ON TORI TRIANGULATIONS ASSOCIATED WITH TWO-DIMENSIONAL CONTINUED FRACTIONS OF CUBIC IRRATIONALITIES.

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Introduction.

A lot of properties of ordinary continued fractions has multidimensional analogies. For instance, H. Tsuchihashi [7] showed the connection between periodic multidimensional continued fractions and multidimensional cusp singularities. The relation between sails of multidimensional continued fractions and Hilbert bases is described by J.-O. Moussafir in the work [6].

In his book [1] dealing with theory of continued fractions V. I. Arnold gives various images of the sails of so-called *two-dimensional generalized golden ratio* continued fraction. In the article [5] E. I. Korkina studied the sales for the simplest two-dimensional continued fractions of cubic irrationalities, whose fundamental region consists of two triangles, three edges and one vertex.

We consider the same model of the multidimensional continued fraction as considered by the authors mentioned above. In the present work we obtain examples of new tori triangulation of the sails for two-dimensional continued fractions of cubic irrationalities for some special families possessing the fundamental regions with more complicated structures.

In §1 we give the necessary definitions and notions. In §2 we investigate the properties of two-dimensional continued fractions constructed with Frobenius operators, further we discuss the relation between the equivalence classes of tori triangulations and cubic extensions of the field of rationals. (The detailed analysis of the properties of cubic extensions for the rational numbers field and their classification is realized by B. N. Delone and D. K. Faddeev in the work [3].) In §3 we describe new examples of tori triangulations.

The author is grateful to professor V. I. Arnold for constant attention for this work and useful remarks.

1. Definitions.

Points of the space $\mathbb{R}^k$ $(k \geq 1)$ whose coordinates are all integers are called *integer points*.

Consider a set of $n+1$ hyperplanes passing through the origin in general position in the space $\mathbb{R}^{n+1}$. The complement to these hyperplanes consists of 2^{n+1} open orthants. Let us choose an arbitrary orthant.

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The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called *the sail.*

The union of all 2^{n+1} sails defined by these hyperplanes of the space r^{n+1} is called *n-dimensional continued fraction* constructed according to the given $n+1$ hyperplanes in general position in $n+1$-dimensional space.

Two n-dimensional sails (continued fractions) are called *equivalent* if there exists a linear integer lattice preserving transformation of the n+1-dimensional space such that it maps one sail (continued fraction) to the other.

To construct the whole continued fraction up to the equivalence relation in one-dimensional case it is sufficiently to know some integer characteristics of one sail (that is to say the integer lengths of the edges and the integer angles between the consecutive edges of one sail).

Conjecture 1. (Arnold) *There exists the collection of integer characteristics of the sail that uniquely up to the equivalence relation determines the continued fraction.*

Let $A \in GL(n+1, \mathbb{R})$ be an operator with real distinct roots. Let us take the n-dimensional subspaces that spans all possible subsets of n linearly independent eigenvectors of the operator A. As far as the eigenvectors are linearly independent, the obtained n+1 hyperspaces are n+1 hyperspaces in general position. The multidimensional continued fraction is constructed with respect to these hyperspaces.

Proposition 1.1. *Continued fractions constructed by some arbitrary operators A and B of the group $GL(n+1, \mathbb{R})$ with distinct real irrational eigenvalues are equivalent iff there exists an integer operator X with the unit determinant such that the operator $\tilde{A}$ obtained from the operator A by means of the conjugation by the operator X commutes with B.*

Proof. Let the continued fractions constructed by the operators A and B of the group $GL(n+1, \mathbb{R})$ with distinct real irrational eigenvalues are equivalent, i. e. there exists linear integer lattice preserving transformation of the space that maps the continued fraction of the operator A to the continued fraction of the operator B (and the orthants of the first continued fraction maps to the orthants of the second one). Under such transformation the operator A conjugates by some integer operator X with the unit determinant. All eigenvalues of the obtained operator A are distinct and real (since the characteristic polynomial of the operator is invariant). As far as the orthants of the first continued fraction maps to the orthants of the second one, the sets of the eigen directions for the operators $\tilde{A}$ and B coincides. Thus, the given operators are diagonalizable together in the same basis and hence they commutes.

Let us prove the converse. Suppose that there exists an integer operator X with the unit determinant such that the operator $\tilde{A}$ obtained from the operator A by means of the conjugation by the operator X commutes with B. Note that the eigenvalues of the operators A and $\tilde{A}$ coincide. Therefore, all eigenvalues of the operator $\tilde{A}$ (just as for the operator B) are real, distinct, and irrational. Let us consider a basis such that the operator $\tilde{A}$ is diagonal in it. Simple verification shows that the operator B is also diagonal in this basis. Hence, the operators $\tilde{A}$ and B define the same orthant decomposition of the $n+1$-dimensional space and the operators corresponding to this continued fractions coincide. It remains to note that a conjugation by an integer operator with the unit determinant corresponds to the linear integer lattice preserving transformation of the $n+1$-dimensional space. □

Further we consider only continued fractions constructed by invertible integer operators of the $n+1$-dimensional space such that their inverse are also integer. The set of such operators form the group denoted by $GL(n+1, \mathbb{Z})$. This group consist of the integer operators with the determinants equal ± 1.

The n-dimensional continued fraction constructed by an operator $A \in GL(n+1, \mathbb{Z})$ with irreducible characteristic polynomial over the field of rationals and real eigenvalues is called *the n-dimensional continued fraction of $(n+1)$-algebraic irrationality.* The cases of $n = 1, 2$ correspond to *one(two)-dimensional continued fractions of quadratic (cubic) irrationalities.*

Let the characteristic polynomial of the operator A be irreducible over the field of rationals and its roots be real and distinct. Consider the integer lattice preserving operators with the unit determinants commuting with A. These operators form an Abelian group. It follows from Dirichlet unity elements theorem (see. [2]) that this group is isomorphic to $\mathbb{Z}^n$ and that its action is free. The factor of a sail under such group action is isomorphic to n-dimensional torus. (For the converse see [4] and [7].) The polyhedron decomposition of n-dimensional torus is defined in the natural way, the affine types of the polyhedra are also defined (in the notion of the affine type we include the number and mutual arrangement of the integer points for the faces of the polyhedron). In the case of two-dimensional continued fractions for cubic irrationalities such decompositions are usually called torus *triangulations.*

By *a fundamental region* of the sail we call a union of sail faces that contains exactly one face from each equivalence class.

2. Conjugacy classes of two-dimensional continued fractions for cubic irrationalities.

Two-dimensional continued fractions for cubic irrationalities constructed by the operators A and $-A$ coincide. So, the study of continued fractions for integer operators with the determinants equal ± 1 reduces to the study of continued fractions for integer operators with the unit determinants (i. e. operators of the group $SL(3, \mathbb{Z})$).

An operator (or a matrix) with the unit determinant

$$A_{m,n} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -m & -n \end{pmatrix},$$

where m and n are arbitrary integers is called *a Frobenius operator (matrix).* Let us note the following: if the characteristic polynomial $\chi_{A_{m,n}(x)}$ is irreducible over the field $\mathbb{Q}$

than the matrix for the left multiplication by the element x operator in the natural basis $\{1, x, x^2\}$ in the field $\mathbb{Q}[x]/\left(\chi_{A_{m,n}(x)}\right)$ coincides with the matrix $A_{m,n}$.

Let the eigenvectors of an arbitrary operator $A \in SL(3, \mathbb{Z})$ be distinct real and irrational. Let e_1 be some integer nonzero vector, $e_2 = A(e_1)$, $e_3 = A^2(e_1)$. Then the matrix of the operator A in the basis (e_1, e_2, ce_3) for some rational c will be Frobenius. However the transition matrix here could be non-integer and the corresponding continued fraction is not equivalent to initial one.

Example 2.1. *The continued fraction constructed by the operator*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -7 & 0 & 29 \end{pmatrix},$$

is not equivalent to the continued fraction constructed by any Frobenius operator with the unit determinant.

Here arises the following interesting question. *How often the continued fractions that don't correspond to Frobenius operators can occur?*

In any case, the family of Frobenius operators possesses some useful properties that allows us to construct the whole families of nonequivalent two-dimensional periodic continued fractions at once. That is extremely actual itself.

It is easy to obtain the following statements.

Statement 2.1. *The set Ω of operators $A_{m,n}$ having all eigenvalues real and distinct is defined by the inequality $n^2m^2 - 4m^3 + 4n^3 - 18mn - 27 \leq 0$. For the eigenvalues of the operators to be irrational it is necessary to subtract extra two perpendicular lines in the integer plane: $A_{a,-a}$ and $A_{a,a+2}$, for all $a \in \mathbb{Z}$.*

Statement 2.2. *For any integers n and m, the two-dimensional continued fractions for the cubic irrationalities constructed by the operators $A_{m,n}$ and $A_{-n,-m}$ are equivalent.*

Further we will consider all statements modulo this symmetry.

Remark. Example 2.3 below shows that some periodic continued fractions of the set Ω are equivalent.

Let us note that there exist nonequivalent two-dimensional periodic continued fractions constructed by operators of the group $GL(n + 1, \mathbb{R})$ whose characteristic polynomials define isomorphic extensions of the rational numbers field. In the following example the operators with equal characteristic polynomials but distinct continued fractions are shown.

Example 2.2. *The operators $(A_{-1,2})^3$ and $A_{-4,11}$ have distinct two-dimensional continued fractions (although their characteristic polynomials coincides).*

At the other hand similar periodic continued fractions may correspond to operators with distinct characteristic polynomials.

Example 2.3. *The operators $A^2_{0,-a}$ and $A_{-2a,-a^2}$ are conjugated by the operator in the group $GL(3, \mathbb{Z})$ and hence the periodic continued fractions (including the torus triangulations) corresponding to the operators $A_{0,-a}$ and $A_{-2a,-a^2}$ are equivalent.*

Let us also note that triangulations for distinct cubic extensions of the field $\mathbb{Q}$ are always nonequivalent.

3. Torus triangulations and fundamental regions for some series of operators $A_{m,n}$

Here we calculate torus triangulations and fundamental domains for several infinite series of Frobenius operators. In this paragraph we consider only the sails containing the point $(0, 0, 1)$ in its convex hull.

The ratio of the Euclidean volume of an integer k-dimensional polyhedron in n-dimensional space to the Euclidean volume of the minimal symplex in the same k-dimensional subspace is called *the integer k-dimensional volume* of the polyhedron (if $k = 1$ — *the integer length* of the segment, if $k = 2$ — *the integer area* of the polygon).
The ratio of the Euclidean distance from the given integer hyperplane (containing an $n{-}1$-dimensional integer sublattice) to the given integer point to the minimal Euclidean distance from the hyperplane to integer points in the complement of this hyperplane is called the corresponding *integer distance.*
By *the integer angle* between two given integer rays (i.e. rays that contain more than one integer point) with the vertex at the same integer point we call the value $S(u, v)/(|u| \cdot |v|)$, where u and v are arbitrary integer vectors passing along the rays and $S(u, v)$ is the integer volume of the triangle with edges u and v.

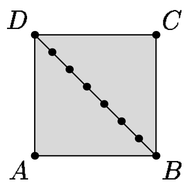
Remark. Our integer volume is always integer (in standard parallelepiped measuring the value will be $k!$ times less). The integer k-dimensional volume of the symplex equals the index of the lattice subgroup generated by its edges having the common vertex.

Since the integer angles of any triangle with all integer vertices can be uniquely restored by the integer lengths of the triangle and its integer volume we will not write the integer angles of triangles below.

Conjecture 2. *The specified invariants distinguish all nonequivalent torus triangulations of two-dimensional continued fractions for the cubic irrationalities.*

In the formulations of Propositions 3.1—3.5 we say only about homeomorphic type for the torus triangulations although in the proof we give the description of the fundamental regions that allows to calculate any other invariant including affine types of the faces. (As an example we calculate integer volumes and distances to faces in Propositions 3.1 and 3.2.) The examples of affine structure of triangulation faces are shown on the figures of the propositions.

Proposition 3.1. *Let $m = b{-}a{-}1$, $n = (a{+}2)(b{+}1)$ $(a, b \geq 0)$, then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to the following one:*



(*on the figure* $b=6$).

Proof. The operators

$$X_{a,b} = A_{m,n}^{-2}, \quad Y_{a,b} = A_{m,n}^{-1}\big(A_{m,n}^{-1} - (b+1)I\big)$$

commutes with the operator $A_{m,n}$ without transposing of the sails (note that the operator $A_{m,n}$ transposes the sails). Here I is the identity element of the group $SL(3,\mathbb{Z})$.

Let us describe the closure for one of the fundamental regions obtaining by the factoring of the sail over the operators $X_{a,b}$ and $Y_{a,b}$. Consider the points $A=(1,0,a{+}2)$, $B=(0,0,1)$, $C=(b{-}a{-}1,1,0)$ and $D=((b{+}1)^2,b{+}1,1)$ of the sail containing the point $(0,0,1)$. Under the operator $X_{a,b}$ action the segment AB maps to the segment DC (the point A maps to the point D and B to C). Under the operator $Y_{a,b}$ action the segment AD maps to the segment BC (the point A maps to the point B and D to C). The integer points $((b{+}1)i,i,1)$, where $i\in\{1,\ldots,b\}$ belong to the interval BD.

As can be easily seen, the integer lengths of the segments AB, BC, CD, DA and BD equal 1, 1, 1, 1 and $b{+}1$ correspondingly; the integer areas of both triangles ABD and BCD equal $b{+}1$. The integer distances from the origin to the plains containing the triangles ABD and BCD equal 1 and $a{+}2$ correspondingly.

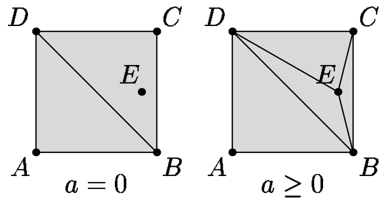
The operators $X_{a,b}$ and $Y_{a,b}$ map the sail to itself, since all their eigenvectors are positive (or in this case it is equivalent to say that values of their characteristic polynomials on negative semi-axis are always negative). Furthermore these operators generates the group of integer operators mapping the sail to itself. This follows from the fact that the triangulation obtaining by the factoring the sail over this operators contains the unique vertex (zero-dimensional face). Hence the torus triangulation has no smaller subperiod. □

Let us show that all vertices for the fundamental domain of the arbitrary periodic continued fraction can be chosen of the closed convex hull of the following five points: the origin; A; $X(A)$; $Y(A)$ and $XY(A)$, where A is the arbitrary zero-dimensional face of the sail, and the operators X and Y generates the group of integer operators mapping the sail to itself.

Consider a tetrahedral angle with the vertex at the origin and edges passing through the points A, $X(A)$, $Y(A)$, and $XY(A)$. The union of all images for this angle under the transformations of the form X^mY^n, where m and n are integers, covers the whole interior of the orthant. Hence all vertices of the sail can be obtained by shifting by operators X^mY^n the vertices of the sail lying in our tetrahedral angle. The convex hull for the integer points of the form $X^mY^n(A)$ is in the convex hull of all integer points for the given orthant at that. Therefore, the boundary of the convex hull for all integer points of the orthant is in the complement to the interior points of the convex hull for the integer points of the form $X^mY^n(A)$. The complement is in the unit of all images for the convex hull of the following points: the origin, A, $X(A)$, $Y(A)$, and $XY(A)$, under the transformations of the form X^mY^n, where m and n are integers.

It is obviously that all points of the constructed polyhedron except the origin lie in the concerned open orthant at that.

Proposition 3.2. *Let $m = -a$, $n = 2a+3$ $(a \geq 0)$, then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to the following one:*



Proof. Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

$$X_a = A_{m,n}^{-2}; \quad Y_a = (2I - A_{m,n}^{-1})^{-1}.$$

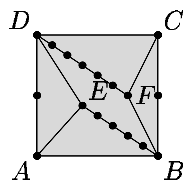
As in the previous case let us make the closure of one of the fundamental regions of the sail (containing the point $(0,0,1)$) that obtains by the factoring over the operators X_a and Y_a. Let $A = (0,0,1)$, $B = (2,1,1)$, $C = (7,4,2)$ and $D = (-a,1,0)$. Besides this points the vertex $E = (3,2,1)$ is in the fundamental region. Under the operator X_a action the segment AB maps to the segment DC (the point A maps to the point D and B — to C). Under the operator Y_a action the segment AD maps to the segment BC (the point A maps to the point B and D — to C).

If $a = 0$ then the integer length of the sides AB, BC, CD and DA equal 1, and the integer areas of the triangles ABD and BCD equal 1 and 3 correspondingly. The integer distances from the origin to the plains containing the triangles ABD and BCD equal 2 and 1 correspondingly.

If $a > 0$ then all integer length of the sides and integer areas of all four triangles equal 1. The integer distances from the origin to the plains containing the triangles ABD, BDE, BCE and CED equal $a+2$, $a+1$, 1 and 1 correspondingly.

Here and below the proofs of the statements on the generators are similar to the proof of the corresponding statements of Proposition 3.1. □

Proposition 3.3. *Let $m = 2a-5$, $n = 7a-5$ $(a \geq 2)$, then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to the following one:*



(on the figure $a = 5$).

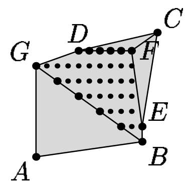
Proof. Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

$$X_a = 2A_{m,n}^{-1} + 7I; \quad Y_a = A_{m,n}^{2}.$$

Let us make the closure of the fundamental regions of the sail (containing the point $(0,0,1)$) that obtains by the factoring over the operators X_a and Y_a. Let $A = (-14, 4, -1)$,

$B=(-1,1-a,7a^2-10a+4)$, $C=(1,5-7a,49a^2-72a+30)$ and $D=(0,0,1)$. Under the operator X_a action the segment AB maps to the segment DC (the point A maps to the point D and the B — to C). Under the operator Y_a action the segment AD maps to the segment BC (the point A maps to the point B and D — to C). Besides this points the vertices $E=(-1,0,2a-1)$ and $F=(0,-a,7a^2-5a+1)$ are in the fundamental region. The interval BE contains $a-2$ integer points, the interval DF — $a-1$, AD and CB — one point for each. □

Proposition 3.4. *Let $m=a-1$, $n=3+2a$ $(a\geq 0)$, then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to the following one:*



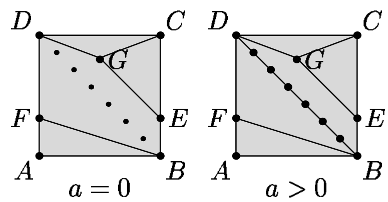
(*on the figure $a=4$*).

Proof. Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

$$X_a=(2I+A_{m,n}^{-1})^{-2};\quad Y_a=A_{m,n}^{-2}.$$

Let us make the closure of one of the fundamental regions of the sail (containing the point $(0,0,1)$) that obtains by the factoring over the operators X_a and Y_a. Consider the points $A=(1,-2a-3,4a^2+11a+10)$, $B=(0,0,1)$, $C=(-4a-11,2a+5,-a-2)$, and $D=(-a-2,0,a^2+3a+3)$. Besides this points, the vertices $E=(-2,1,0)$, $F=(-2a-3,a+1,1)$ and $G=(0,-1-a,2a^2+5a+4)$ are in the fundamental region. The intervals BG and DF contains a integer points each. Interior of the pentagon $BEFDG$ contains exactly $(a+1)^2$ integer points of the form: $(-j,-i+j,(2a+3)i-(a+2)j+1)$, where $1\leq i\leq a+1$, $1\leq j\leq 2i-1$. Under the operator X_a action the segment AB maps to the segment DC (the point A maps to the point D and B to C). Under the operator Y_a action the broken line AGD maps to the broken line BEC (the point A maps to the point B, the point G maps to the point E, and the point D — to the point C). □

Proposition 3.5. *Let $m=-(a+2)(b+2)+3$, $n=(a+2)(b+3)-3$ $(a\geq 0,\ b\geq 0)$, then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to the following one:*



(*on the figure $b=5$*).

Proof. Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

$$X_{a,b} = ((b+3)I - (b+2)A_{m,n}^{-1})A_{m,n}^{-2}; \quad Y_{a,b} = A_{m,n}^{-2}.$$

Let us make the closure of one of the fundamental regions of the sail (containing the point $(0,0,1)$) that obtains by the factoring over the operators $X_{a,b}$ and $Y_{a,b}$. Let the points $A = (b^2{+}3b{+}3, b^2{+}2b{-}a{+}1, a^2b{+}3a^2{+}4ab{+}b^2{+}6a{+}5b{+}4)$, $B = (b^2{+}5b{+}6, b^2{+}4b{+}4)$, $C = (-ab{-}2a{-}2b{-}1, 1, 0)$ and $D = (0,0,1)$. The interval BD contains $b{+}1$ integer points. Besides this points the vertices $E = (b{+}4, b{+}3, b{+}2)$, $F = (b{+}2, b{+}1, a{+}b{+}2)$ and $G = (1,1,1)$ are in the fundamental region. Under the operator $X_{a,b}$ action the segment AB maps to the segment DC (the point A maps to the point D and the point B — to the point C). Under the operator $Y_{a,b}$ action the broken line AFD maps to the broken line BEC (the point A maps to the point B, the point F maps to the point E, and the point D — to the point C). □

Note that the generators of the subgroup of operators commuting with the operator $A_{m,n}$ preserving the sails can be expressed with the operators $A_{m,n}$ and $\alpha I + \beta A_{m,n}^{-1}$, where α and β are nonzero integers.

It turns out that in general case the following statement holds: the determinants of the matrices for the operators $\alpha I + \beta A_{m,n}^{-1}$ and $\alpha I + \beta A_{m+k\beta,n+k\alpha}^{-1}$ are equivalent. In particular, if the absolute value of the determinant of the matrix for the operator $\alpha I + \beta A_{m,n}^{-1}$ is unit, then the absolute value of the determinant of the matrix for the operator $\alpha I + \beta A_{m+k\beta,n+k\alpha}^{-1}$ is also unit for an arbitrary integer k.

Seemingly, torus triangulations for the other sequences of operators $A_{m_0+\beta s, n_0+\alpha s}$, where $s \in \mathbb{N}$, (besides considered in the propositions 3.1—3.5) have much in common (for example, number of polygons and their types).

Note that the numbers α and β for such sequences satisfy the following interesting property. Since

$$|\alpha I + \beta A_{m,n}^{-1}| = \alpha^3 + \alpha^2\beta m - \alpha\beta^2 n + \beta^3,$$

we have the following. There exist integers n and m such that $|\alpha^3 + \alpha^2\beta m - \alpha\beta^2 n + \beta^3| = 1$ iff $\alpha^3 - 1$ is divisible by β and $\beta^3 - 1$ is divisible by α, or $\alpha^3 + 1$ is divisible by β and $\beta^3 + 1$ is divisible by α.

For instance, the corresponding pairs (α, β) for $10 \geq \alpha \geq \beta \geq -10$ (besides described in the propositions 3.1—3.5) are listed here: $(3,2)$, $(7,-2)$, $(9,-2)$, $(9,2)$, $(7,-4)$, $(9,4)$, $(9,5)$, $(9,7)$.

In conclusion we show the table with squares filled with torus triangulations of the sails constructed in this work whose convex hulls contain the point with the coordinates $(0,0,1)$ (see Fig. 1). The torus triangulation for the sail of the two-dimensional continued fraction for the cubic irrationality, constructed by the operator $A_{m,n}$ is shown in the square sited at the intersection of the n-th string and the m-th column. If one of the roots of characteristic polynomial for the operator equals 1 or -1, then we mark the square (m,n) with the sign $*$ or $\#$ respectively. The squares that correspond to the operators which

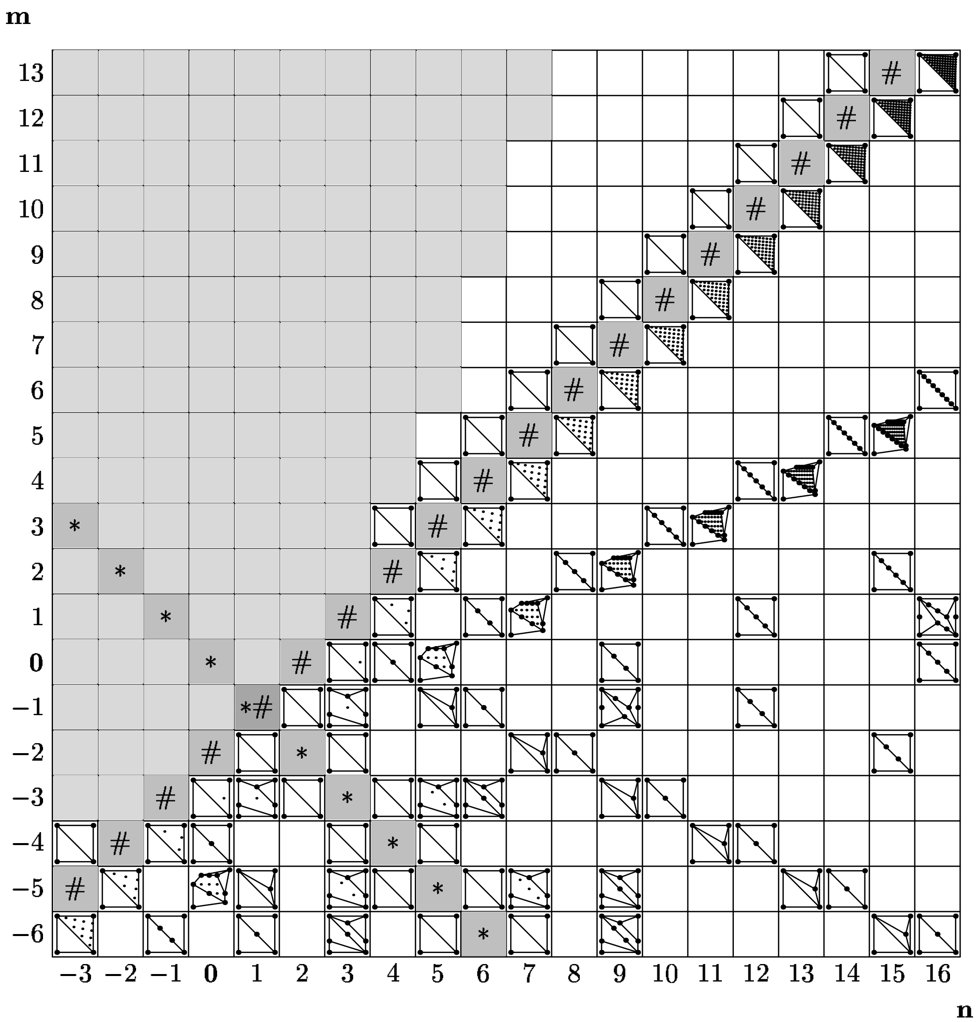


FIGURE 1. Torus triangulations for operators $A_{m,n}$.

characteristic polynomial has two complex conjugate roots we paint over with light gray color.

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