Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

An Introduction to Holomorphic Dynamics

(with particular focus on transcendental functions)

L. Rempe

Department of Mathematical Sciences,
University of Liverpool

Liverpool, December 2006

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Outline

1. Introduction
   - Basic notation
   - Basic concepts
   - Some examples
2. Normal families
   - Definition
   - Theorems of Picard and Montel
3. Definition of Fatou and Julia sets
4. Basic Properties
   - Periodic points
   - Fatou components
   - Properties of the Julia set
5. $J(f) \neq \emptyset$
   - Periodic points
   - Escaping points

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Outline

1. Introduction
   - Basic notation
   - Basic concepts
   - Some examples
2. Normal families
   - Definition
   - Theorems of Picard and Montel
3. Definition of Fatou and Julia sets
4. Basic Properties
   - Periodic points
   - Fatou components
   - Properties of the Julia set
5. $J(f) \neq \emptyset$
   - Periodic points
   - Escaping points

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Outline

1. Introduction
   - Basic notation
   - Basic concepts
   - Some examples
2. Normal families
   - Definition
   - Theorems of Picard and Montel
3. Definition of Fatou and Julia sets
4. Basic Properties
   - Periodic points
   - Fatou components
   - Properties of the Julia set
5. $J(f) \neq \emptyset$
   - Periodic points
   - Escaping points

Outline

1. Introduction
   - Basic notation
   - Basic concepts
   - Some examples
2. Normal families
   - Definition
   - Theorems of Picard and Montel
3. Definition of Fatou and Julia sets
4. Basic Properties
   - Periodic points
   - Fatou components
   - Properties of the Julia set
5. $J(f) \neq \emptyset$
   - Periodic points
   - Escaping points

Outline

1. Introduction
   - Basic notation
   - Basic concepts
   - Some examples
2. Normal families
   - Definition
   - Theorems of Picard and Montel
3. Definition of Fatou and Julia sets
4. Basic Properties
   - Periodic points
   - Fatou components
   - Properties of the Julia set
5. $J(f) \neq \emptyset$
   - Periodic points
   - Escaping points

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic notation

- $\mathbb{C}$ is the complex plane.
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- A function $f : U \to \hat{\mathbb{C}}$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\hat{\mathbb{C}}$. (I.e., f is holomorphic outside of a discrete set of poles.)
- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.
- A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (entire or meromorphic) is called *transcendental* if it does not extend continuously to $\hat{\mathbb{C}}$. (I.e., f has an essential singularity at ∞.)
- (Otherwise, it is a polynomial or a rational function.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic notation

- $\mathbb{C}$ is the complex plane.
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- A function $f : U \to \hat{\mathbb{C}}$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\hat{\mathbb{C}}$. (I.e., f is holomorphic outside of a discrete set of poles.)
- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.
- A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (entire or meromorphic) is called *transcendental* if it does not extend continuously to $\hat{\mathbb{C}}$. (I.e., f has an essential singularity at ∞.)
- (Otherwise, it is a polynomial or a rational function.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic notation

- $\mathbb{C}$ is the complex plane.
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- A function $f : U \to \hat{\mathbb{C}}$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\mathbb{C}$. (I.e., f is holomorphic outside of a discrete set of poles.)
- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.
- A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (entire or meromorphic) is called *transcendental* if it does not extend continuously to $\hat{\mathbb{C}}$. (I.e., f has an essential singularity at ∞.)
- (Otherwise, it is a polynomial or a rational function.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic notation

- $\mathbb{C}$ is the complex plane.
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- A function $f : U \to \hat{\mathbb{C}}$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\mathbb{C}$. (I.e., f is holomorphic outside of a discrete set of poles.)
- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.
- A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (entire or meromorphic) is called *transcendental* if it does not extend continuously to $\hat{\mathbb{C}}$. (I.e., f has an essential singularity at ∞.)
- (Otherwise, it is a polynomial or a rational function.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic notation

- $\mathbb{C}$ is the complex plane.
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- A function $f : U \to \hat{\mathbb{C}}$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\mathbb{C}$. (I.e., f is holomorphic outside of a discrete set of poles.)
- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.
- A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (entire or meromorphic) is called *transcendental* if it does not extend continuously to $\hat{\mathbb{C}}$. (I.e., f has an essential singularity at ∞.)
- (Otherwise, it is a polynomial or a rational function.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic notation

- $\mathbb{C}$ is the complex plane.
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- A function $f : U \to \hat{\mathbb{C}}$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\mathbb{C}$. (I.e., f is holomorphic outside of a discrete set of poles.)
- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.
- A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (entire or meromorphic) is called *transcendental* if it does not extend continuously to $\hat{\mathbb{C}}$. (I.e., f has an essential singularity at ∞.)
- (Otherwise, it is a polynomial or a rational function.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Holomorphic dynamics

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Suppose also that f is nonconstant and nonlinear.

For simplicity, we will assume that f is either a polynomial or transcendental. (Don't consider rational functions.)

That is, we have the following cases:

- Polynomials $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto z^2$;
- Entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$; e.g. $z \mapsto \exp(z)$;
- Transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$; e.g. $z \mapsto \tan(z)$.

We want to study f as a dynamical system. That is, we are interested in the behavior of

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic objects

$$f : \mathbb{C} \to \hat{\mathbb{C}}$$

- **Fatou set** $F(f)$: set of points near which the iterates have "stable" behavior.
- **Julia set** $J(f)$: complement of the Fatou set; locus of "chaotic" behavior.
- **Escaping set**

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic objects

$$f : \mathbb{C} \to \hat{\mathbb{C}}$$

- **Fatou set** $F(f)$: set of points near which the iterates have "stable" behavior.
- **Julia set** $J(f)$: complement of the Fatou set; locus of "chaotic" behavior.
- **Escaping set**

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Basic objects

$$f : \mathbb{C} \to \hat{\mathbb{C}}$$

- **Fatou set** $F(f)$: set of points near which the iterates have "stable" behavior.
- **Julia set** $J(f)$: complement of the Fatou set; locus of "chaotic" behavior.
- **Escaping set**

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

A simple example

For example, when $f(z) = z^2$: points with $|z| < 1$ will converge to a stable equilibrium at zero under iteration. Points with $|z| > 1$ will converge to ∞.

On the unit circle, there are points nearby which converge to 0 and points which converge to ∞, so the behavior is *not* stable there. So we should have:

$$J(f) = \{|z| = 1\};$$
$$F(f) = \mathbb{C} \setminus J(f) = \{|z| \neq 1\};$$
$$I(f) = \{|z| > 1\}.$$

Note: on $J(f)$, the map acts by angle doubling.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

A simple example

For example, when $f(z) = z^2$: points with $|z| < 1$ will converge to a stable equilibrium at zero under iteration. Points with $|z| > 1$ will converge to ∞.

On the unit circle, there are points nearby which converge to 0 and points which converge to ∞, so the behavior is *not* stable there. So we should have:

$$J(f) = \{|z| = 1\};$$
$$F(f) = \mathbb{C} \setminus J(f) = \{|z| \neq 1\};$$
$$I(f) = \{|z| > 1\}.$$

Note: on $J(f)$, the map acts by angle doubling.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

A simple example

For example, when $f(z) = z^2$: points with $|z| < 1$ will converge to a stable equilibrium at zero under iteration. Points with $|z| > 1$ will converge to ∞.

On the unit circle, there are points nearby which converge to 0 and points which converge to ∞, so the behavior is *not* stable there. So we should have:

$$J(f) = \{|z| = 1\};$$

$$F(f) = \mathbb{C} \setminus J(f) = \{|z| \neq 1\};$$

$$I(f) = \{|z| > 1\}.$$

Note: on $J(f)$, the map acts by angle doubling.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

A simple example

For example, when $f(z) = z^2$: points with $|z| < 1$ will converge to a stable equilibrium at zero under iteration. Points with $|z| > 1$ will converge to ∞.

On the unit circle, there are points nearby which converge to 0 and points which converge to ∞, so the behavior is *not* stable there. So we should have:

$$J(f) = \{|z| = 1\};$$
$$F(f) = \mathbb{C} \setminus J(f) = \{|z| \neq 1\};$$
$$I(f) = \{|z| > 1\}.$$

Note: on $J(f)$, the map acts by angle doubling.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

A simple example

For example, when $f(z) = z^2$: points with $|z| < 1$ will converge to a stable equilibrium at zero under iteration. Points with $|z| > 1$ will converge to ∞.

On the unit circle, there are points nearby which converge to 0 and points which converge to ∞, so the behavior is *not* stable there. So we should have:

$$J(f) = \{|z| = 1\};$$
$$F(f) = \mathbb{C} \setminus J(f) = \{|z| \neq 1\};$$
$$I(f) = \{|z| > 1\}.$$

Note: on $J(f)$, the map acts by angle doubling.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

A simple example

For example, when $f(z) = z^2$: points with $|z| < 1$ will converge to a stable equilibrium at zero under iteration. Points with $|z| > 1$ will converge to ∞.

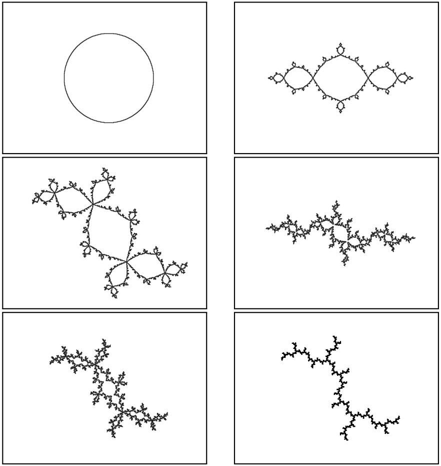
On the unit circle, there are points nearby which converge to 0 and points which converge to ∞, so the behavior is *not* stable there. So we should have:

$$J(f) = \{|z| = 1\};$$
$$F(f) = \mathbb{C} \setminus J(f) = \{|z| \neq 1\};$$
$$I(f) = \{|z| > 1\}.$$

Note: on $J(f)$, the map acts by angle doubling.

Some Julia sets

$$z \mapsto z^2 + c$$

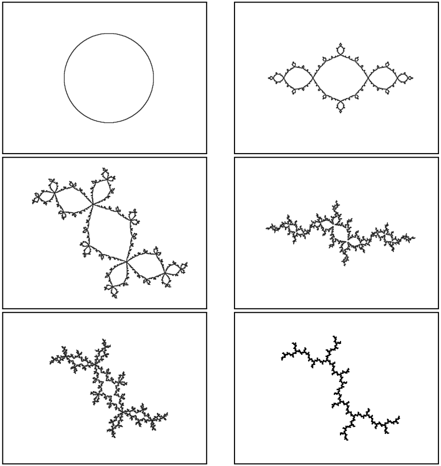


- The Julia set J is compact.
- The escaping set is a completely invariant component of the Fatou set (called the *basin of infinity*).

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

$$z \mapsto z^2 + c$$

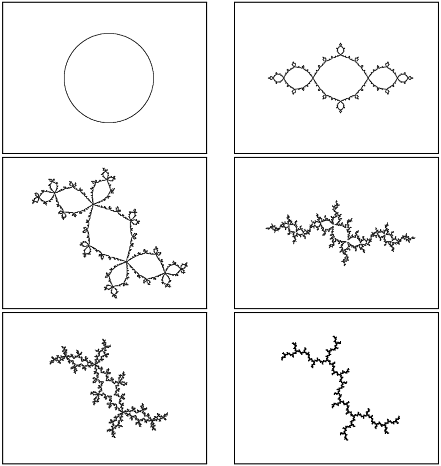


- The Julia set J is compact.
- The escaping set is a completely invariant component of the Fatou set (called the *basin of infinity*).

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

$$z \mapsto z^2 + c$$

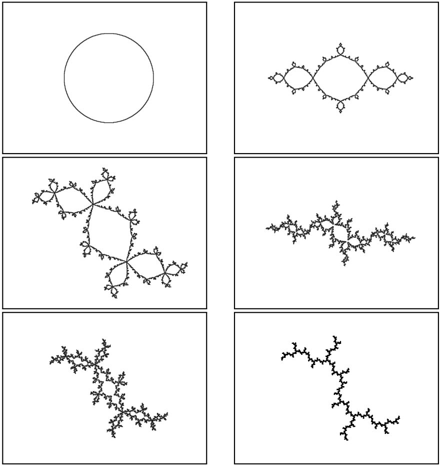


- The Julia set J is compact.
- The escaping set is a completely invariant component of the Fatou set (called the *basin of infinity*).

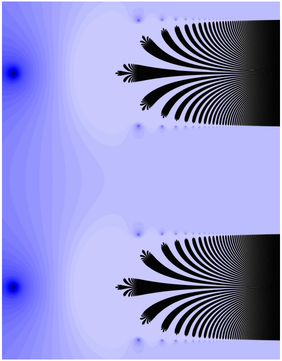
Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

$z \mapsto z^2 + c$



$z \mapsto \exp(z) - 2$

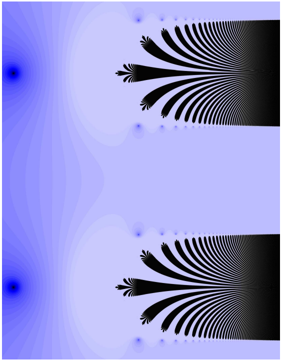


Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

- J is an uncountable union of Jordan arcs $g : [0, \infty) \to \mathbb{C}$ with $g(t) \to \infty$. We call $g((0, \infty))$ a *ray* and $g(0)$ the *endpoint* of that ray.
- The set E of all endpoints has Hausdorff dimension 2, but the union R of all rays has Hausdorff dimension 1.
- E is totally disconnected, but $E \cup \{\infty\}$ is connected.

$z \mapsto \exp(z) - 2$

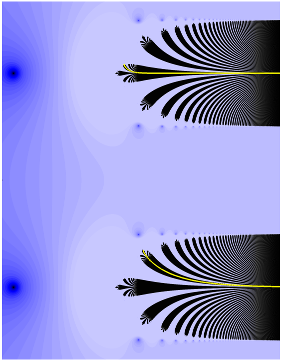


Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

- J is an uncountable union of Jordan arcs $g : [0, \infty) \to \mathbb{C}$ with $g(t) \to \infty$. We call $g((0, \infty))$ a *ray* and $g(0)$ the *endpoint* of that ray.
- The set E of all endpoints has Hausdorff dimension 2, but the union R of all rays has Hausdorff dimension 1.
- E is totally disconnected, but $E \cup \{\infty\}$ is connected.

$z \mapsto \exp(z) - 2$

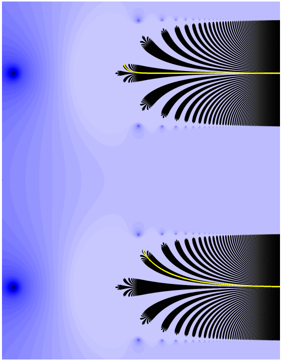


Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

- J is an uncountable union of Jordan arcs $g : [0, \infty) \to \mathbb{C}$ with $g(t) \to \infty$. We call $g((0, \infty))$ a *ray* and $g(0)$ the *endpoint* of that ray.
- The set E of all endpoints has Hausdorff dimension 2, but the union R of all rays has Hausdorff dimension 1.
- E is totally disconnected, but $E \cup \{\infty\}$ is connected.

$z \mapsto \exp(z) - 2$

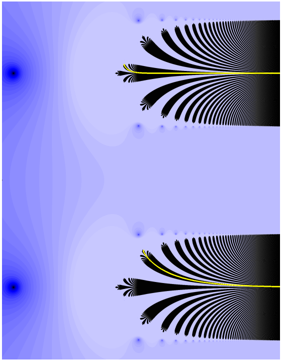


Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

- J is an uncountable union of Jordan arcs $g : [0, \infty) \to \mathbb{C}$ with $g(t) \to \infty$. We call $g((0, \infty))$ a *ray* and $g(0)$ the *endpoint* of that ray.
- The set E of all endpoints has Hausdorff dimension 2, but the union R of all rays has Hausdorff dimension 1.
- E is totally disconnected, but $E \cup \{\infty\}$ is connected.

$z \mapsto \exp(z) - 2$

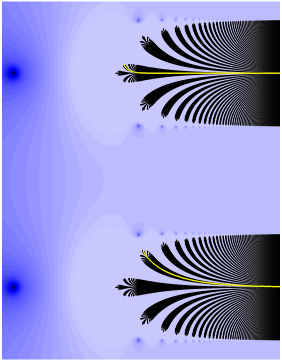


Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

- The set R of rays is completely contained in the escaping set $I(f)$.
- Some endpoints belong to $I(f)$; others do not.

$z \mapsto \exp(z) - 2$

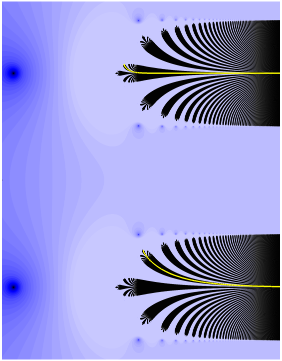


Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some Julia sets

- The set R of rays is completely contained in the escaping set $I(f)$.
- Some endpoints belong to $I(f)$; others do not.

$z \mapsto \exp(z) - 2$



Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Locally uniform convergence

Let f_n be a family of holomorphic (or meromorphic) functions defined on some open set U.

Recall that we say that (f_n) converges *locally uniformly* to a function f if the sequence converges uniformly on every compact subset of U.

(For example, the sequence $f_n(z) = z/n$ converges locally uniformly to $f(z) = 0$ on $\mathbb{C}$.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Locally uniform convergence

Let f_n be a family of holomorphic (or meromorphic) functions defined on some open set U.

Recall that we say that (f_n) converges *locally uniformly* to a function f if the sequence converges uniformly on every compact subset of U.

(For example, the sequence $f_n(z) = z/n$ converges locally uniformly to $f(z) = 0$ on $\mathbb{C}$.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Locally uniform convergence

Let f_n be a family of holomorphic (or meromorphic) functions defined on some open set U.

Recall that we say that (f_n) converges *locally uniformly* to a function f if the sequence converges uniformly on every compact subset of U.

(For example, the sequence $f_n(z) = z/n$ converges locally uniformly to $f(z) = 0$ on $\mathbb{C}$.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Normality

A family $\mathcal{F}$ of holomorphic or meromorphic functions on U is *normal* (on U) if every sequence of functions in $\mathcal{F}$ contains a locally uniformly convergent subsequence.

We say that $\mathcal{F}$ is normal *in a point* z if z has an open neighborhood on which $\mathcal{F}$ is normal.

Arzela-Ascoli: Normality is a local property.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Normality

A family $\mathcal{F}$ of holomorphic or meromorphic functions on U is *normal* (on U) if every sequence of functions in $\mathcal{F}$ contains a locally uniformly convergent subsequence.

We say that $\mathcal{F}$ is normal *in a point* z if z has an open neighborhood on which $\mathcal{F}$ is normal.

Arzela-Ascoli: Normality is a local property.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Normality

A family $\mathcal{F}$ of holomorphic or meromorphic functions on U is *normal* (on U) if every sequence of functions in $\mathcal{F}$ contains a locally uniformly convergent subsequence.

We say that $\mathcal{F}$ is normal *in a point z* if z has an open neighborhood on which $\mathcal{F}$ is normal.

Arzela-Ascoli: Normality is a local property.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some facts about normality

Theorem (Riemann)

Suppose f is holomorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f is bounded near z_0, then z_0 is a removable singularity.

Theorem (Liouville)

Any bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Theorem (Montel)

Any uniformly bounded family of holomorphic functions is normal.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Some facts about normality

Theorem (Riemann)

Suppose f is holomorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f is bounded near z_0, then z_0 is a removable singularity.

Theorem (Liouville)

Any bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Theorem (Montel)

Any uniformly bounded family of holomorphic functions is normal.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Some facts about normality

Theorem (Riemann)

Suppose f is holomorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f is bounded near z_0, then z_0 is a removable singularity.

Theorem (Liouville)

Any bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Theorem (Montel)

Any uniformly bounded family of holomorphic functions is normal.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Some facts about normality

Theorem (Picard)

Suppose f is meromorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f omits three values in the Riemann sphere (e.g., f never takes the values 0, 1 and ∞), then z_0 is a removable singularity.

Theorem (Picard)

Any meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ which omits three values is constant.

Theorem (Montel)

A family of meromorphic functions which all omit the same three values is normal.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Some facts about normality

Theorem (Picard)

Suppose f is meromorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f omits three values in the Riemann sphere (e.g., f never takes the values 0, 1 and ∞), then z_0 is a removable singularity.

Theorem (Picard)

Any meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ which omits three values is constant.

Theorem (Montel)

A family of meromorphic functions which all omit the same three values is normal.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Some facts about normality

Theorem (Picard)

Suppose f is meromorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f omits three values in the Riemann sphere (e.g., f never takes the values 0, 1 and ∞), then z_0 is a removable singularity.

Theorem (Picard)

Any meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ which omits three values is constant.

Theorem (Montel)

A family of meromorphic functions which all omit the same three values is normal.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Definition of Fatou and Julia sets

- $F(f) := \{z \in \mathbb{C} : (f^n)$ is defined and normal near $z\}$;
- $J(f) := \mathbb{C} \setminus F(f)$.

Definition of Fatou and Julia sets

- $F(f) := \{z \in \mathbb{C} : (f^n) \text{ is defined and normal near } z\}$;
- $J(f) := \mathbb{C} \setminus F(f)$.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Definition of Fatou and Julia sets

- $F(f) := \{z \in \mathbb{C} : (f^n)$ is defined and normal near $z\}$;
- $J(f) := \mathbb{C} \setminus F(f)$.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Periodic points

- $z \in \mathbb{C}$ is *periodic* if $f^n(z) = z$.
- A periodic point is *attracting* if $|(f^n)'(z)| < 1$.
  (Attracting points are in the Fatou set.)
- A periodic point is *repelling* if $|(f^n)'(z)| > 1$.
  (Repelling points are in the Julia set.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Periodic points

- $z \in \mathbb{C}$ is *periodic* if $f^n(z) = z$.
- A periodic point is *attracting* if $|(f^n)'(z)| < 1$.
  (Attracting points are in the Fatou set.)
- A periodic point is *repelling* if $|(f^n)'(z)| > 1$.
  (Repelling points are in the Julia set.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Periodic points

- $z \in \mathbb{C}$ is *periodic* if $f^n(z) = z$.
- A periodic point is *attracting* if $|(f^n)'(z)| < 1$. (Attracting points are in the Fatou set.)
- A periodic point is *repelling* if $|(f^n)'(z)| > 1$. (Repelling points are in the Julia set.)

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Periodic points

- $z \in \mathbb{C}$ is *periodic* if $f^n(z) = z$.
- A periodic point is *attracting* if $|(f^n)'(z)| < 1$. (Attracting points are in the Fatou set.)
- A periodic point is *repelling* if $|(f^n)'(z)| > 1$. (Repelling points are in the Julia set.)

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Periodic points

- $z \in \mathbb{C}$ is *periodic* if $f^n(z) = z$.
- A periodic point is *attracting* if $|(f^n)'(z)| < 1$. (Attracting points are in the Fatou set.)
- A periodic point is *repelling* if $|(f^n)'(z)| > 1$. (Repelling points are in the Julia set.)

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Fatou components

Components of the Fatou set have several possible types:

- Attracting basins (possibly at ∞, for polynomials);
- parabolic basins;
- *Siegel disks:* simply connected domains on which f^k is conjugate to an irrational rotation;
- *Herman rings:* doubly connected domains on which f^k is conjugate to an irrational rotation (not possible for polynomials and entire functions);
- *Baker domains:* domains on which the iterates tend to an essential singularity (not possible for polynomials and rational functions);
- a preimage component of a domain of one of these types;
- *Wandering domains:* not possible for polynomials and rational functions.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Examples of Baker and wandering domains

$$f(z) = z + 1 + \exp(-z).$$

$$f(z) = z + \sin(2\pi z)$$

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Examples of Baker and wandering domains

$$f(z) = z + 1 + \exp(-z).$$

$$f(z) = z + \sin(2\pi z)$$

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Properties of the Julia set

- $J(f)$ **is nonempty.** (In fact, $J(f)$ is infinite).
- If $z \in \hat{\mathbb{C}}$ (with at most two exceptions), then the set

$$\{w \in \mathbb{C} : f^k(w) = z \text{ for some } k\}$$

  accumulates on the whole Julia set.
- $J(f)$ has no isolated points.
- $J(f)$ is uncountable.
- $J(f)$ is the closure of the set of repelling periodic points.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Properties of the Julia set

- $J(f)$ **is nonempty.** (In fact, $J(f)$ is infinite).
- If $z \in \hat{\mathbb{C}}$ (with at most two exceptions), then the set

$$\{w \in \mathbb{C} : f^k(w) = z \text{ for some } k\}$$

accumulates on the whole Julia set.
- $J(f)$ has no isolated points.
- $J(f)$ is uncountable.
- $J(f)$ is the closure of the set of repelling periodic points.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Properties of the Julia set

- $J(f)$ **is nonempty.** (In fact, $J(f)$ is infinite).
- If $z \in \hat{\mathbb{C}}$ (with at most two exceptions), then the set

  $$\{w \in \mathbb{C} : f^k(w) = z \text{ for some } k\}$$

  accumulates on the whole Julia set.
- $J(f)$ has no isolated points.
- $J(f)$ is uncountable.
- $J(f)$ is the closure of the set of repelling periodic points.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Properties of the Julia set

- $J(f)$ **is nonempty.** (In fact, $J(f)$ is infinite).
- If $z \in \hat{\mathbb{C}}$ (with at most two exceptions), then the set

$$\{w \in \mathbb{C} : f^k(w) = z \text{ for some } k\}$$

  accumulates on the whole Julia set.
- $J(f)$ has no isolated points.
- $J(f)$ is uncountable.
- $J(f)$ is the closure of the set of repelling periodic points.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Properties of the Julia set

- $J(f)$ **is nonempty.** (In fact, $J(f)$ is infinite).
- If $z \in \hat{\mathbb{C}}$ (with at most two exceptions), then the set

$$\{w \in \mathbb{C} : f^k(w) = z \text{ for some } k\}$$

  accumulates on the whole Julia set.
- $J(f)$ has no isolated points.
- $J(f)$ is uncountable.
- $J(f)$ is the closure of the set of repelling periodic points.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

$J(f) \neq \emptyset$

Theorem

$J(f)$ contains infinitely many points.

The most difficult case is that of $f : \mathbb{C} \to \mathbb{C}$ entire and transcendental.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

$J(f) \neq \emptyset$

Theorem

$J(f)$ contains infinitely many points.

The most difficult case is that of $f : \mathbb{C} \to \mathbb{C}$ entire and transcendental.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

$J(f) \neq \emptyset$

Theorem

$J(f)$ contains infinitely many points.

The most difficult case is that of $f : \mathbb{C} \to \mathbb{C}$ entire and transcendental.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Existence of periodic points

Lemma

If f has infinitely many periodic points, then $J(f)$ is infinite.

Lemma

f^2 has a fixed point.

Proof: Apply Picard's theorem to $z \mapsto \dfrac{f^2(z) - z}{f(z) - z}$.

Lemma

Suppose f has a fixed point at 0. If f only has finitely many fixed points, then f has infinitely many zeros.

Proof: Apply Picard's theorem to $f(z)/z$.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Existence of periodic points

Lemma

If f has infinitely many periodic points, then $J(f)$ is infinite.

Lemma

f^2 has a fixed point.

Proof: Apply Picard's theorem to $z \mapsto \dfrac{f^2(z) - z}{f(z) - z}$.

Lemma

Suppose f has a fixed point at 0. If f only has finitely many fixed points, then f has infinitely many zeros.

Proof: Apply Picard's theorem to $f(z)/z$.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Existence of periodic points

Lemma

If f has infinitely many periodic points, then $J(f)$ is infinite.

Lemma

f^2 has a fixed point.

Proof: Apply Picard's theorem to $z \mapsto \dfrac{f^2(z) - z}{f(z) - z}$.

Lemma

Suppose f has a fixed point at 0. If f only has finitely many fixed points, then f has infinitely many zeros.

Proof: Apply Picard's theorem to $f(z)/z$.

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Existence of periodic points

Lemma

If f has infinitely many periodic points, then $J(f)$ is infinite.

Lemma

f^2 has a fixed point.

Proof: Apply Picard's theorem to $z \mapsto \dfrac{f^2(z) - z}{f(z) - z}$.

Lemma

Suppose f has a fixed point at 0. If f only has finitely many fixed points, then f has infinitely many zeros.

Proof: Apply Picard's theorem to $f(z)/z$.

Holomorphic Dynamics

L. Rempe

Introduction
Basic notation
Basic concepts
Some examples

Normal families
Definition
Theorems of Picard and Montel

Definition of Fatou and Julia sets

Basic Properties
Periodic points
Fatou components
Properties of the Julia set

$J(f) \neq \emptyset$
Periodic points
Escaping points

Existence of periodic points

Lemma

If f has infinitely many periodic points, then $J(f)$ is infinite.

Lemma

f^2 has a fixed point.

Proof: Apply Picard's theorem to $z \mapsto \dfrac{f^2(z) - z}{f(z) - z}$.

Lemma

Suppose f has a fixed point at 0. If f only has finitely many fixed points, then f has infinitely many zeros.

Proof: Apply Picard's theorem to $f(z)/z$.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

Theorem (Eremenko)

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and transcendental. Then the escaping set $I(f)$ is nonempty.

(In fact, Eremenko even proves $J(f) \cap I(f) \neq \emptyset$. However, our proof will not yield this directly.)

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

A consequence of Bohr's theorem

Theorem (Bohr)

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and transcendental. Let R be sufficiently large, and let

$$D := \{z \in \mathbb{C} : |z| < R\}.$$

Then $f(D)$ contains a circle $\{|z| = \tilde{R}\}$ of radius $\tilde{R} \geq 2R$.

(As an example, consider $f(z) = \exp(z)$.)

We will prove this theorem next time, using a normal family argument.

A consequence of Bohr's theorem

Theorem (Bohr)

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and transcendental. Let R be sufficiently large, and let

$$D := \{z \in \mathbb{C} : |z| < R\}.$$

Then $f(D)$ contains a circle $\{|z| = \tilde{R}\}$ of radius $\tilde{R} \geq 2R$.

(As an example, consider $f(z) = \exp(z)$.)

We will prove this theorem next time, using a normal family argument.

Introduction
Basic notation
Basic concepts
Some examples
Normal families
Definition
Theorems of Picard and Montel
Definition of Fatou and Julia sets
Basic Properties
Periodic points
Fatou components
Properties of the Julia set
$J(f) \neq \emptyset$
Periodic points
Escaping points

A consequence of Bohr's theorem

Theorem (Bohr)

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and transcendental. Let R be sufficiently large, and let

$$D := \{z \in \mathbb{C} : |z| < R\}.$$

Then $f(D)$ contains a circle $\{|z| = \widetilde{R}\}$ of radius $\widetilde{R} \geq 2R$.

(As an example, consider $f(z) = \exp(z)$.)

We will prove this theorem next time, using a normal family argument.