DESCRIPTION OF MY RESEARCH UNTIL MAY 2008.

OLEG KARPENKOV

Contents

In this small paper we give an outline of results obtained by the author in lattice geometry (including theory of multidimensional continued fractions) and theory of energies functionals of knots and graphs.

1. Multidimensional continued fractions

1.1. **Introduction.** The problem of generalizing ordinary continued fractions to the higher-dimensional case was posed by C. Hermite [17] in 1839. A large number of attempts to solve this problem lead to the birth of several different remarkable theories of multidimensional continued fractions. We consider the geometrical generalization of ordinary continued fractions to the multidimensional case presented by F. Klein in 1895 and published by him in [30] and [31].

Consider a set of $n+1$ hyperplanes of $\mathbb{R}^{n+1}$ passing through the origin in general position. The complement to the union of these hyperplanes consists of 2^{n+1} open orthants. Let us choose an arbitrary orthant.

Definition 1.1. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the *sail*. The set of all 2^{n+1} sails of the space $\mathbb{R}^{n+1}$ is called the *n-dimensional continued fraction* associated to the given $n+1$ hyperplanes in general position in $(n+1)$-dimensional space.

Two n-dimensional continued fractions are said to be *equivalent* if there exists a linear transformation that preserves the integer lattice of the $(n+1)$-dimensional space and maps the sails of the first continued fraction to the sails of the other.

Multidimensional continued fractions in the sense of Klein have many connections with other branches of mathematics. For example, J.-O. Moussafir [40] and O. N. German [16] studied the connection between the sails of multidimensional continued fractions and Hilbert bases. In [49] H. Tsuchihashi found the relationship between periodic multidimensional continued fractions and multidimensional cusp singularities, which generalizes the relationship between ordinary continued fractions and two-dimensional cusp singularities. M. L. Kontsevich and Yu. M. Suhov discussed the statistical properties of the boundary of a random multidimensional continued fraction in [32]. The combinatorial topological generalization of Lagrange theorem was obtained by E. I. Korkina in [34] and its algebraic generalization by G. Lachaud [37].

V. I. Arnold presented a survey of geometrical problems and theorems associated with one-dimensional and multidimensional continued fractions in his article [6] and his book [3]). For the algorithms of constructing multidimensional continued fractions, see the papers of R. Okazaki [42], J.-O. Moussafir [41] and the author [22].

E. Korkina in [33] and [35] and G. Lachaud in [37], [38], A. D. Bruno and V. I. Parusnikov in [11], [44], and [45], the author in [20] and [21] produced a large number of fundamental domains for periodic algebraic two-dimensional continued fractions. A nice collection of two-dimensional continued fractions is given in the work [10] by K. Briggs.

1.2. **Polygonal faces of continued fractions.** It follows from the definition that multidimensional continued fractions are polyhedral (finite or infinite) surfaces. Here arises a question of *a good face description for such polyhedral surface*. One-dimensional faces, i.e. segments, are integer-linear equivalent iff the number of their integer inner points coincide. So the one-dimensional faces are in a natural one-to-one correspondence with positive integers.

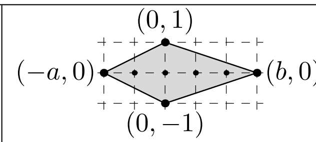
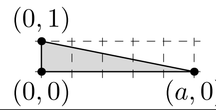
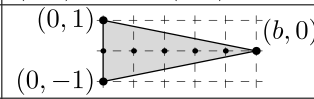
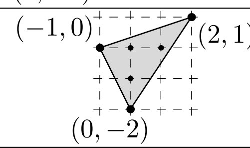
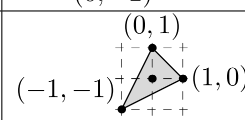
For the two-dimensional faces it was known that any convex polygon is a face of some continued fraction on unit integer distance to the origin. I have found the list of all convex polygons that can be on integer distance to the origin greater than one.

By $(a_1, \ldots, a_k)$ in $\mathbb{R}^m$ for $k < m$ we denote the point $(a_1, \ldots, a_k, 0, \ldots, 0)$. It turns out that it is possible to implicitly describe all integer-affine classes of multistory completely empty convex three-dimensional marked pyramids.

Theorem 1.2. [24] *Any multistory completely empty convex three-dimensional marked pyramid is integer-affine equivalent exactly to one of the marked pyramids from the list "M-W".*

It was known only the following statement on compact two-dimensional faces contained in planes on the integer distance to the origin greater than one. *Such faces are either triangles or quadrangles* (see the work [3] by J.-O. Moussafir).

1.3. **Gauss-Kuzmin face distribution and Möbius measure.** For the first time the statement on statistics of numbers as elements of ordinary continued fractions was formulated by K. F. Gauss in his letters to P. S. Laplace (see in [15]). This statement was proven further by R. O. Kuzmin [36], and further was proven one more time by P. Lévy [39]. Further investigations in this direction were made by E. Wirsing in [50]. (A basic notions of theory of ordinary continued fractions is described in the books [18] by A. Ya. Hinchin and [3] by V. I. Arnold.) In 1989 V. I. Arnold generalized statistical problems to the case of one-dimensional and multidimensional continued fractions in the sense of Klein, see in [5] and [4].

The list **"M-W"**	Parameters	Coords. of the vertex	Coordinates of the base	Integer-affine type of the base
$M_{a,b}$	$b{\geq}a{\geq}1$	$(0,0,0)$	$(2,-1,0)$, $(2,-a{-}1,1)$, $(2,-1,2)$, $(2,b{-}1,1)$	
$T^{\xi}_{a,r}$	$a{\geq}1$, $r{\geq}2$, $0{<}\xi{\leq}r/2$, $\gcd(\xi,r){=}1$	$(0,0,0)$	$(\xi,r{-}1,-r)$, $(a{+}\xi,r{-}1,-r)$, $(\xi,r,-r)$	
U_b	$b{\geq}2$	$(0,0,0)$	$(2,1,b{-}1)$, $(2,2,-1)$, $(2,0,-1)$	
V		$(0,0,0)$	$(2,-2,1)$, $(2,-1,-1)$, $(2,1,2)$	
W		$(0,0,0)$	$(3,0,2)$, $(3,1,1)$, $(3,2,3)$	

One-dimensional case was studied in details by M. O. Avdeeva and B. A. Bykovskii in the works [1] and [2]. In two-dimensional and multidimensional cases V. I. Arnold formulated many problems on statistics of sail characteristics of multidimensional continued fractions such as an amount of triangular, quadrangular faces and so on, such as their integer areas, and length of edges, etc. A major part of these problems is open nowadays, while some are almost completely solved.

M. L. Kontsevich and Yu. M. Suhov in their work [32] proved the existence of the mentioned above statistics. I wrote explicitly a natural Möbius measure of the manifold of all n-dimensional continued fractions in the sense of Klein and introduced the corresponding integral formulae for the statistics (see [27] for more details).

1.4. **Algorithmic aspects for algebraic multidimensional continued fractions.** A multidimensional periodic algebraic continued fraction is a set of infinite polyhedral sails, that contain an infinite number of faces. The quotient of any sail under the Dirichlet group action is isomorphic to an n-dimensional torus. The algebraic periodicity of the polyhedron allows to reconstruct the whole continued fraction knowing only the fundamental domain. Moreover, any fundamental domain contains only a finite number of faces of the whole algebraic periodic continued fraction. Hence we are faced with the problem of finding a good algorithm that enumerates all the faces for this domain.

There were no algorithm for constructing multidimensional continued fractions until T. Shintani's work [46] in 1976. Let F be a totally real algebraic field of degree n. We take all different embeddings of F into $\mathbb{R}$ and denote them by φ_i, $i=1,\ldots,n$ (there are exactly n different embeddings, since F is totally real). Consider the following embedding of F into $\mathbb{R}^n$. For an arbitrary element x of F we suppose

$$x \to (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)).$$

T. Shintani considered the action of the group of all totally positive elements for the ring of integers of F (by component-wise multiplication by totally positive integers x_+) on $\mathbb{R}^n_+$ for the described embedding of F. He proved that the fundamental domain for this action is the union of a finite number of simplicial cones of special type. (Note that if we take some other order for the embeddings $\varphi_{i'}$, then the fundamental domains will be integer-linear equivalent to the fundamental domains for the embeddings considered above.) The statement of T. Shintani and its proof is actually the basis for the construction of one-dimensional continued fractions. Following T. Shintani's work, E. Thomas and A. T. Vasques obtained several fundamental domains for the two-dimensional case in [48]. Finally, R. Okazaki presented a method that permits to construct fundamental domains for fields of arbitrary degree in his article [42]. E. Korkina in [33], [35] and G. Lachaud in [37] produced an infinite number of fundamental domains for periodic algebraic two-dimensional continued fractions. The method used for constructing fundamental domains of multidimensional continued fractions in these papers was inductive. The method produces the fundamental domain face by face, verifying that each new face does not lie in the same orbit with some face constructed before. Applying the method, one can find the fundamental domain in finitely many steps.

Later on J. O. Moussafir developed an essentially different approach in his work [41]. It works for an arbitrary (not necessary periodic) continued fraction and computes any bounded part of an infinite polyhedron. The approach is based on deduction. One produces a conjecture on the face structure for a big part of the continued fraction, then it remains to prove that any conjectured face is indeed a face of the part. This method can be also applied to the case of periodic continued fractions.

Finally, we described a new advanced deductive construction adapted especially to fundamental domains of periodic continued fractions. The construction involves a method for conjecturing the structure of the fundamental domain and an algorithm testing whether the conjectured domain is indeed fundamental. The main advantage of the algorithm is the following: the number of "false" vertices of our approximation is much smaller than the number of "false" vertices of the approximation in the method of J. O. Moussafir (so that the computational time is considerable reduced). This algorithm substantially uses the periodicity of the continued fraction and hence it is impossible to apply it to non-periodic continued fractions.

For the two-dimensional case I have proven the following statement.

Suppose we have a conjecture on the structure of the fundamental domain for some sail of a two-dimensional periodic continued fraction. Let this domain contain N faces of all dimensions. The test of the conjecture (the algorithm) requires no more than CN^4 additions, multiplications and comparisons of two integers, where C is a universal constant that does not depend on N.

All previous verification algorithms work exponential time with respect to N.

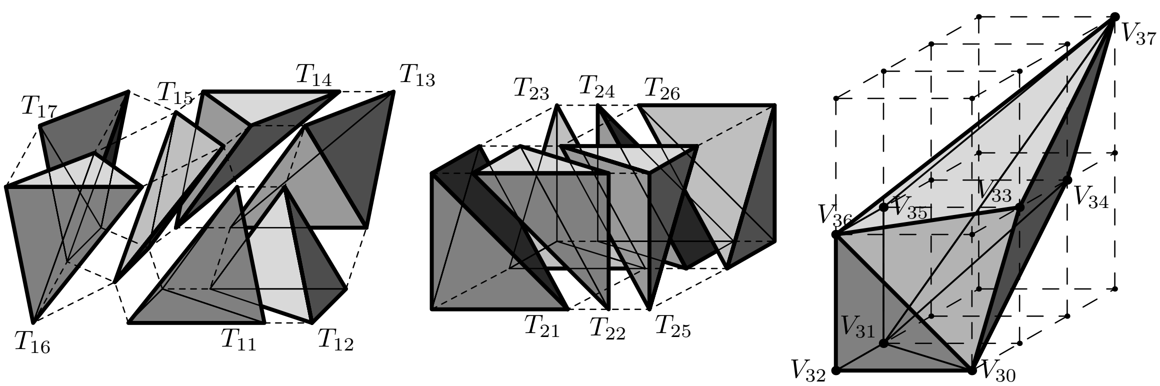
Using the present algorithm, the author both generalized almost all known simple examples and series of examples of fundamental domains constructed before, and found a lot of new examples and series (see [20] and [21]). Using these examples, the author found the complete list of all two-dimensional periodic continued fractions constructed by matrices of small norm ($|*| \le 6$) up to the integer-linear equivalence relation, see [21]. By the norm of a matrix, here we mean the sum of the absolute values of all its coefficients.

1.5. **Simplest continued fractions.** The problem of investigation of the simplest n-dimensional continued fraction for $n \geq 2$ was posed by V. Arnold. The answer for the case of $n = 2$ can be found in the works of E. Korkina [35] and G. Lachaud [38]. I have studied the case of $n = 3$ in [23]. I constructed three examples of three-dimensional continued fractions that for many reasons (such as additional symmetries, simplicity of the fundamental domains, characteristic polynomials of special types) seems to be the simplest examples tree-dimensional continued fractions.

Denote by $A_{a,b,c,d}$ the following integer operator

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

On Figure 1 we show one of the fundamental domains for each of the operators $A_{1,-3,0,4}$, $A_{1,-4,1,4}$, and $A_{-1,-3,1,3}$. We indicate with dotted lines how to glue the faces to obtain the combinatorial scheme of the described fundamental domains.



$A_{1,-3,0,4}$ $A_{1,-4,1,4}$ $A_{-1,-3,1,3}$

Figure 1. Three examples of three-dimensional continued fractions.

2. Lattice-regular polygons and polytopes

The second aim of my study in lattice geometry was to investigate symmetric convex polytopes in all dimensions (see [25]).

The study of convex lattice polytopes is actual in lattice geometry (see, for example [8], [9], [19], [47]), in geometry of toric varieties (see [14], [28], [43]) and multidimensional continued fractions (see [3], [35], [24], [38], [41]).

Consider an n-dimensional real vector space. Let us fix a full-rank lattice in it. A *convex polytope* is a convex hull of a finite number of points. A hyperplane π is said to be *supporting* for a (closed) convex polytope P, if the intersections of P and π is not empty, and the whole polytope P is contained in one of the closed half-spaces bounded by π. An intersection of any polytope P with any its supporting hyperplane is called a *face* of the polytope. Zero- and one-dimensional faces are called *vertices* and *faces*.

Consider an arbitrary n-dimensional convex polytope P. An arbitrary unordered $(n{+}1)$-tuple of faces containing the whole polytope P, some its hyperface, some hyperface of this hyperface, and so on (up to a vertex of P) is called a *face-flag* for the polytope P.

A convex polytope is said to be lattice if all its vertices are lattice points. A lattice polytope is called *lattice-regular* if for any two its face-flags there exist a lattice-affine transformation preserving the polytope and taking one face-flag to the other.

Definitions and formulation of the main result of my paper [25]**.** Let us fix some basis of lattice vectors $\overline{e}_i$ for $i=1,\ldots,n$ generating the lattice in $\mathbb{R}^n$. Denote by O the origin in $\mathbb{R}^n$.

Consider arbitrary non-zero integers $n_1,\ldots,n_k$ for $k\ge 2$. By $\gcd(n_1,\ldots,n_k)$ we denote the greatest common divisor of the integers n_i, where $i=1,\ldots,k$. We write that $a\equiv b(\operatorname{mod} c)$ if the reminders of a and b modulo c coincide.

Let Q be an arbitrary lattice polytope with the vertices $A_i=O+\overline{v}_i$ (where $\overline{v}_i$ — lattice vectors) for $i=1,\ldots,m$, and t be an arbitrary positive integer. The polygon P with the vertices $B_i=O+t\overline{v}_i$ for $i=1,\ldots,m$ is said to be the *t-multiple* of the polygon Q.

Definition 2.1. A lattice polytope P is said to be *elementary* if for any integer $t>1$ and any lattice polytope Q the polytope P is not lattice-congruent to the t-multiple of the lattice polytope Q.

We will use the following notation.

Symplices. For any $n>1$ we denote by $\{3^{n-1}\}_p^L$ the n-dimensional symplex with the vertices:

$$V_0=O, \qquad V_i=O{+}\overline{e}_i, \text{ for } i=1,\ldots,n{-}1, \quad \text{and} \quad V_n=(p{-}1)\sum_{k=1}^{n-1}\overline{e}_k+p\overline{e}_n.$$

Cubes. Any lattice cube is generated by some lattice point P and a n-tuple of linearly independent lattice vectors $\overline{v}_i$:

$$\Big\{P+\sum_{i=1}^{n}\alpha_i\overline{v}_i\Big|0\le\alpha_i\le 1, i=1,\ldots,n\Big\}$$

We denote by $\{4,3^{n-2}\}_1^L$ for any $n\ge 2$ the lattice cube with a vertex at the origin and generated by all basis vectors.

By $\{4,3^{n-2}\}_2^L$ for any $n\ge 2$ we denote the lattice cube with a vertex at the origin and generated by the first $n{-}1$ basis vectors and the vector $\overline{e}_1+\overline{e}_2+\ldots+\overline{e}_{n-1}+2\overline{e}_n$.

By $\{4,3^{n-2}\}_3^L$ for any $n\ge 3$ we denote the lattice cube with a vertex at the origin and generated by the vectors: $\overline{e}_1$, and $\overline{e}_1+2\overline{e}_i$ for $i=2,\ldots,n$.

Generalized octahedra. We denote by $\{3^{n-2},4\}_1^L$ for any $n\ge 2$ the lattice generalized octahedron with the vertices $O\pm\overline{e}_i$ for $i=1,\ldots,n$.

By $\{3^{n-2},4\}_2^L$ for any positive n we denote the lattice generalized octahedron with the vertices $O\pm\overline{e}_i$ for $i=1,\ldots,n{-}1$, and $O\pm\big(\overline{e}_1+\overline{e}_2+\ldots+\overline{e}_{n-1}+2\overline{e}_n\big)$.

By $\{3^{n-2},4\}_3^L$ for any positive n we denote the lattice generalized octahedron with the vertices O, $O-\overline{e}_1$, $O-\overline{e}_1-\overline{e}_i$ for $i=2,\ldots,n$, and e_i for $i=2,\ldots,n$.

A segment, octagons, and 24-sells. Denote by $\{\}^L$ the lattice segment with the vertices O and $O+\overline{e}_1$.

By $\{6\}_1^L$ we denote the hexagon with the vertices $O\pm\overline{e}_1$, $O\pm\overline{e}_2$, $O\pm(\overline{e}_1-\overline{e}_2)$.

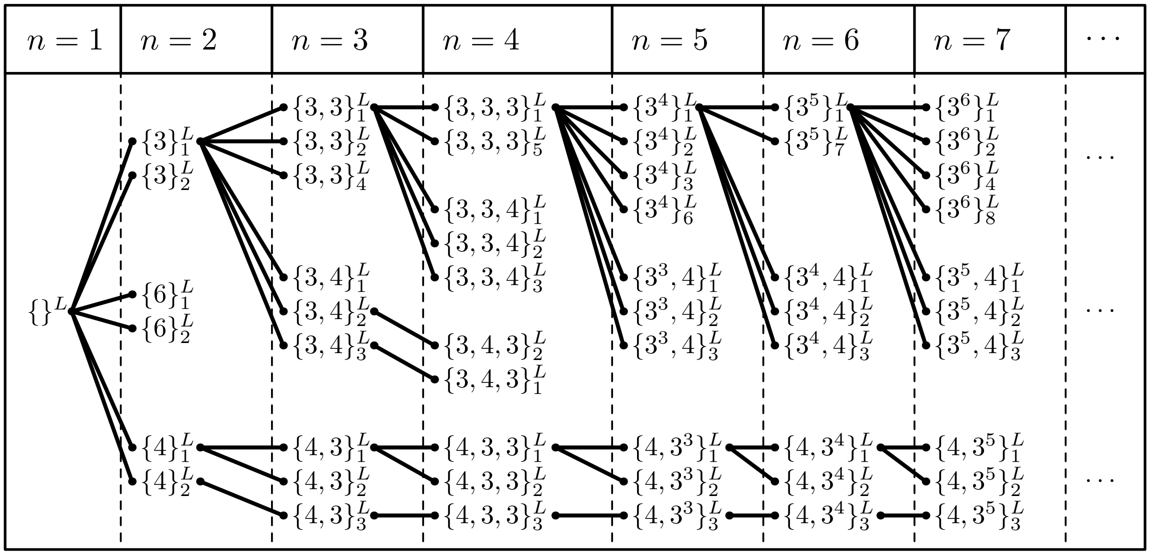


FIGURE 2. The adjacency diagram for the elementary lattice-regular convex lattice polytopes.

By $\{6\}_2^L$ we denote the hexagon with the vertices $O\pm(2\overline{e}_1+\overline{e}_2)$, $O\pm(\overline{e}_1+2\overline{e}_2)$, $O\pm(\overline{e}_1-\overline{e}_2)$. By $\{3,4,3\}_1^L$ we denote the 24-sell with 8 vertices of the form

$$O\pm 2(\overline{e}_2+\overline{e}_3+\overline{e}_4),\quad O\pm 2(\overline{e}_1+\overline{e}_2+\overline{e}_4),\quad O\pm 2(\overline{e}_1+\overline{e}_3+\overline{e}_4),\quad O\pm 2\overline{e}_4,$$

and 16 vertices of the form

$$O\pm(\overline{e}_2+\overline{e}_3+\overline{e}_4)\pm(\overline{e}_1+\overline{e}_2+\overline{e}_4)\pm(\overline{e}_1+\overline{e}_3+\overline{e}_4)\pm\overline{e}_4.$$

By $\{3,4,3\}_2^L$ we denote the 24-sell with 8 vertices of the form

$$O\pm 2(\overline{e}_1+\overline{e}_2+\overline{e}_3+\overline{e}_4),\quad O\pm 2(\overline{e}_1-\overline{e}_2+\overline{e}_3+\overline{e}_4),$$
$$O\pm 2(\overline{e}_1+\overline{e}_2-\overline{e}_3+\overline{e}_4),\quad O\pm 2(\overline{e}_1+\overline{e}_2+\overline{e}_3-\overline{e}_4),$$

and 16 vertices of the form

$$O\pm(\overline{e}_1+\overline{e}_2+\overline{e}_3+\overline{e}_4)\pm(\overline{e}_1-\overline{e}_2+\overline{e}_3+\overline{e}_4)\pm(\overline{e}_1+\overline{e}_2-\overline{e}_3+\overline{e}_4)\pm(\overline{e}_1+\overline{e}_2+\overline{e}_3-\overline{e}_4).$$

Theorem on enumeration of convex elementary lattice-regular lattice polytopes.

Theorem 2.2. [25] *Any elementary lattice-regular convex lattice polytope is lattice-congruent to some polytope of the following list.*

List of the polygons.
Dimension 1: *the segment* $\{\}^L$.
Dimension 2: *the triangles* $\{3\}_1^L$ *and* $\{3\}_2^L$;
the squares $\{4\}_1^L$ *and* $\{4\}_2^L$;
the octagons $\{6\}_1^L$ *and* $\{6\}_2^L$.
Dimension 3: *the tetrahedra* $\{3,3\}_i^L$, *for* $i=1,2,4$;
the octahedra $\{3,4\}_i^L$, *for* $i=1,2,3$;
the cubes $\{4,3\}_i^L$, *for* $i=1,2,3$.
Dimension 4: *the symplices* $\{3,3,3\}_1^L$ *and* $\{3,3,3\}_5^L$;

the generalized octahedra $\{3,3,4\}_i^L$*, for* $i=1,2,3$*;*
the 24-sells $\{3,4,3\}_1^L$ *and* $\{3,4,3\}_2^L$*;*
the cubes $\{4,3,3\}_i^L$*, for* $i=1,2,3$.
Dimension n (n>4): *the symplices* $\{3^{n-1}\}_i^L$ *where positive integers* i *are divisors of* $n+1$*;*
the generalized octahedra $\{3^{n-2},4\}_i^L$*, for* $i=1,2,3$*;*
the cubes $\{4,3^{n-2}\}_i^L$*, for* $i=1,2,3$.

All polytopes of this list are lattice-regular. Any two polytopes of the list are not lattice-congruent to each other.

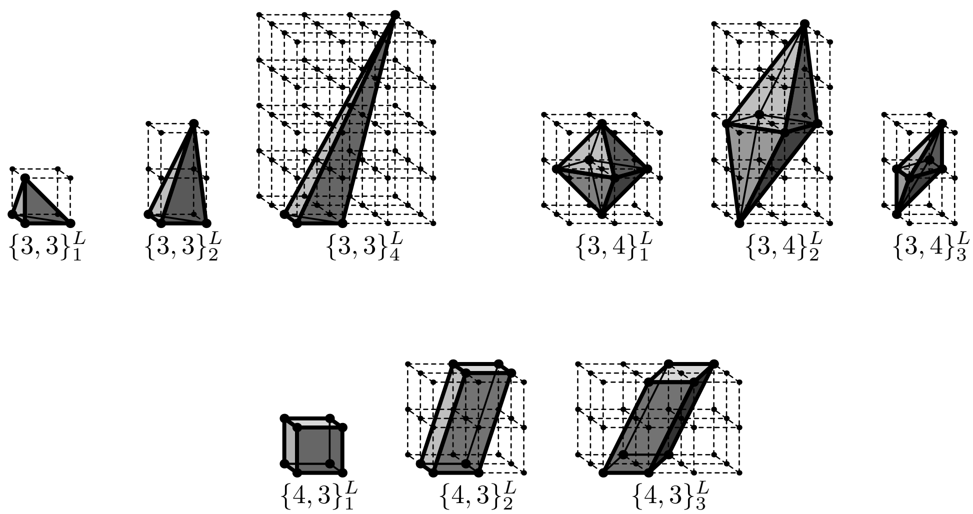


FIGURE 3. Three-dimensional lattice-regular polytopes.

On Figure 2 we show the adjacency diagram for the elementary lattice-regular convex lattice polygons of dimension not exceeding 7. Lattice-regular lattice three-dimensional polygons of different (nine) types are shown on Figure 3.

3. Basic notions of integer trigonometry

3.1. **Introduction.** Consider a two-dimensional (or even n-dimensional) oriented real vector space and fix some full-rank lattice in it. In [26] we investigates *geometry of lattice* in the following sense. *Objects* of this geometry are lines containing lattice points, polygons with lattice vertices, rays and so on. A natural *transformation group* here is the group of affine lattice preserving transformations of the plane.

It turns out that 'discrete' lattice geometry is as rich as 'continuous' Euclidean geometry.

In the work [26] I developed the trigonometric theory for lattice geometry. We define *lattice tangent* to be equal to special Hirzebruch-Jung continued fraction. These continued fractions has never been considered as trigonometric functions before. We proved the formulas for summation of lattice tangents, found a lattice analog for the Euclidean theorem on sum of angles in the triangle, obtained the description of lattice triangles. To establish all the listed results we found an elegant generalization of the classical geometrical interpretation of continued fractions with integer positive elements to the case of continued fractions with integer (not necessary positive) elements.

Now we formulate one of the results of the my work [26].

3.2. **Integer triangles.** The study of lattice angles is an essential part of modern lattice geometry. Invariants of lattice angles are used in the study of lattice convex polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the works of F. Klein [30], V. I. Arnold [3], E. Korkina [35], M. Kontsevich and Yu. Suhov [32], G. Lachaud [38]).

In this subsection we describe lattice triangles up to the lattice-affine equivalence relation. The classification problem of convex lattice polygons becomes now classical. There is still no a good description of convex lattice polygons. It is only known that the number of such polygons with lattice area bounded from above by n growths exponentially in $n^{1/3}$, while n tends to infinity (see the works of V. Arnold [7], and of I. Bárány and A. M. Vershik [9]).

Lattice polygons and polytopes of the lattice geometry are in the limelight of complex projective toric varieties (see for more information the works of V. I. Danilov [12], G. Ewald [13], T. Oda [43], and W. Fulton [14]). For instance, the result of this subsection gives the corresponding global relations for the toric singularities of projective toric varieties associated to integer-lattice triangles.

Necessary definitions. For any positive integer n and a point $A(x, y)$ denote by nA the point with the coordinates (nx, ny). A polygon $nA_0 \ldots nA_k$ is called *n-homothetic* to the polygon $P = A_0 \ldots A_k$ and denoted by nP. Polygons P_1 and P_2 are said to be integer-homothetic if there exist positive integers m_1 and m_2 such that m_1P_1 is integer-equivalent to m_2P_2.

Let us expand the set of rationals with operations $+$ and $1/*$ by the element ∞ end denote this expansion by $\overline{\mathbb{Q}}$. We say that $q \pm \infty = \infty$, $1/0 = \infty$, $1/\infty = 0$ (the expressions $\infty \pm \infty$ are not defined).

For any finite sequence of integers $(a_0, a_1, \ldots, a_n)$ we associate an element

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots) \ldots)) \in \overline{\mathbb{Q}}$$

and denote it by $]a_0, a_1, \ldots, a_n[$. If the elements of the sequence $a_1, \ldots, a_n$ are positive, then the expression for q is called the *ordinary continued fraction*.

Proposition. *For any rational there exists a unique ordinary continued fraction with odd number of elements.*

Let us consider for $q_i \in \mathbb{Q}$, $i = 1, \ldots, k$ the ordinary continued fractions with odd number of elements: $q_i =]a_{i,0}, a_{i,1}, \ldots, a_{i,2n_i}[$. Denote by $]q_1, q_2, \ldots, q_k[$ the element

$$]a_{1,0}, a_{1;1}, \ldots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \ldots, a_{2,2n_2}, \ldots a_{k,0}, a_{k,1}, \ldots, a_{k,2n_k}[\in \overline{\mathbb{Q}}.$$

Integer tangents. An *integer length* of the segment AB (denoted by $\mathrm{l}\ell(AB)$) is the ratio of its Euclidean length and the minimal Euclidean length of integer vectors with vertices in AB. An *integer* (*non-oriented*) *area* of the polygon P is the doubled Euclidean area of the polygon, it is denoted by $\mathrm{lS}(P)$.

Consider an arbitrary integer angle $\angle ABC$. The boundary of the convex hull of the set of all integer points except B in the convex hull of the angle $\angle ABC$ is called the *sail* of the orthant. The sail of the angle is a finite broken line with the first and the last vertices on different edges of the angle. Let us orient the broken line in the direction from the ray BA to the ray

BC and denote its vertices: $A_0, \ldots, A_{m+1}$. Denote $a_i = \mathrm{l}\ell(A_iA_{i+1})$ for $i = 0, \ldots, m$, and also $b_i = \mathrm{lS}(A_{i-1}A_iA_{i+1})$ for $i = 1, \ldots, m$. The following rational is called the *integer tangent of the angle* $\angle ABC$:

$$]a_0, b_1, a_1, b_2, a_2, \ldots, b_m, a_m[, \quad \text{we denote:} \quad \mathrm{ltan}\,\angle ABC.$$

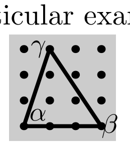
Formulation of the theorem. In Euclidean geometry on the plane the existence condition for the triangle with given angles can be written with tangents of angles in the following way. There exists a triangle with angles α, β, and γ iff $\tan(\alpha+\beta+\gamma) = 0$ and $\tan(\alpha+\beta) \notin [0; \tan\alpha]$ (without lose of generality, here we suppose that α is acute). Let us show the integer analog of the last statement.

Theorem 3.1. [26] **a).** *Let α_0, α_1, and α_2 be an ordered triple of integer angles. There exists an oriented integer triangle with the consecutive angles integer-equivalent to the angles α_0, α_1, and α_2 iff there exists $j \in \{0, 1, 2\}$ such that the angles $\alpha = \alpha_j$, $\beta = \alpha_{j+1 (\mathrm{mod}\, 3)}$, and $\gamma = \alpha_{j+2 (\mathrm{mod}\, 3)}$ satisfy the following conditions:*

i) $]\,\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta, -1, \mathrm{ltan}\,\gamma[\, = 0$; *ii)* $]\,\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta[\, \notin [0; \mathrm{ltan}\,\alpha]$.

b). *Two integer triangles with the same sequences of integer tangents are integer-homothetic.*

Note that for the conditions of the theorem one should take ordinary continued fractions with odd number of elements for tangents of angles. Let us illustrate the theorem with the following particular example:



$\mathrm{ltan}\,\alpha = 3 =]3[$;
$\mathrm{ltan}\,\beta = 9/7 =]1, 3, 2[$;
$\mathrm{ltan}\,\gamma = 3/2 =]1, 1, 1[$.

i) $]3, -1, 1, 3, 2, -1, 1, 1, 1[\, = 0$;
ii) $]3, -1, 1, 3, 2[\, = -3/2 \notin [0; 3]$.

References

[1] M. O. Avdeeva, V. A. Bykovskii, *Solution of Arnold's problem on Gauss-Kuzmin statistics*, Preprint, Vladivostok, Dal'nauka, (2002).
[2] M. O. Avdeeva, *On statistics of incomplete quotients of finite continued fractions*, Func. an. and appl., v. 38(2004), n. 2, pp. 1–11.
[3] V. I. Arnold, *Continued fractions*, M.: Moscow Center of Continuous Mathematical Education, (2002).
[4] V. I. Arnold, *Higher Dimensional Continued Fractions*, Regular and Chaotic Dynamics, v. 3(1998), n3, pp. 10–17.
[5] *Arnold's problems*, Phasis, M., (2000).
[6] V. I. Arnold, *Preface*, Amer. Math. Soc. Transl., v. 197(2), (1999), pp. ix–xii.
[7] V. I. Arnold, *Statistics of integer convex polygons*, Func. an. and appl., v. 14(1980), n. 2, pp. 1–3.
[8] I. Bárány J.-M. Kantor, *On the number of lattice free polytopes*, European J. Combin. v. 21(1), 2000, pp. 103–110.
[9] I. Bárány, A. M. Vershik, *On the number of convex lattice polytopes*, Geom. Funct. Anal. v. 2(4), 1992, pp. 381–393.
[10] K. Briggs, *Klein polyhedra*, http://www.btexact.com/people/briggsk2/klein-polyhedra.html, (2002).
[11] A. D. Bryuno, V. I. Parusnikov, *Klein polyhedrals for two cubic Davenport forms*, Mathematical notes, 56(4), (1994), pp. 9–27.
[12] V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk, v. 33(1978), n. 2, pp. 85–134.
[13] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Grad. Texts in Math. v. 168, Springer-Verlag, New York, 1996.
[14] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies; Princeton University Press, v. 131(1993).
[15] C. F. Gauss, Recherches Arithmétiques, Blanchard, 1807, Paris.
[16] O. N. German, *Sails and Hilbert Bases*, Proc. of Steklov Ins. Math, v. 239(2002), pp. 88-95.

[17] C. Hermite, *Letter to C. D. J. Jacobi*, J. Reine Angew. Math. vol. 40, (1839), p. 286.
[18] A. Ya. Hinchin, *Continued fractions*, M.: FISMATGIS, (1961).
[19] J.-M. Kantor, K. S. Sarakaria *On primitive subdivisions of an elementary tetrahedron*, Pacific J. of Math. v. 211(1), 2003, pp. 123–155.
[20] O. Karpenkov, *On tori decompositions associated with two-dimensional continued fractions of cubic irrationalities*, Func. an. and appl., v. 38(2004), no 2, pp. 28–37.
[21] O. Karpenkov, *On two-dimensional continued fractions for integer hyperbolic matrices with small norm*, Uspehi Mat. Nauk, vol. 59(2004), no. 5, pp. 149–150.
[22] O. Karpenkov, *On some new approach to constructing periodic continued fractions*, preprint n 12, Laboratoire de Mathématiques Discrètes du C.N.R.S., Luminy (2064), http://iml.univ-mrs.fr/editions/preprint2004/files/karpenkov.pdf.
[23] O. Karpenkov, *Three examples of three-dimensional continued fractions in the sense of Klein*, C. R. Acad. Sci. Paris, Ser. B, vol. 343(2006), pp. 5–7.
[24] O. Karpenkov, *Completely empty pyramids on integer lattices and two-dimensional faces of multidimensional continued fractions*, to appear in Monatshefte für Mathematik (published online: 13 June 2007). http://arxiv.org/abs/math.NT/0510482.
[25] O. Karpenkov, *Classification of lattice-regular lattice convex polytopes*, Functional Analisys and Other Mathematics, vol. 1(2006), n. 1, pp.31–54.
[26] O. Karpenkov, *Elementary notions of lattice trigonometry*, to appear in Math. Scand.
[27] O. Karpenkov, *On invariant Mobius measure and Gauss-Kuzmin face distribution*, to appear in the volume dedicated to the 70-th anniversary of prof. V. I. Arnold. http://arxiv.org/abs/math.NT/0610042.
[28] A. G. Khovanskii, A. Pukhlikov, *Finitely additive measures of virtual polytopes*, Algebra and Analysis, v. 4(2), 1992, pp. 161–185; translation in St.Petersburg Math. J., v. 4(2), 1993, pp. 337–356.
[29] A. G. Khovanskii, A. Pukhlikov, *A Riemann-Roch theorem for integrals and sums of quasipolynomials over virtual polytopes*, Algebra and Analysis, v. 4(4), 1992, pp. 188–216; translation in St. Petersburg Math. J., v. 4(4), 1993, pp. 789–812.
[30] F. Klein, *Ueber einegeometrische Auffassung der gewöhnliche Kettenbruchentwicklung*, Nachr. Ges. Wiss. Göttingen Math-Phys. Kl., 3, (1895), 352-357.
[31] F. Klein, *Sur une représentation géométrique de développement en fraction continue ordinaire*, Nouv. Ann. Math. 15(3), (1896), pp. 327–331.
[32] M. L. Kontsevich and Yu. M. Suhov, *Rtatistics or Klein Polyledra and Multidimensional Continued Fracthons*, Amer. Math. Soc. Transl., v. 197(7), (1899).
[33] E. I. Korkina, *The simplest 2-dimensional continued fraction*, International Geometrical Colloquium, Moscow 1993.
[34] E. I. Korkina, *La périodicité des fractions continues multidimensionelles*, C. R. Ac. Sci. Paris, v. 319(1914), pp. 277–780.
[35] E. I. Korkina, *Twz-dimensional continued fractions. The simplest examples*, Proceedings of V. A. Steklov Math. Ins., v. 209(1995), pp. 143–166.
[36] R. O. Kuzmin, *On a problem of Gauss* Dokl. Akad. Nauk SSSR Ser A(1928), pp. 675–380.
[37] G. Lachaud, *Polyèdre d'Arnold et voile d'un cône simplicial: analogues du thèoreme de Lagrange*, C. R. Ac. Sci. Paris, v. 717(7943), pp. 711–716.
[38] G. Lachaud, *Voiles et polyhedres de Klein*, Act. Sci. Ind., 176 pp., Hermann, 2002.
[39] P. Lévy, *Sur les lois de probabilité dont dépendent les quotients complets et incomplets d'une fraction continue*, Bull. Soc. Math. 57, 1929, 178–194.
[40] J.-O. Moussafir, *Sales and Hilbert bases*, Func. an. and appl., v. 34(2000), n. 2, pp. 43–49.
[41] J.-O. Moussafir, Voiles et Polyédres de Klein: Geomstrie, Algorithmes et Statistiques, docteur en sciences thése, Université Paris IX - Dauphine, (2000), see also http://www.ceremade.dauphine.fr/~msfr/articles_msfr/these.ps.gz.
[42] R. Okazaki, *On an effective determination of a Shintani's decomposition of the cone* $\mathbb{R}^n_+$, J. Math. Kyoto Univ., v33-4(1995), pp. 1051–1070.
[43] T. Oda, *Convex bodies and Algebraic Geometry, An Introduction to the Theory of Toric Varieties*, Springer-Verlag, Survey in Mathemayics, 15(1988).

[44] V.I. Parusnikov, *Klein's polyhedra for the third extremal ternary cubic form*, preprint 137 of Keldysh Institute of the RAS, Moscow, (1995).
[45] V.I. Parusnikov, *Klein's polyhedra for the sevefth extremal cubic form*, preprint 79 of Keldysh Institute of the RAS, Moscow, (1993).
[46] T. Shintani, *On evaluation of zeta functions of totally real algebraic number fields at nonpositive integers*, J. Fac. Sci. Univ. Tokyo Sect. IA v.23(1986), pp. 333–117.
[47] B. Reznick, *Lattice point symplices*, Discrete Math., v. 60(1986), pp. 219–242.
[48] E. Thomas and A. T. Vasques, *On the resolution of cusp singularities and the Shintani decomposition in totally real cubic number fields*, Math. Ann. v.247(1980), pp. 1–20.
[49] H. Tsuchihashi, *Higher dimensional analogues of periodic continued fractions and cusp singularities*, Tohoku Math. Journ. v. 35(1973), pp. 176–393.
[50] E. Wirsing *On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces*, Acta Arithmetica, v. 24(1974), pp 507–528.

4. Short description of my main scientific interests in theory of energies functionals of knots.

1. Ordinary energy functionals. The study of knot energies was initiated by the work of Moffatt (1969) [12], and was developed by him in [13] following Arnold's work [2].

A functional on the space of all knots is said to be an energy functional if it is bounded below, C^1-continuous, and tends to infinity while the knot tends to the "knot" with a point of a double transversal self-intersection. Gradient flows of such functionals bring arbitrary knots to some so-called *perfect* critical knots. Besides, for some energies it is conjectured, that the corresponding perfect critical knots are unique for any connected component of the space of all knots. So there is a hope that the set of perfect knots is a complete knot invariant.

The first discrete energy of knots were produced by W. Fukuhara in 1988, for the details see his work [4]. Many articles were dedicated to general theory of knot energies: [1], [6], [14]. In [8] I found an integral equations on knots with critical functionals of energies, for some certain class of energies. I introduce Mm-energy of knots in [9]. This energy is good for numerical calculation of geometrical shapes of critical knots.

2. Möbius energy. One of the most beautiful and significant energy functionals is Möbius energy. Möbius energy was discovered by J. O'Hara [5] in 1991. Further investigations of Möbius energy properties were made by M. H. Freedman, Z. -H. He, and Z. Wang in [3]. Particularly, the authors introduced variational principles for Möbius energy and found some upper estimates for the minimal possible energy of knots with the given crossing number in their work. Conformal properties of Möbius energy allow us to calculate explicitly some critical values for toric knots, see the work [11]. In my work [10] the notion of Möbius energy is generalized to the case of the embedded graphs.

3. A few words about tensegrities. At this moment I am involved i the project on *tensegrities*.

Take a graph in a Euclidean space and replace some of its edges by strings and the others by rods. Such rod-string configuration is called a tensegrity if it is rigid. Tensegrities are a natural generalization of hinge mechanisms. Their first appearance was in art architecture, in 1968, when Kenneth Snelson erected his 60 feet high Needle Tower. Tensegrities are also used in the study of biological structures [7], engineering, theory of deployable constructions, etc. B. Roth and W. Whiteley founded the mathematical theory of tensegrities [15].

Assume we have fixed a combinatorial graph structure and are now considering the configuration space of all tensegrities corresponding to the graph. It turns out that the configuration space admits a structure of a smooth manifold. Two tensegrities are said to be similar if the set of rod edges of the first tensegrity coincides with the set of rod edges of the second. The configuration space of all tensegrities for a given graph consists of a certain number of connected components corresponding to similar tensegrities. These components are separated by special strata. The latter are responsible for structural transitions and may be distinguished by their types. We make efforts to detect and classify these strata starting with basic planar configurations.

References

[1] A. Abrams, J. Cantarella, J. H. G. Fu, M. Ghomi, and R. Howard, *Circles minimize most knot energies.* http://arXiv.org/abs/math/0105138.

[2] V. I. Arnold, *The asymptotic Hopf invariant and its applications*, in Proc. Summer School in Differential Equations, Erevan, Armenian SSR Academy of Science (1974); English translation in Selecta Math. Sov. 5(1986), 527-345.

[3] M. H. Freedman, Z.-H. He, and Z. Wang *Möbius energy of knots and unknots*, Ann. of Math. (2)139 (1994), no. 1, pp. 1–50.

[4] W. Fukuhara, *Energy of a knot*, The fête of topology, Academic Press, (1988), pp 443–451.

[5] J. O'Hara, *Energy of a knot*, Topology 38(1991), no. 2, pp. 241–247.

[6] J. O'Hara, *Family of energy functionals of knots*, Topology Appl. 08(1992), no. 2, pp. 147–161.

[7] D. E. Ingber, *Cellular tensegrity:defining new rules of biological design that govern the cytoskeleton*, Journal of Cell Science, 104(1993), pp. 613–627.

[8] O. Karpenkov, *Energy of a knot: variational principles* Russian J. of Math. Phys. 9(2002), no. 3, pp. 275–287.

[9] O. Karpenkov, *Energy of a knot: some new aspects*, Fundamental Mathematics Today, IUM, MCCME, 2003, pp. 214–223.

[10] O. Karpenkov, *Möbius enerby of graphs*, Math. Notes, vol. 79(2006), no. 1, pp. 134–138.

[11] D. Kim, R. Kusner, *Torus Knots Extremizing the Möbius Energy* Experimental Mathematics, Vol. 2 (1993), no. 1, pp. 1–9.

[12] H. K. Moffatt, *The degree of knottedness of tangled vortex lines*, J. Fluid Mech. Vol. 35(1969), pp. 117–129.

[13] H. K. Moffatt, *Magnetostatic equilibria and analogous Euler flows of arbitrary complex topology. Part 1. Fundamentals*, J. Fluid Mech. Vol. 35(1985), pp. 359–378.

[14] J. M. Sullivan, *Approximating ropelength by energy functions* Physical Knots: Knotting, Linking, and Folding Geometric Objects in $\mathbb{R}^3$, J. A. Calvo, K. Millett, and E. Rawdon eds., Contemp. Math. 2004, Amer. Math. Soc., Providence, RI (2002), pp. 181–186.

[15] W. Whiteley, *Rigidity and scene analysis*, in J. E. Goodman and J. O'Rourke, editors, Handbook of Discrete and Computational Geometry, chapter 49, pages 893-916. CRC Press, New York, 1997.

E-mail address, Oleg Karpenkov: `karpenk@math.leidenuniv.nl`