CLASSIFICATION OF THREE-DIMENSIONAL MULTISTORY COMPLETELY EMPTY CONVEX MARKED PYRAMIDS. [1]

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UDK: 511.9

In this note we announce the answer to a question in lattice geometry and its application to the theory of multidimensional continued fractions. The author is grateful to professor V. I. Arnold for constant attention to this work and for useful remarks.

Definitions. A point of $\mathbb{R}^m$ is called *integer* if all its coordinates are integers. Consider a convex polygon M whose vertices are all integer, and an integer point A from the complement to the plane containing M. A cone with the vertex A and the base M is said to an *integer convex marked pyramid* (with marked point A). An integer convex marked pyramid is said to be *completely empty* if it does not contain integer points different from A and from the integer points of the base M. Two sets are called *integer-affine* (*integer-linearly*) *equivalent* if there exists an affine (linear) transformation of $\mathbb{R}^m$ preserving the set of all integer points, and transforming the first set to the second. A plane is called *integer* if it is integer-affine equivalent to some plane passing through the origin and containing the sublattice of the integer lattice, and the rank of the sublattice is equivalent to the dimension of the plane.

Consider some integer two-dimensional plane and an integer point in the complement to this plane. Let the Euclidean distance from the given point to the given plane equal l. The minimal value of nonzero Euclidean distances from integer points of the span of the the given plane and the given point to the plane is denoted by l_0. The ratio l/l_0 is said to be the *integer distance* from the given integer point to the given integer plane. An integer pyramid is called *multistory* if the integer distance from the vertex of this pyramid to its base plane is greater that one.

Definition of multidimensional continued fraction in the sense of Klein. Consider arbitrary $n+1$ hyperplanes in $\mathbb{R}^{n+1}$ that intersect at the unique point: in the origin. Assume also that all the given planes do not contain any integer point different from the origin. The complement to these hyperplanes consists of 2^{n+1} open orthants. Consider any such orthant. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the *sail* of the orthant. The set of all 2^{n+1} sails is called the *n-dimensional continued fraction* constructed according to the given $n+1$ hyperplanes. The intersection of some hyperplane with the sail is said to be *k-dimensional face of the sail* if it is contained in some k-dimensional plane and is homeomorphic to the k-dimensional disc. (See. [1] and [2].)

Formulation of the main statements. By $(a_1, \ldots, a_k)$ in $\mathbb{R}^m$ for $k < m$ we denote the point $(a_1, \ldots, a_k, 0, \ldots, 0)$. It turns out that it is possible to implicitly describe all integer-affine classes of multistory completely empty convex three-dimensional marked pyramids.

Theorem 1. *Any multistory completely empty convex three-dimensional marked pyramid is integer-affine equivalent exactly to one of the marked pyramids from the list "M-W".*

[1] Supported by SS-1972.2003.1 project.

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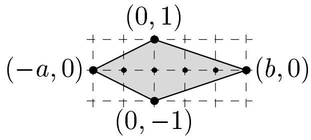
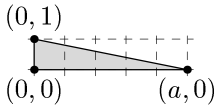
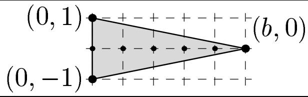
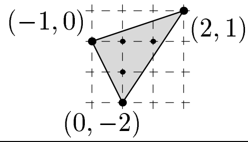
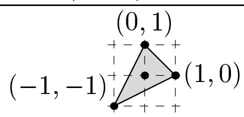
<table>
<tr><th>The list
"M-W"</th><th>Parameters</th><th>Coords. of the vertex</th><th>Coordinates of the base</th><th>Integer-affine type of the base</th></tr>
<tr><td>$M_{a,b}$</td><td>$b \geq a \geq 1$</td><td>$(0,0,0)$</td><td>$(2,-1,0)$,
$(2,-a-1,1)$,
$(2,-1,2)$, $(2,b-1,1)$</td><td>
</td></tr>
<tr><td>$T^{\xi}_{a,r}$</td><td>$a \geq 1$, $r \geq 2$,
$0 < \xi \leq r/2$,
$\gcd(\xi, r) = 1$</td><td>$(0,0,0)$</td><td>$(\xi, r-1, -r)$,
$(a+\xi, r-1, -r)$,
$(\xi, r, -r)$</td><td>
</td></tr>
<tr><td>U_b</td><td>$b \geq 2$</td><td>$(0,0,0)$</td><td>$(2,1,b-1)$, $(2,2,-1)$,
$(2,0,-1)$</td><td>
</td></tr>
<tr><td>V</td><td></td><td>$(0,0,0)$</td><td>$(2,-2,1)$,
$(2,-1,-1)$,
$(2,1,2)$</td><td>
</td></tr>
<tr><td>W</td><td></td><td>$(0,0,0)$</td><td>$(3,0,2)$, $(3,1,1)$,
$(3,2,3)$</td><td>
</td></tr>
</table>

Note that up to this moment the following statement on compact two-dimensional faces (of the sails for multidimensional continued fractions) contained in planes on the integer distance to the origin greater than one was known. Such faces are either triangles or quadrangles (see the work [3] by J.-O. Moussafir). We give a complete integer-linear classification of such faces.

Corollary 2. *Any compact two-dimensional face of a sails of a multidimensional continued fraction contained in a plane at an integer distance to the origin greater than one is integer-linear equivalent exactly to one of the faces with the vertices of the following list:*

— $(2,-1,0)$, $(2,-a-1,1)$, $(2,-1,2)$, $(2,b-1,1)$, *where* $b \geq a \geq 1$*;*

— $(\xi, r-1, -r)$, $(a+\xi, r-1, -r)$, $(\xi, r, -r)$, *where* $a \geq 1$, *and* ξ *and* r *are relatively prime, and* $r \geq 2$ *and* $0 < \xi \leq r/2$*;*

— $(2,1,b-1)$, $(2,2,-1)$, $(2,0,-1)$, *where* $b \geq 2$*;*

— $(2,-2,1)$, $(2,-1,-1)$, $(2,1,2)$ *and* $(3,0,2)$, $(3,1,1)$, $(3,2,3)$.

All triangular faces of the list are realizable by sails of any dimension greater than one. All quadrangular faces of the list are realizable by sails of any dimension greater than two and are not realizable by sails of dimension two.

REFERENCES

[1] V. I. Arnold, *Continued fractions*, M., Moscow Center of Continues Mathematical Education, (2002). [2] O. N. Karpenkov, *On tori decompositions associated with two-dimensional continued fractions of cubic irrationals*, Func. an. appl., vol. 38(2), pp. 28-37, (2004). [3] J.-O. Moussafir, *Voiles et Polyèdres de Klein: Géométrie, Algorithmes et Statistiques*, thèse de docteur sciences, Université Paris IX - Dauphine, (2000)