 0$ is finite dimensional, and all but finitely many factors are points. Thus,

$$Teich(\mathcal{F}) \cong \prod_{0 \leq n \leq \infty} Teich^n(\mathcal{F})$$

is a contractible complex Banach manifold. □

Clearly, $\dim Teich^m(\mathcal{F}) = 0$ for large finite m, and

$$Teich^\infty(\mathcal{F}) \hookrightarrow \varprojlim Teich^n(\mathcal{F}^n)$$

hence $\dim_{\mathbf{C}} Teich^\infty(\mathcal{F}) \leq \delta(\mathcal{F})$. This bound does not require Theorems 4 or 5, as a closer analysis shows

$$Teich^\infty(\mathcal{F}) \hookrightarrow \varprojlim Teich^n_\Omega(\mathcal{F}^n).$$

Consequently, we still obtain a bound on $\dim_{\mathbf{C}} Teich_\Omega(\mathcal{F})$, and this is all we need in the next section to prove no wandering domains.

Finite dim gives reduction to linefields

Corollary 11 *Let $\mathcal{F}$ be a finite type tower. Then $Teich^J(\mathcal{F})$ has finite ergodic decomposition.*

Conjecture 1 *Let $\mathcal{F}$ be a finite type tower with no affine part. Then $J_+(\mathcal{F})$ supports no invariant linefield.*

There are really two parts to the conjecture, the first concerning J_+ of finite type map, appropriately generalizing the standard conjecture for rational maps, the second concerning J_∞ of an infinite height tower.

Heuristically, each parabolic basin should decrease the expected dimension of $Teich(f)$ by 1. Let $p(f)$ be the number of parabolic basins.

Conjecture 2 *Let $f : W \rightarrow X$ be a finite type iterable analytic map with empty affine part. Then*

$$\dim_{\mathbf{C}} Teich^{+}(f) + p(f) \leq \delta(f).$$

In the algebraic setting of rational maps, the corresponding conjecture would assert that the loci of maps with parabolic points lie in general position. This seems to be unknown even with the avaliable algebraic techniques. The Conjecture would follow from a more refined contraction principle involving parabolics.

Assuming Conjecture 2, we obtain a sharp dimension bound.

Conjecture 3 *Let $\mathcal{F}$ be a finite type tower with base f and empty affine and elementary parts. Then* $\dim_{\mathbf{C}} Teich^{+}(\mathcal{F}) \leq \delta(f)$.

Proof (assuming Conjecture 2): As above, it suffices to establish the bound for finite height towers. In view of 9, we may assume $height(\mathcal{F}) \geq 2$. Let r be the base of $\mathcal{RF}$; then

$$\dim_{\mathbf{C}} Teich^{+}(\mathcal{F}) \leq \dim_{\mathbf{C}} Teich^{1}(f) + \delta(f)$$

by induction. Now

$$\dim_{\mathbf{C}} Teich^{1}(f) = \dim_{\mathbf{C}} Teich^{+}(f) - \dim_{\mathbf{C}} Teich(X_f, S_f)$$

where

$$\dim_{\mathbf{C}} Teich(X_f, S_f) = \#S_f - 3p(f).$$

The sphere components of X_f contribute $2p(f)$ poles with finite forward r-orbit; as every other point of S_f is the image of an infinite forward singular f-orbit in $Bas(f)$,

$$\delta(r) < \#S(f) - 2p(f),$$

and Conjecture 2 yields the bound

$$\dim_{\mathbf{C}} Teich^{+}(\mathcal{F}) \leq [\delta(f) - p(f)] - [\#S_f - 3p(f)] + \delta(r) \leq \delta(f).$$

□

4.5 Dynamical Consequences

We say that an annulus component A of $\mho(\mathcal{F})$ is *infinitely wrapped* if $\deg g \to \infty$ as g increases in $\mathcal{F}[A]$.

Lemma 88 *Let $\mathcal{F}$ be a complete finite type tower, U a fixed component of $\Omega(\mathcal{F})$ containing an annulus component of $\mho(\mathcal{F})$ with $ht_{\mathcal{F}}(A) = ht_{\mathcal{F}}(U)$. Then U is a type I superattracting domain if A is infinitely wrapped, and a Siegel disc or Hermann ring otherwise.*

In this section we prove the No Wandering Domains Theorem for finite type towers. The two main ingredients are the finiteness of $\dim Teich(\mathcal{F})$ and a dynamical argument concerning annular components of $\mho(\mathcal{F})$.

Lemma 89 *Let $\mathcal{F}$ be a finite type tower, V a wandering component of $\mho(\mathcal{F})$. Then V is an infinitely wrapped annulus.*

Proof: If not, then by Lemma 1, $V_\infty = \lim_{\rightarrow} g_* V$ is a Riemann surface. Consequently, $Teich(V_\infty)$ injects continuously into $Teich(\mathcal{F})$. By Theorem 6, $\dim Teich(V_\infty) < \infty$, so V_∞ is a finite type surface. As $\pi_1(V_\infty)$ is finitely generated, it follows by Lemma 1 that $g_* V$ has finite type for sufficiently large g. By hypothesis, there are infinitely many $g_* V$ occupying distinct components of X; but X has only finitely many components. □

Let A be an annulus component of $\mho(\mathcal{F})$, $\Gamma \subseteq A$ a circle. For $g \in \mathcal{F}[A]$, $A_g = g(A)$ is an annulus component, $\Gamma_g = g(\Gamma)$ is a circle in A_g, and $g_{|\Gamma} : \Gamma \to \Gamma_g$ is a covering space of degree $\deg g$.

Lemma 90 *Let $\mathcal{F}$ be a finite type tower on X, A an infinitely wrapped annulus component of $\mho(\mathcal{F})$, $\Gamma \subseteq A$ a circle. Then $diam(\Gamma_g) \to 0$ as g increases in $\mathcal{F}[A]$.*

Proof: Without loss of generality, we may assume X connected. Remove a finite set of marked points to obtain a hyperbolic Riemann surface X^*. Then $\ell_{X^*}(\Gamma_g) < \ell_{A_g}(\Gamma_g) = \ell_g$ by Schwarz' Lemma, and $\ell_X(\Gamma_g) = O(\ell_{X^*}(\Gamma_g))$ by Lemma ??. As $\ell_g = (\deg h)\ell_{gh}$ for $h \in \mathcal{F}[A_g]$, $\ell_g \to 0$ as g increases in $\mathcal{F}[A]$. Consequently, $diam(\Gamma_g) < \ell_X(g) \to 0$. □

Lemma 91 *Let X be a compact surface endowed with a constant curvature metric. Then there exists ϵ such that for any simple closed curve γ in X*

with $diam|\gamma| < \epsilon$*, exactly one component* B *of* $X - |\gamma|$ *is a simply connected region with* $diam(B) < \epsilon$*.*

Proof: Let $\epsilon = \min(\iota(X), \frac{1}{2}diam(X))$, where $\iota(X) > 0$ is the injectivity radius of X. Given γ with $diam|\gamma| < \epsilon$, fix $x \in |\gamma|$ and let D be the ϵ ball about x, so that $|\gamma| \subseteq D$. D is isometric to its lifts in $\tilde{X}$, thus homeomorphic to a disc and geodesically convex. By the Jordan Curve Theorem, $D - |\gamma|$ has two components, of which the bounded one B is simply connected, and $diam(B) = diam|\gamma|$ by the geodesic convexity of D. Let A be the other component of $X - |\gamma|$. Then $diam(A) > \epsilon$, as otherwise $diam X < 2\epsilon$. Moreover, if A is simply connected then X is a sphere. □ In view of Lemma 91, for sufficiently large $g \in \mathcal{F}[A]$, Γ_g bounds a unique small disc B_g with $diam(B_g) = diam(\Gamma_g) \to 0$. Let C_g be the complementary component of $X - \Gamma_g$.

Lemma 92 *Assume in the above setting that* $\mathcal{F}$ *is complete. Then for sufficiently large* $g \in \mathcal{F}[A]$*, every* $h \in \mathcal{F}[A_g]$ *extends to* $\check{h} \in \mathcal{F}[A_g \cup B_g]$*, where* $\check{h}_{|B_g} : B_g \to B_{hg}$ *is a branched cover of the same degree as* h*; in particular,* $B_g \subseteq \Omega(\mathcal{F}^n)$ *for* $n < \hbar(A)$*.*

Proof: We may assume without loss of generality that X is connected and $ht_{\mathcal{F}}(A) = \hbar_{\mathcal{F}}(A)$. As f has finite type, $B_g \cap S(f)$ for g beyond some θ_0 is either empty or a single interior point. If $f \circ g \succeq \gamma_0$, the unique component of $f^{-1}(B_{fg})$ whose boundary meets Γ_g is a disc D_g with $\partial D_g = \Gamma_g$. We

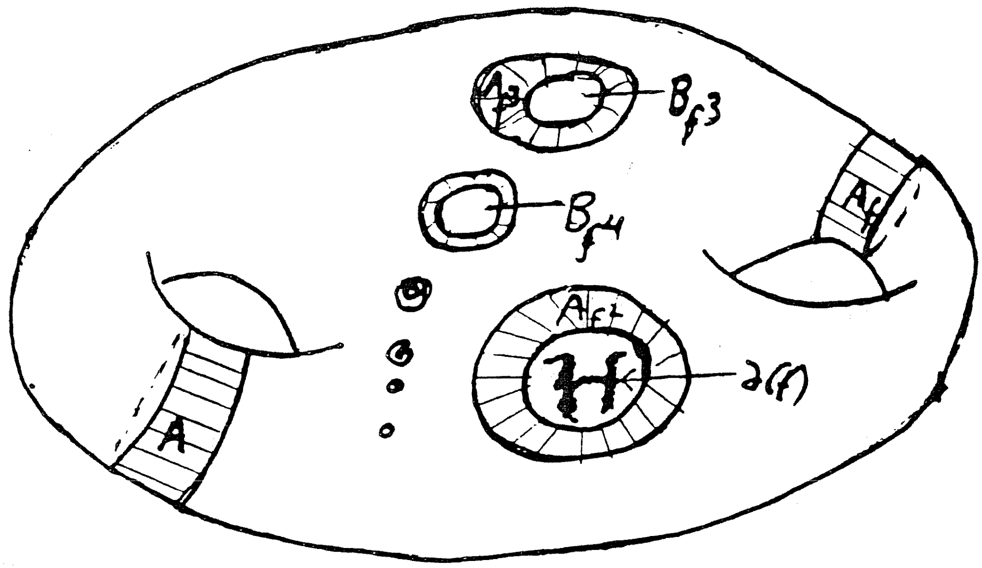

Figure 4.2: Wandering annuli I

claim that $D_g = B_g$ for all g beyond some θ_1. Otherwise, $D_{g_k} = C_{g_k}$ for some cofinal increasing sequence $g_k \in \mathcal{F}[A]$, and X is a sphere. Furthermore, if $B_{g_i} \cap B_{g_j} = \emptyset$, then $X = C_{g_i} \cup C_{g_j} \subseteq W$, so f is rational; but then $X = f(C_{g_i}) \cup f(C_{g_j}) = B_{fg_i} \cup B_{fg_j}$, which is impossible as the B_g are small. Consequently, $B_{g_i} \subset\subset B_{g_j}$ or vice-versa. As $diam(B_g) \to 0$, there is a subsequence $\alpha_\ell = g_{k_\ell}$ with $B_{\alpha_{\ell+1}} \subset\subset B_{\alpha_\ell}$ for all ℓ. But then $C_{\alpha_\ell} \subset\subset C_{\alpha_{\ell+1}}$, so $B_{f\alpha_\ell} = f(C_{\alpha_\ell}) \subset\subset f(C_{\alpha_{\ell+1}}) = B_{f\alpha_{\ell+1}}$, contradicting $diam(B_g) \to 0$. It follows for $g \succeq \theta_1$ that every $h \in \mathcal{F}[A_g]$ with $ht_{\mathcal{F}}(h) = 1$ extends as claimed.

Now assume inductively that every $\alpha \in \mathcal{F}[A_g]$, where $ht_{\mathcal{F}}(\alpha) \leq n < \hbar_{\mathcal{F}}(A)$ extends in the desired fashion for $g \succeq \theta_1$. For large $\alpha \in \mathcal{F}^n[A_g]$, $\Gamma_{\alpha g}$ lies in an attracting or parabolic domain U of $\Omega(\mathcal{F}^n)$ with $ht_{\mathcal{F}^n}(U) = n$. Any increasing sequence $\beta_k \in Fix_{\mathcal{F}}(U)$ converges locally uniformly to the associated fixed point. Consequently, $diam(\Gamma_{\beta\alpha g}) \to 0$ as β increases in $Fix_{\mathcal{F}}(U)$. Moreover, for sufficiently large β, $\Gamma_{\beta\alpha g}$ lies in a small simply connected absorbing region $V \subseteq U$, hence $B_{\beta\alpha} \subseteq V$. It follows by induction that $B_g \subseteq \Omega(\mathcal{F}^n)$ for $g \succeq \theta_1$.

Continuing, for sufficiently large $\beta \in Fix_{\mathcal{F}}(U)$, $B_{\beta\alpha g}$ lies in some region $R \subseteq V$, where R is a band surrounding the fixed point if U is a type I attracting domain, and a vertical strip otherwise. By Lemma 40, there exist arbitrarily small $\gamma \in \mathcal{F}[R]$ with $ht_{\mathcal{F}}(\gamma) = n+1$, such that $\gamma : R \to \gamma(R)$ is a covering space with image in a small disc about a parabolic or repelling fixed point of $\mathcal{F}^n$. Moreover, if $\deg \gamma > 1$, then $\gamma(R)$ is a band surrounding a repelling point. As $B_{\gamma\beta\alpha g} \subseteq \Omega(\mathcal{F}^n)$, it follows that $\Gamma_{\gamma\beta\alpha g}$ is homotopically trivial in $\gamma(R)$, hence $B_{\gamma\beta\alpha g} \subseteq \gamma(R)$ as $\gamma(R)$ is small. The component of $\gamma^{-1}(B_{\gamma\beta\alpha g})$ with boundary meeting $\Gamma_{\beta\alpha g}$ is a disc, necessarily $B_{\beta\alpha g}$, and thus $\gamma_{|B_{\beta\alpha g}} : B_{\beta\alpha g} \to B_{\gamma\beta\alpha g}$ is a homeomorphism.

Suppose $h \in \mathcal{F}[A_g]$ with $ht_{\mathcal{F}}(h) = n+1$. We claim that h extends in the desired fashion. Assume without loss of generality that h is primitive. As $\gamma \circ \beta \circ \alpha \preceq h$ for sufficiently small γ, $h = \delta \circ \gamma \circ \beta \circ \alpha$ where $ht_{\mathcal{F}}(\delta) \leq n$. By induction, h extends as claimed. □

Lemma 93 *For sufficiently large $g \in \mathcal{F}[A]$, if $h \in \mathcal{F}[A_g]$ and $\deg(h) > 1$, then there is a largest $j \in \mathcal{F}[A]$ with $j \preceq h$ and $\deg j = 1$. Furthermore, if $\mathcal{F}$ is complete, then B_{jg} contains a critical value of f.*

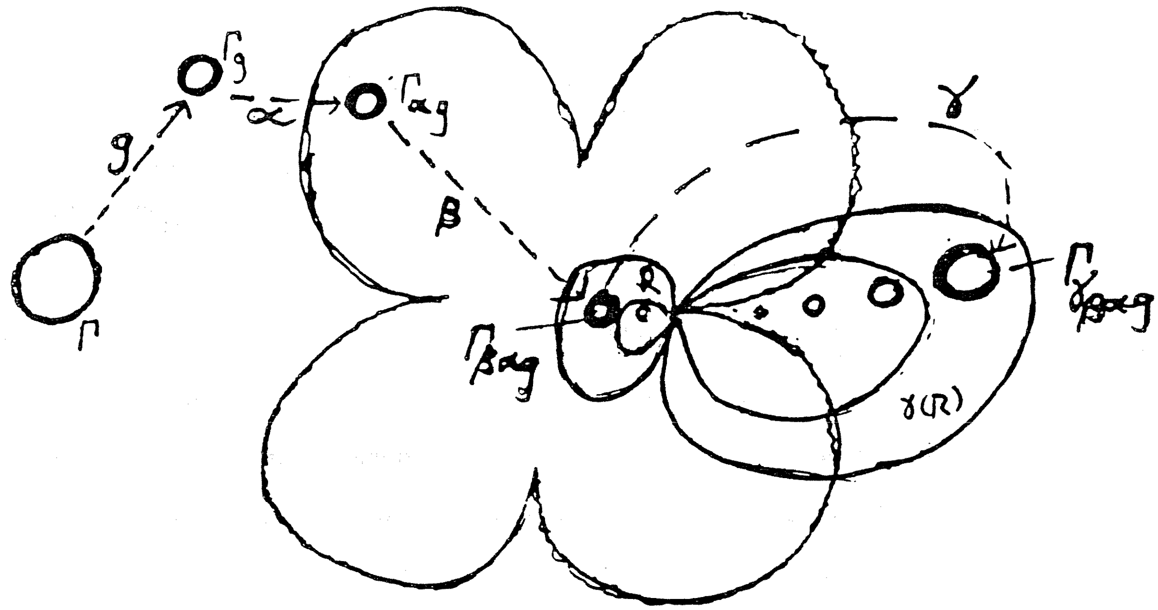

Figure 4.3: Wandering annuli II

Proof: Proceeding by induction on $n = ht_{\mathcal{F}}(h)$, we may assume without loss of generality that h is primitive. For $n = 1$ there is nothing to show. If $n > 1$, write $h = \delta \circ \gamma \circ \beta$ where $ht_{\mathcal{F}}(\beta), ht_{\mathcal{F}}(\delta) < n$ and $ht_{\mathcal{F}}(\gamma) = n$. As shown above, we may choose β and δ so that $\deg \gamma = 1$. Thus, either $\deg \beta$ or $\deg \beta$ is greater than 1, and the assertion follows by induction. □

Proposition 23 *Let $\mathcal{F}$ be a complete finite type tower on X, A an infinitely wrapped annulus component of $\mho(\mathcal{F})$. Then for large $g \in \mathcal{F}[A]$, A_g lies in a type I superattracting domain of height $\hbar_{\mathcal{F}}(A)$.*

Proof: Assume without loss of generality that $ht_{\mathcal{F}}(A) = \hbar_{\mathcal{F}}(A)$. In view of Lemma 93, there is a cofinal sequence $g_k \in \mathcal{F}[A]$ with each B_{g_k} containing one of the finitely many critical values of f. Consequently, as $diam(B_g) \to$

0, there exist $g \in \mathcal{F}[A]$ and $h \in \mathcal{F}[A_g]$ with $B_{hg} \subset\subset B_g$. Let U be the component of $\Omega(\mathcal{F})$ containing A_g. By Lemma 92 and Schwarz' Lemma, B_g contains an attracting or superattracting fixed point x of $\mathcal{F}^n$, where $n = ht_{\mathcal{F}}(h)$. Thus, $U_{[n]}$ is a fixed component of $\Omega(\mathcal{F}^n)$. But $ht_{\mathcal{F}}(A_g) = ht_{\mathcal{F}}(x) = n$, hence $U = U_{[n]}$ by a further appeal to Lemma 92. The conclusion follows by Lemma 88. □

Theorem 7 *Let $\mathcal{F}$ be a finite type tower. Then $\Omega(\mathcal{F})$ has no wandering component.*

Proof: By Lemma ??, if U is a wandering component of $\Omega(\mathcal{F})$ then $A = U - \mathcal{M}(\mathcal{F})$ is a wandering component of $\mho(\mathcal{F})$. In view of Lemma 89, A is an infinitely wrapped annulus. But then U is stable by Proposition 23. □

In view of Corollary 3, we conclude:

Corollary 12 *Let $\mathcal{F}$ be a finite type tower. Then every component of $\Omega(\mathcal{F})$ has finite height.*

Proposition 24 *Let X be a compact complex 1-manifold, $f : W \to X$ a finite type iterable analytic map. Assume that every essential component is a sphere or torus. Then the set of non-repelling periodic points of f is finite. Consequently, all but finitely many grand orbits in $\pi_0(\Omega(f))$ either escape or contain Siegel discs.*

Corollary 13 *Let $\mathcal{F}$ be a finite type tower. Then all but finitely many grand orbits in $\pi_0(\Omega(\mathcal{F}))$ are Siegel discs of height 1.*

Proof: By Proposition 24, $\Omega(r_k)$ has a finite number of periodic components as soon as $k \geq 1$. $\square$

Topological Dimension

We repeatedly use the following trivial fact: If $f : X \to Y$ is continuous and $V \subseteq Y$ is open, then $f(\partial f^{-1}(V)) \subseteq \partial V$.

Lemma 94 *Let X and Y be topological spaces, $f : X \to Y$ continuous and injective. Then* $\dim X \leq \dim Y$.

Proof: We may assume without loss of generality that $d = \dim Y < \infty$, and proceed by induction on finite d. The case $d = -1$ is vacuous as $X = Y = \emptyset$. Assume $d \geq 0$, and inductively assume the dimension inequality for $n < d$. Suppose U is open, $x \in U \subseteq X$. As f is injective, we may choose open $V \subseteq Y$ with $x \in f^{-1}(V) \subseteq U$ and $\dim \partial V < d$. Then $f_{|\partial f^{-1}(V)} : \partial f^{-1}(V) \to \partial V$ is injective, hence $\dim \partial f^{-1}(V) \leq \dim \partial V < d$ by inducton. $\square$

Lemma 95 *Let X and Y be non-empty topological spaces of finite dimension, $f : X \to Y$ continuous. If $V \subseteq Y$ is open and* $\dim \partial V < \dim Y$ *then* $\dim \partial f^{-1}(V) < \dim X$.

Proof: Let $g = f_{|\partial f^{-1}(V)} : \partial f^{-1}(V) \to \partial V$. We proceed by induction on $d = \dim Y$. For $d = 0$, $\partial V = \emptyset$, hence $\partial f^{-1}(V) = \emptyset$ and thus $\dim f^{-1}(V) = -1 < \dim X$. Assume the claim for $d - 1$. Given $x \in \partial f^{-1}(V)$ and open

$U \subseteq \partial f^{-1}(V)$ containing x, choose open $W \subseteq \partial V$ with $x \in g^{-1}(W) \subseteq U$ and $\dim \partial W < \dim \partial V < \dim X$ as X is finite dimensional. □

Lemma 96 *Let $\{f_{\alpha\beta} : X_\alpha \to X_\beta : \alpha \geq \beta\}$ be an inverse system of topological spaces. Then*

$$\dim \varprojlim X_\alpha \leq \limsup \dim X_\alpha.$$

Proof: Let $Y = \lim_{\leftarrow} X_\alpha$ with canonical maps $\pi_\alpha : Y \to X_\alpha$, and let $d = \limsup \dim X_\alpha$. If $d = \infty$, there is nothing to prove; we proceed by induction on finite d. Passing to a cofinal subsystem, we may assume every X_α has dimension d. If $d = -1$ then every $X_\alpha = \emptyset$, hence $Y = \emptyset$ and $\dim Y = -1$. Assume the claim holds for $d-1$, and suppose $U \subseteq Y$ is open, $y \in U$. There exist γ and open $V_\gamma \subseteq X_\gamma$ with $y \in \pi_\gamma^{-1}(V_\gamma) \subseteq U$ and $\dim \partial V_\gamma \leq d-1$. Set $V = \pi_\gamma^{-1}(V_\gamma)$, $V_\alpha = f_{\alpha\gamma}^{-1}(V_\gamma)$ for $\alpha > \gamma$. By construction, V and the V_α are open sets, $\pi_\alpha(\partial V) \subseteq \partial V_\alpha$ for $\alpha \geq \gamma$, and $f_{\alpha\beta}(\partial V_\alpha) \subseteq \partial V_\beta$ for $\alpha \geq \beta \geq \gamma$. Moreover, $\dim \partial V_\alpha \leq d-1$ by Lemma 95. Thus,

$$\{f_{\alpha\beta|\partial V_\alpha} : \partial V_\alpha \to \partial V_\beta \;:\; \alpha \geq \beta \geq \gamma\}$$

is an inverse system, and there is a canonical continuous map

$$j : \partial V \to \varprojlim \partial V_\alpha.$$

By naturality, composing with the canonical map $\lim_{\leftarrow} \partial V_\alpha \to \lim_{\leftarrow} X_\alpha$ gives the inclusion of ∂V into Y, hence j is injective. By induction,

$$\dim \partial V \leq \dim \varprojlim \partial V_\alpha \leq d-1,$$

hence $\dim Y \leq d$. □

Proposition 25 *Let M be a topological n-manifold, $0 \leq n \leq \infty$. Then* $\dim M = n$.

Bibliography

[1] I.N. Baker, Repulsive fixpoints of entire functions, *Math. Z.* **104** (1968), 252-256.

[2] I.N. Baker, J. Kotus, L. Yinian, Iterates of meromorphic functions: I, *J. Erg. Thy. and Dyn. Sys.* **11** (1991), 241-248.

[3] P. Battacharyya, Iteration of analytic functions, Ph.D. thesis, University of London, 1969.

[4] N. Bourbaki, Éléments de Mathématique, Topologie Générale 2, Paris, Hermann, 1966.

[5] S. Bullett and C. Penrose, Geometry and topology of iterated correspondences: An illustrated survey, IHES preprint 1992.

[6] R. Devaney and L. Keen, Dynamics of meromorphic maps: Maps with polynomial Schwarzian derivative, *Ann. Sci. Ec. Norm. Sup* 4^e Ser. **22** (1989), 55-79.

[7] A. Douady and J. Hubbard, Étude dynamique des polynômes complexes I & II, Publ. Math. Orsay, 1984-5.

[8] A. Douady and J. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. Ec. Norm. Sup.* 4^e Ser. **18** (1985), 287-344.

[9] A. Douady and J. Hubbard, A proof of Thurston's topological characterization of rational functions, Mittag-Leffler Insitute preprint.

[10] C. Earle and C. McMullen, Quasiconformal isotopies, in *Holomorphic Functions and Moduli I*, New York, Springer-Verlag MSRI **10**, 1988, 143-154.

[11] A. Epstein, L. Keen, and C. Tresser, The set of maps

$$F_{A,b} : x \leadsto x + A + \frac{b}{2\pi}\sin(2\pi x)$$

with any given rotation interval is contractible, Preprint.

[12] A. Eremenko and M. Lyubich, Dynamical properties of some classes of entire functions, Stony Brook IMS preprint 1990/4.

[13] F. Gardiner, *Teichmüller Theory and Quadratic Differentials*, New York, Wiley-Interscience, 1987.

[14] L. Goldberg and L. Keen, A finiteness theorem for a dynamical class of entire functions, *J. Erg. Thy. and Dyn. Sys.* **6** (1986), 183-192.

[15] X. Gomez-Mont, Transversal holomorphic structures, *J. Diff. Geo.* **15** (1980), 161-185.

[16] W. Hayman, *Meromorphic Functions*, Oxford, Oxford University Press, 1964.

[17] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, Princeton University Press, 1948.

[18] L. Keen, Dynamics of holomorphic self-maps of $\mathbf{C}^*$, in *Holomorphic Functions and Moduli I*, New York, Springer-Verlag MSRI **10**, 1988, 9-30.

[19] S. Kerckhoff and W. Thurston, Non-continuity of the action of the modular group at Bers' boundary of Teichmuller space, *Inv. Math.* **100** (1990), 25-47.

[20] J. Kotus, Iterated holomorphic maps on the puncutured plane, Polish Academy of Sciences preprint.

[21] J. Kotus, Iterates of meromorphic functions, Preprint.

[22] P. Lavaurs, Systèmes dynamiques holomorphes: explosion de points periodiques paraboliques, Thèse de doctorat de l'Université de Paris-Sud, Orsay, France, 1989.

[23] O. Lehto and K. Virtannen, *Quasiconformal Mappings in the Plane*, Berlin and New York, Springer-Verlag, 1973.

[24] C. McMullen, Families of rational maps and iterative root-finding algorithms, *Ann. of Math.*, **125** (1987), 467-493.

[25] C. McMullen, Automorphism of rational maps, in *Holomorphic Functions and Moduli I*, New York, Springer-Verlag MSRI **10**, 1988, 31-60.

[26] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions, Preprint.

[27] C. McMullen, Amenability, Poincaré series and quasiconformal maps, *Inv. Math.* **97** (1989), 95-127.

[28] C. McMullen, Iteration on Teichmüller space, *Inv. Math.* **99** (1990), 425-454.

[29] C. McMullen, Rational maps and Kleinian groups, *1990 ICM Proceedings*, Springer-Verlag, 1990, 889-899.

[30] J. Milnor, Dynamics in one complex variable: Introductory lectures, Stony Brook IMS preprint 1990/5.

[31] E. Moise, *Geometric Topology in Dimensions 2 and 3*, Berlin and New York, Springer-Verlag **48**, 1977.

[32] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, New York, Wiley-Interscience, 1988.

[33] R. Nevanlinna, *Analytic Functions*, Berlin and New York, Springer-Verlag, 1970.

[34] H. Radström, On the iteration of analytic functions, *Math. Scand.* **1** (1953), 85-92.

[35] M. Shishikura, On the quasi-conformal surgery of rational functions, *Ann. Sci. Ec. Norm. Sup.* 4^e Ser. **20** (1987), 1-29.

[36] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set, Stony Brook IMS preprint 1991/7.

[37] D. Sullivan, Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou-Julia problem on wandering domains, *Ann. of Math.* **122** (1985), 401-418.

[38] D. Sullivan, Quasiconformal homeomorphisms and dynamics III: Topological conjugacy classes of analytic endomorphisms, Preprint.

[39] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures, in *Mathematics into the Twenty-first Century II*, Providence, AMS Centennial Publications, (1992), 417-467.

[40] W. Thurston, *The Geometry and Topology of Three Manifolds*, Lecture notes, Princeton University (1979).

[41] W. Thurston, On the combinatorics and dynamics of iterated rational maps, Preprint.