ENERGY OF A KNOT: SOME NEW ASPECTS.

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1. Introduction

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and $\tau : S^1 \longrightarrow \mathbb{R}^3$ be a smooth knot. We will assume that $\tau(t)$ is the arc length parametrization. Denote by $D(t_1, t_2)$ the length of the minimal sub arc between t_1 and t_2 on the circle. Let $|*|$ denote the absolute value of vectors in $\mathbb{R}^3$.

Following [1], we denote by

$$E(\tau) = E_f(\tau) = \iint\limits_{S^1\times S^1} f(|\tau(t_1) - \tau(t_2)|, D(t_1, t_2))dt_1 dt_2$$

the energy of the knot τ, where $f(\rho, \alpha)$ satisfies the following conditions:

1) $f(\rho, \alpha) \in C^{1,1}(U)$, where $U = \{(\rho, \alpha)|0 < \rho \leq \alpha, \alpha \leq \pi\}$;

2) there exist the following limits:

$$\lim_{\substack{(\rho,\alpha)\in U\\ \rho\to 0, \rho/\alpha\to 1}} f(\rho, \alpha), \qquad \lim_{\substack{(\rho,\alpha)\in U\\ \rho\to 0, \rho/\alpha\to 1}} \frac{\partial f(\rho, \alpha)}{\partial \rho}, \qquad \lim_{\substack{(\rho,\alpha)\in U\\ \rho\to 0, \rho/\alpha\to 1}} \frac{\partial f(\rho, \alpha)}{\partial \rho}.$$

Almost all energies are not homothety invariant, so we will consider only knots of length 2π.

The energy of a knot is not an invariant of the topological class of this knot. If we make a smooth perturbation of a knot, its energy smoothly changes. We will consider energies with the following important properties. The energy is always positive. When a knot crossing tends to a double point, the energy tends to infinity. So every topological class of knots has a representative with the minimal value of energy. This knot is called a *normal form* of the class. It is unknown whether each class has a unique normal form or not, i.e., whether the normal form for some energy is an invariant of the topological class or not. The normal forms satisfy the variational equations considered below.

Some energies have a physical meaning. For example $f = 1/(|\tau(t_1) - \tau(t_2)|)$ is the energy of a charged knot. Unfortunately, this energy is always infinite. As long as the charged knot does not break there must be some other forces which save the knot. Let us consider a model of such a restriction:

$$f = \frac{(D^2(t_1, t_2))}{|\tau(t_1) - \tau(t_2)|}.$$

For this energy we will develop our variational principles.

The study of knot energies began with the work of Moffatt (1969) [6], and was developed by him in [7] following Arnold's work [2]. The first steps in studying properties of the energies of knots were made by O'Hara [8, 9, 10] and the first variational principles for polygons in space were studied by Fukuhara [4].

The paper is organized as follows. We start in Section 2 with the definitions and formulations of the variational principles. We show that any extremal knot τ satisfies certain variational equations and discuss the corollaries of this variational principles. In Section 3 we represent Mm-energy. The

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definition of this energy differs with one regarded above. Nevertheless besides its own properties Mm-energy has some similar with Möbius energy properties. In Section 4 we consider Möbius energy of a knot. We prove some inequality for the energy of a normal form of the connected sum of two knots.

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2. Variational principles and corollaries

In this section we will work mostly with knots of fixed length 2π. So let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and let $\tau : S^1 \longrightarrow \mathbb{R}^3$ denote some smooth knot of length 2π. Let $\tau(t)$ be the arc length parametrization.

By $\kappa(t)$ we denote the curvature at t and $R(t) = 1/\kappa(t)$, the radius of curvature at t.

Definition 2.1. Given a smooth knot $\tau : S^1 \longrightarrow \mathbb{R}^3$ and a point $t_0 \in S^1$, a *locally perturbed knot* is a knot (denoted by $\tau_{t_0,\varepsilon}$) such that

a) $|\tau(t) - \tau_{t_0,\varepsilon}(t)| < \varepsilon^2$ if $D(t_0, t) \le \varepsilon$ and $\tau(t) = \tau_{t_0,\varepsilon}(t)$ if $D(t_0, t) > \varepsilon$;
b) $|\kappa(t) - \kappa_{t_0,\varepsilon}(t)| < \varepsilon$ for $D(t_0, t) < \varepsilon$;
c) $\tau_{t_0,\varepsilon}(t_0 + \lambda) = \tau_{t_0,\varepsilon}(t_0) + \lambda\dot\tau_{t_0,\varepsilon}(t_0) + (\lambda^2/2)\ddot\tau_{t_0,\varepsilon}(t_0) + o(\varepsilon^2)$ if $D(t_0, t_0 + \lambda) \le \varepsilon$.

Note that at the points $t_0 - \varepsilon$ and $t_0 + \varepsilon$ the curvature is not restricted.

The length of the knot $\tau_{t_0,\varepsilon}$ can change, but we regard knots of length 2π only. One of the ways to solve this problem is to consider the restriction of the set of locally perturbed knots to the set of knots of constant length 2π, but this definition is unsatisfactory. Indeed, let a knot τ in some neighborhood of the point t_0 be a piece of a straight line. Then the set of locally perturbed knots at the point t_0 of length 2π consists of the knot τ only.

We will extend this set in the following way.

Definition 2.2. Let the length of $\tau_{t_0,\varepsilon}$ be $(1 + \delta)2\pi$. The *locally perturbed length 2π knot* $\tilde\tau_{t_0,\varepsilon}$ is the knot obtained from $\tau_{t_0,\varepsilon}$ by homothety with coefficient $1/(1 + \delta)$ and center at the origin. We also say that the knot $\tilde\tau$ is *associated* with the knot τ.

Consider any $\tau_{t_0,\varepsilon}$. We will show later that $\delta = c_1\varepsilon^3 + \circ(e^3)$. Thus by Definition 2.1 we have

$$|\tau_{t_0,\varepsilon}(t_1) - \tau_{t_0,\varepsilon}(t_2)| = |\tau(t_1) - \tau(t_2)| + c_2(t_1, t_2)\varepsilon^2 + o(e^2)$$

if $D(t_0, t_1) < \varepsilon$ or $D(t_0, t_2) < \varepsilon$. Then we may conclude that

$$E(\tau_{t_0,\varepsilon}) = E(\tau) + c_3\varepsilon^3 + o(\varepsilon^3) \quad \text{and} \quad E(\tilde\tau_{t_0,\varepsilon}) = E(\tau) + c_4\varepsilon^3 + o(\varepsilon^3).$$

The coefficients c_3 and c_4 of the term ε^3 will be called the *variation* and denoted by $Var(\tau_{t_0,\varepsilon})$ and $Var(\tilde\tau_{t_0,\varepsilon})$ respectively.

Now all is prepared for the definition of a locally extremal point of a knot.

Definition 2.3. Any $t_0 \in S^1$ is called *locally extremal point* of τ if $Var(\tilde\tau_{t_0,\varepsilon}) = 0$ for each locally perturbed knot $\tilde\tau_{t_0,\varepsilon}$ of length 2π.

Definition 2.4. The knot τ is said to be *locally extremal* if all its points are locally extremal.

Let us find necessary and sufficient conditions for the point t_0 be locally extremal. We denote the vector product of two vectors a and b by $[a, b]$. By (a, b, c) we denote the mixed product (oriented volume) of the vectors a, b and c. Let $\dot\tau(t)$ be the velocity vector and $\ddot\tau(t)$ be the acceleration vector.

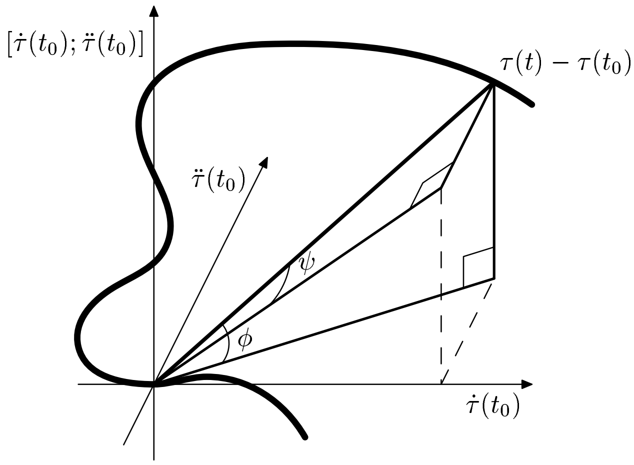

FIGURE 1. The geometric interpretation of $\psi(t_0,t)$ and $\phi(t_0,t)$.

Now we define the functions $\Psi(t_0,t)$ and $\Phi(t_0,t)$.

$$\Psi(t_0,t)=\begin{cases}\left(\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|},\frac{\ddot\tau(t_0)}{|\ddot\tau(t_0)|},\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|}\right) & \text{, if } \ddot\tau(t_0)\neq 0;\\ \left(\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|},\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|}\right) & \text{, if } \ddot\tau(t_0)=0.\end{cases}$$

$$\Phi(t_0,t)=\begin{cases}\left(\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|},\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|},\left[\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|},\frac{\ddot\tau(t_0)}{|\ddot\tau(t_0)|}\right]\right) & \text{, if } \ddot\tau(t_0)\neq 0;\\ 0 & \text{, if } \ddot\tau(t_0)=0.\end{cases}$$

Note that $|\dot\tau(t_0)|=1$ and $|\tau(t)-\tau(t_0)|\neq 0$ if $t\neq t_0$. Thus Ψ and Φ are well defined.

We also remark that $\Psi(t_0,t)=\sin\psi(t_0,t)$, where $\psi(t_0,t)$ is the angle between the vector $\tau(t)-\tau(t_0)$ and the oriented plane spanning of $\dot\tau(t_0)$ and $\ddot\tau(t_0)$. The function Φ has a similar representation: $\Phi(t_0,t)=\sin\phi(t_0,t)$, where $\phi(t_0,t)$ is the angle between the vector $\tau(t)-\tau(t_0)$ and the oriented plane spanning of $\dot\tau(t_0)$ and $[\dot\tau(t_0),\ddot\tau(t_0)]$. (See Fig. 1). These angles can be either positive or negative.

Theorem 2.1. *Let τ be a smooth knot. The point t_0 is a locally extremal point of τ if and only if the following conditions hold:*

$$V_1(t_0):=\frac{2}{3R(t_0)}\left(4\int\limits_{S^1}\Big(f+R(t_0)\Phi(t_0,t)\frac{\partial f}{\partial\rho}\Big)dt-\frac{1}{\pi}\iint\limits_{S^1\times S^1}\Big(2f+D(t_1,t_2)\frac{\partial f}{\partial\rho}+|\tau(t_1)-\tau(t_2)|\frac{\partial f}{\partial\alpha}\Big)dt_1dt_2+2\iint\limits_{A}\frac{\partial f}{\partial\alpha}dt_1dt_2\right)=0;$$

$$V_2(t_0):=\frac{4}{3R(t_0)}\int\limits_{S}\frac{\partial f}{\partial\rho}\Psi(t_0,t)dt=0.$$

Here $A\subset S^1\times S^1$ is the set of points (t_1,t_2) such that $D(t_1,t_2)=D(t_1,t_0)+D(t_0,t_2)$.

The proof of Theorem 2.1 see in [5].

Corollary 2.1. *A knot τ is locally extremal if and only if almost all of its points are locally extremal, i.e.,*

$$\int\limits_{S^1} \Big(V_1^2(t) + V_2^2(t) \Big) dt = 0.$$

In [1] it is shown that the circle is not always the global maximum, or the global minimum for the energy considered. However the circle is a locally extremal knot for any energy E satisfying the conditions 1), 2) of the Introduction.

Corollary 2.2. *The circle is always a locally extremal knot.*

The proof of Corollary 2.2 is given in [5].

Now let us say a few worlds about Möbius energy which is (in the version from [3])

$$f_M = \frac{1}{|\tau(t_1) - \tau(t_2)|^2} - \frac{1}{D^2(t_1, t_2)}.$$

It has many remarkable properties (see [8] and [3]). Möbius energies of homothetic knots are equal. This energy is invariant for Möbius transformations (see also Section 4). The variational equations and the gradient flow equation of Möbius energy was studied in [3].

Unfortunately, for Möbius energy, the variation Var is always infinite, and this mean that we can not perturb the knot in the way considered above.

The main property of Möbius energy is as follows. When a knot crossing tends to a double point, the energy tends to infinity. The energy is always positive. So every topological type of knot has a representative with minimal value of energy, some normal form.

Notice that the main part of Möbius energy is $1/|\tau(t_1) - \tau(t_2)|^2$. The other part $1/D^2(t_1, t_2)$ is only a normalization that makes the integral convergent. So let us make another normalization of the "main part" of Möbius energy. In this case we often lose the invariance for Möbius transformations. Let us consider the following energy:

$$\tilde{f} = \frac{D^3(x, y)}{|\tau(x), \tau(y)|^2}.$$

It is easily seen that this energy on one hand has the above property and on the other we can use our variational principles. Note also that such an energy is the same for homothetic knots.

Corollary 2.3. *We present V_1 and V_2 for this energy:*

$$V_1(t_0) = \frac{2}{3R(t_0)} \Bigg(4 \int\limits_{S^1} \Big(\frac{|\tau(t) - \tau(t_0)|^3}{D(t, t_0)^2} \Big(1 - 2\frac{R(t_0)}{D(t, t_0)} \Phi(t_0, t) \Big) \Big) dt -$$

$$\frac{3}{\pi} \iint\limits_{S^1 \times S^1} \frac{|\tau(t_2) - \tau(t_1)|^3}{D(t_2, t_1)^2} dt_1 dt_2 + 6 \iint\limits_{A} \frac{|\tau(t_2) - \tau(t_1)|^2}{D(t_2, t_1)^2} dt_1 dt_2 \Bigg);$$

$$V_2(t_0) = -\frac{8}{3R(t_0)} \int\limits_{S} \frac{|\tau(t_1) - \tau(t_2)|^3}{D(t_0, t)^3} \Psi(t_0, t) dt.$$

3. Definition and some basic properties of Mm-energy

In this section we define the Mm-energy of a knot. The nature of this energy differs from the energies considered in the previous sections.

Let us fix some point t_0 on the circle and define the real number $f_{Mm}(t_0)$. Consider the map $\rho_{t_0} : S^1 \longrightarrow \mathbb{R}$ such that $\rho_{t_0}(t) = |\tau(t) - \tau(t_0)|$. Let us note that the map τ is smooth. Hence ρ_{t_0} is

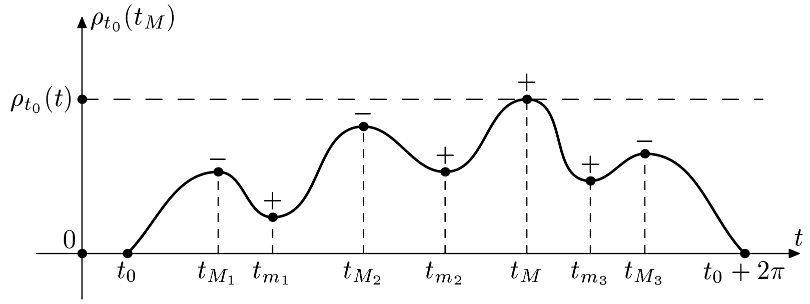

FIGURE 2. The function ρ_{t_0}.

also smooth except for one point t_0. If the number of maximums and minimums is finite, then we define the function f_{Mm} as follows:

$$f_{Mm}(t_0) = \frac{1}{\rho_{t_0}(t_M)} + \sum_{t_{m_i}\in U_1} \frac{1}{\rho_{t_0}(t_{m_i})} - \sum_{t_{M_j}\in U_2} \frac{1}{\rho_{t_0}(t_{M_j})},$$

where t_M is one of the points where the function ρ_{t_0} achieves its global maximum; U_1 is the set of all points of the circle, except the point t_0, where the function ρ_{t_0} has local minimums; U_2 is the set of all points of the circle, except the point t_M, where the function ρ_{t_0} has local maximums (see Fig. 2). Here we suppose $t_0 < t_* < t_0 + 2\pi$. In the case of an infinite number of maximums and minimums we make a small smooth perturbation $\tilde{\rho}_{t_0}$ so that the number of minimums and maximums becomes finite. Now we can calculate the value of $\tilde{f}_{Mm}(t_0)$ for the function $\tilde{\rho}_{t_0}$ as it was made before. Finally we define the $f_{Mm}(t_0)$ as the limit of $\tilde{f}_{Mm}(t_0)$ in the C^∞-topology.

Now we define the Mm-energy.

Definition 3.1. We call *Mm-energy* of the given knot the following number:

$$E_{Mm}(\tau) = \int_{S^1} f_{Mm}(t)dt,$$

if the integral converges.

Remark 3.1. Consider some small smooth perturbation of a knot. Then for any point t_0 of the circle the function ρ_{t_0} is also perturbed in a smooth way. At a generic point four possible modifications in the sums of f_{Mm} can occur: small changes of the values of the maximums and minimums; the death of one maximum and of the neighboring minimum; conversely, the birth of one maximum and minimum at some point; a local maximum close to the global maximum can become the global maximum. In all these cases the variation of the resulting f_{Mm} is small. This is the reason why the Mm-energy depends on small perturbations of knots continuously.

Further we formulate the basic properties of Mm-energy.

Proposition 3.1. *The Mm-energy is greater than or equal to* 2.

Consider the sum

$$f_{Mm}(t_0) = \frac{1}{\rho_{t_0}(t_M)} + \sum_{t_{m_i}\in U_1} \frac{1}{\rho_{t_0}(t_{m_i})} - \sum_{t_{M_j}\in U_2} \frac{1}{\rho_{t_0}(t_{M_j})}$$

We can fix the ordering of the minimums and the maximums in the standard way:

$$t_0 < t_{M_1} < t_{m_1} < \ldots < t_{M_k} < t_{m_k} < t_M < t_{m_{k+1}} < t_{M_{k+1}} < \ldots < t_{m_n} < t_{M_n} < t_0 + 2\pi.$$

Then we have

$$f_{Mm}(t_0)=\sum_{i=0}^{k}\Big(\frac{1}{\rho_{t_0}(t_{m_i})}-\frac{1}{\rho_{t_0}(t_{M_i})}\Big)+\frac{1}{\rho_{t_0}(t_M)}+\sum_{i=k+1}^{n}\Big(\frac{1}{\rho_{t_0}(t_{m_i})}-\frac{1}{\rho_{t_0}(t_{M_i})}\Big)\geq$$
$$0+\frac{1}{\rho_{t_0}(t_M)}+0=\frac{1}{\rho_{t_0}(t_M)}.$$

Finally, note that the length of the knot is 2π, hence the function $\rho_{t_0}(t_M)$ is smaller than or equal to π. Therefore

$$E_{Mm}(\tau)=\int\limits_{S^1}f_{Mm}(t)dt\leq\int\limits_{S^1}\frac{1}{\rho_t(t_{M_t})}dt\leq\int\limits_{S^1}\frac{1}{\pi}dt=\frac{2\pi}{\pi}=2.$$

This completes the proof of Proposition 3.1.

Proposition 3.2. *The Mm-energy is an invariant of homothety.*

Suppose τ is a knot of length 2π and $\tilde{\tau}$ is a homothetic knot of length $2l\pi$, where l is the coefficient of homothety. Then $d\tilde{t}=ldt$ and $\tilde{\rho}(\tilde{t})=l\rho(t)$ for any t, and so $\tilde{f}_{Mm}(\tilde{t})=f_{Mm}(t)/l$. Thus we obtain

$$E_{Mm}(\tilde{\tau})=\int\limits_{S^1}\tilde{f}_{Mm}(\tilde{t})d\tilde{t}=\int\limits_{S^1}\frac{f_{Mm}(t)}{l}ldt=\int\limits_{S^1}f_{Mm}(t)dt=E_{Mm}(\tau).$$

Proposition 3.2 is proven.

So we can consider knots without any restriction on their lengths.

Proposition 3.3. *When two branches of the knot tends to a double crossing, the Mm-energy tends to infinity.*

Consider a smooth family $\{\tau_\lambda|\lambda\in[0,1]\}$ such that τ_0 is a smooth knot with double crossing and $\tau_\lambda,\lambda\neq 0$ is a smooth knot without any double crossing. For every ε we can choose a sufficiently small λ satisfying the following conditions: there exist two points t_1 and t_2 with $|t_1-t_2|<\varepsilon^2$ such that the functions ρ_{t_1} and ρ_{t_2} have global minima at the points t_2 and t_1 correspondingly; and the ball $B_{\varepsilon,p}$ of radius ε with center at the midpoint p of the segment $[\tau_\lambda(t_1),\tau_\lambda(t_2)]$ has only two connected components of a knot τ_λ inside.

The family is smooth, hence the curvature of all knots is bounded by some N. If $\varepsilon<1/N$, then every point t of the knot τ_λ inside the ball $B_{\varepsilon/2,p}$ has one extremum (i.e., the global minimum) of the function ρ_t inside the ball $B_{\varepsilon,p}$, and every point t of this knot inside the ball $B_{\varepsilon,p}$ has no more than one extremum (i.e., the global minimum) of ρ_t inside the ball $B_{\varepsilon,p}$. Let us estimate the energy inside the ball $B_{\varepsilon,p}$.

$$E_{Mm}(\tau_\lambda\cap B_{\varepsilon,p})>4\int\limits_{\frac{\varepsilon^2}{2}}^{\frac{\varepsilon}{2}}\frac{1}{t+\frac{\varepsilon^2}{2}}dt=4\ln(t+\frac{\varepsilon^2}{2})\Big|_{\frac{\varepsilon^2}{2}}^{\frac{\varepsilon}{2}}=4\ln\frac{\frac{\varepsilon}{2}+\frac{\varepsilon^2}{2}}{\varepsilon^2}>4\ln\frac{2}{\varepsilon}.$$

The other terms (we ignore the global minimum of ρ_t) of the function f_{Mm} changes in a smooth way, hence the Mm-energy grows to infinity.

Therefore Mm-energy separates knots from different topological classes.

The following property is an essential property of Mm-energy.

Proposition 3.4. *The Mm-energy is well defined for piecewise smooth knots with obtuse angles.*

If some point t is "near" the angle then the function ρ_t is monotone function in some neighborhood of the vertex of an angle and hence there are no minima or maxima of ρ_t in this neighborhood.

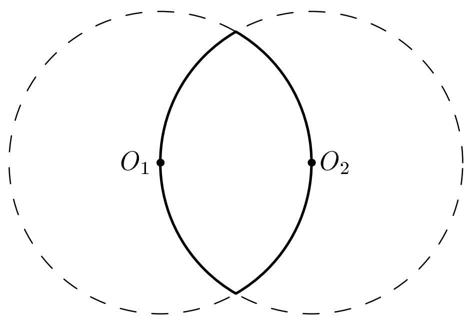

FIGURE 3. Mm-energy of this knot is $2\ln(\frac{7+4\sqrt{3}}{3})$.

In particular, the Mm-energy is well defined for piecewise linear knots with obtuse angles. So we can consider piecewise linear approximations of smooth knots and take the restriction to the set of piecewise linear knots. This property allows us to develop computer experiments in calculating normal forms for Mm-energies of topological classes of knots and the values of Mm-energies for this normal forms.

Now we calculate Mm-energy for some knots. First we find the Mm-energy of the circle τ_0

$$E_{Mm}(\tau_0) = \int\limits_{S^1} \frac{1}{2} dt = \pi.$$

Unfortunately the circle is not the normal form for the class of trivial knots. An example of the trivial knot with Mm-energy less than π is shown on Figure 3. This knot is a union of two arcs of the circle. Direct calculations shows that the Mm-energy of this knot is $2\ln(\frac{7+4\sqrt{3}}{3}) \approx 3.070607 < \pi$.

Computer experiments provide upper bounds for the Mm-energies of the normal forms for some topological classes (see the table behind).

CLASSES OF KNOTS	THE UPPER BOUNDS FOR THE ENERGIES OF NORMAL FORMS
the class of the circle	3.044012
the class of the trefoil	13.152759
the class of the figure-eight	19.450447
the class of 5_1	26.498108
the class of 5_2	27.168222
the class of 6_1	34.469191
the class of 6_2	35.466138
the class of 6_3	37.683129
the class of the connected sum of right and left trefoils	25.734616
the class of the connected sum of two right trefoils	26.748901

4. Möbius energy of the connected sum of knots.

In this section we consider only the standard Möbius energy

$$E(\tau) = \iint\limits_{S^1\times S^1} f_M dt_1 dt_2 = \iint\limits_{S^1\times S^1} \Big(\frac{1}{|\tau(t_1)-\tau(t_2)|^2} - \frac{1}{D^2(t_1,t_2)}\Big) dt_1 dt_2.$$

Denote the topological class of the knot τ by $[\tau]$ and the minimal energy for the class $[\tau]$ (the energy of the normal form of this class) by $E_{[\tau]}$. Let also $[\tau_1+\tau_2]_i$ denote all possible classes for the connected sums of the classes $[\tau_1]$ and $[\tau_2]$. From now we fix the orientations of the summands τ_1 and τ_2. This mean that we choose some class of the connected sum i.

We give some restriction for the energy of the normal form of the connected sum.

Theorem 4.1. *Let $[\tau_1]$ and $[\tau_2]$ be classes of knots. Then the following inequality holds:*

$$E_{[\tau_1+\tau_2]_i} \le E_{[\tau_1]} + E_{[\tau_2]} - 4.$$

In the proof of the Theorem 4.1, we use a nice property of Möbius energy. Möbius energy is invariant for Möbius transformations. Here we recall the theorem from [3].

Theorem 4.2 (Freedman, He, Wang)**.** *Let τ be a knot in $\mathbb{R}^3$ and let T be a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. The following statements hold:*
(i) if $T\circ\tau \subseteq \mathbb{R}^3$, then $E(T\circ\tau) = E(\tau)$;
(ii) if $T\circ\tau$ passes through ∞, then $E(T\circ\tau)=E(\tau)-4$.

Let $\varepsilon<\pi$; then we define the function $\chi_\varepsilon : [-\pi,\pi)\to\mathbb{R}$ as follows:

$$\chi_\varepsilon = \begin{cases} 1 & ,|t|<\frac{\varepsilon}{2} \\ 1-\frac{1}{2}\exp\Big(\big(\frac{4|t|}{\varepsilon}-2\big)^{-2}\Big) & ,\frac{\varepsilon}{2}\le|t|<\frac{3\varepsilon}{4} \\ \frac{1}{2}\exp\Big(\big(4-\frac{4|t|}{\varepsilon}\big)^{-2}\Big) & ,\frac{3\varepsilon}{4}\le|t|<\varepsilon \\ 0 & ,|t|\ge\varepsilon \end{cases}.$$

Further we will consider a function χ_ε as the function defined on the circle.

Let τ be a C^1_1 knot (i.e., there exist the derivative of τ and this derivative belongs to Lipschitz class), t_0 a point of this knot and r the radius of curvature at the point t_0. Then in a small neighborhood of t_0 in some orthogonal coordinates τ can be expressed

$$\tau(t)=\bigg(r\sin\Big(\frac{t-t_0}{r}\Big),\quad r\cos\Big(\frac{t-t_0}{r}\Big)+f(t-t_0),\quad g(t-t_0)\bigg),$$

where $f(t-t_0)=o((t-t_0)^2)$ and $g(t-t_0)=o((t-t_0)^2)$. Now we are ready to define τ_ε.

$$\tau_\varepsilon(t)=\tau(t)-\bigg(0,\quad f(t-t_0)\chi_\varepsilon(t-t_0),\quad g(t-t_0)\chi_\varepsilon(t-t_0)\bigg).$$

Lemma 4.1. *For any $\delta>0$ there exists some small $\varepsilon>0$ such that $|E(\tau)-E(\tau_\varepsilon)|<\delta$.*

Direct calculations show that $\chi'_\varepsilon(t)\le O(\varepsilon^{-1})$ and $\chi''_\varepsilon(t)\le O(\varepsilon^{-2})$. Thus we can obtain

$$\begin{array}{l} f\chi_\varepsilon \le O(\varepsilon^3)O(1)=O(\varepsilon^3); \\ g\chi_\varepsilon \le O(\varepsilon^3)O(1)=O(\varepsilon^3); \\ (f\chi_\varepsilon)'=f'\chi_\varepsilon+f\chi'_\varepsilon \le O(\varepsilon^2)O(1)+O(\varepsilon^3)O(\varepsilon^{-1})\le O(\varepsilon^2); \\ (g\chi_\varepsilon)'=g'\chi_\varepsilon+g\chi'_\varepsilon \le O(\varepsilon^2)O(1)+O(\varepsilon^3)O(\varepsilon^{-1})\le O(\varepsilon^2); \end{array}$$

Therefore the knot τ is the limit in the C^1_1-topology of the knots τ_ε as ε tends to 0. Möbius energy is a smooth functional from the set of C^1_1 knots in the C^1_1-topology (see [3] for the proof of this fact). Hence we can find an ε satisfying the condition of the lemma.

Lemma 4.1 is proven.

Now we consider some class of smooth maps $\gamma:\mathbb{R}\to U\subset\mathbb{R}^3$ without self-intersections, where U is described below. Consider some straight line ℓ and two point O_1 and O_2 on it. We denote the distance between O_1 and O_2 by l. Let r_1 and r_2 be two positive real numbers such that $r_1+r_1<l$. We define U as the union of two open balls B_1 and B_2 of radii r_1 and r_2 centered at O_1 and O_2 and of

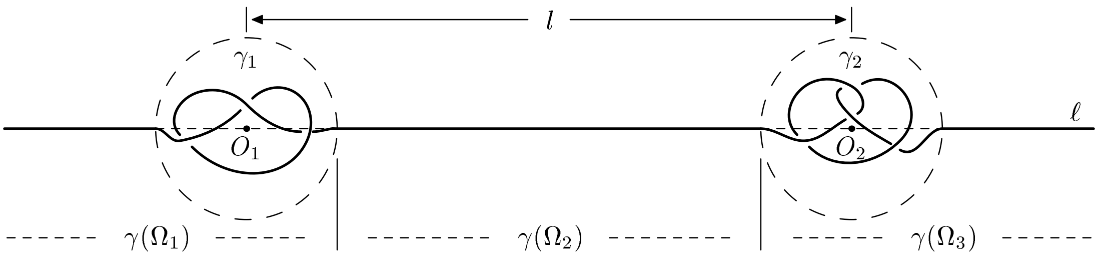

FIGURE 4. The long double knot γ.

the straight line ℓ. The map γ sends bijectively some segment and two rays to the set $\ell \setminus (B_1 \cup B_2)$. Inside the balls the map γ is smooth and has no any self-intersections. We denote the map restricted to B_1 and B_2 by γ_1 and γ_2 (see Fig. 4). Let also $\gamma(t)$ be a unit length parametrization.

Definition 4.1. We call a map from the class described above a *long double knot.*

Consider some one-parametric family of long double knots $\gamma(l)$ with fixed radii of the balls r_1 and r_2, and the fixed functions γ_1 and γ_2. The parameter of this family is $l = |O_2 - O_1| > r_1 + r_2$. Denote by q_1 and q_2 the length of the curves γ_1 and γ_2. Let also γ^- be the long double knot with the function γ_1 in the first ball and the straight segment in the second. Similarly, let γ^+ be the knot with the function γ_2 in the second ball and the straight segment in the first. We denote by Ω_2 the preimage of the central segment, and by Ω_1 and Ω_3 the connected components of $\mathbb{R} \setminus \Omega_2$ (see fig. 4).

Lemma 4.2. *For any $\varepsilon > 0$ there exists an $l > r_1 + r_2$ such that*

$$E\big(\gamma(l)\big) < E(\gamma^-) + E(\gamma^+) + \varepsilon.$$

Note that

$$E\big(\gamma(l)\big) = \Big(\iint\limits_{(\Omega_1\cup\Omega_2)\times(\Omega_1\cup\Omega_2)} + \iint\limits_{(\Omega_1\cup\Omega_2)\times\Omega_3} + \iint\limits_{\Omega_3\times(\Omega_1\cup\Omega_2)} + \iint\limits_{\Omega_3\times\Omega_3} \Big) f_M dt_1 dt_2 =$$

$$\iint\limits_{(\Omega_1\cup\Omega_2)\times(\Omega_1\cup\Omega_2)} f_M dt_1 dt_2 + \iint\limits_{(\Omega_2\cup\Omega_3)\times(\Omega_2\cup\Omega_3)} f_M dt_1 dt_2 - \iint\limits_{\Omega_2\times\Omega_2} f_M dt_1 dt_2 +$$

$$\iint\limits_{\Omega_1\times\Omega_3} f_M dt_1 dt_2 + \iint\limits_{\Omega_3\times\Omega_1} f_M dt_1 dt_2 <$$

$$E(\gamma^-) + E(\gamma^+) - 0 + \iint\limits_{\Omega_1\times\Omega_3} f_M dt_1 dt_2 + \iint\limits_{\Omega_3\times\Omega_1} f_M dt_1 dt_2.$$

Let us estimate the last two integrals.

$$\iint\limits_{\Omega_1\times\Omega_3} f_M dt_1 dt_2 = \iint\limits_{\Omega_3\times\Omega_1} f_M dt_1 dt_2 M < \int\limits_{-\infty}^{r_1 - l/2} \int\limits_{l/2 - r_2}^{+\infty} \Big(\frac{1}{(t_1 - t_2 - q_1 - q_2)^2} - \frac{1}{(t_1 - t_2)^2} \Big) dt_1 dt_2 =$$

$$\ln\Big(\frac{l - r_1 - r_2}{l - r_1 - r_2 - q_1 - q_2} \Big) = \ln\Big(1 + \frac{q_1 + q_2}{l - r_1 - r_2 - q_1 - q_2} \Big) \overset{l \to +\infty}{\longrightarrow} 0.$$

Therefore for any $\varepsilon > 0$ the desired l exists. Lemma 4.2 is proven.

Now we prove Theorem 4.1.

Let τ_1 and τ_2 be the normal forms in the classes $[\tau_1]$ and $[\tau_2]$. Take any $\delta > 0$. We fix some t_1 and t_2. By Lemma 4.1 there exist two knots τ_{1_ε} and τ_{2_ε} with the small arcs in some neighborhood of this two points, such that

$$|E(\tau_{1_\varepsilon}) - E(\tau)| < \delta \quad \text{and} \quad |E(\tau_{2_\varepsilon}) - E(\tau)| < \delta.$$

Consider the Möbius transformations T_1 and T_2 sending the points t_1 and t_2 of the knots τ_{1_ε} a τ_{2_ε} to infinity. The arcs in the neighborhood of t_1 and t_2 map to the rays of the same straight line. Therefore we can combine $T_1 \circ \tau_{1_\varepsilon}$ and $T_1 \circ \tau_{2_\varepsilon}$ to obtain the long double knot.

By Theorem 4.2 we have:

$$E(T_1 \circ \tau_{1_\varepsilon}) = E(\tau_{1_\varepsilon}) - 4 \quad \text{and} \quad E(T_2 \circ \tau_{2_\varepsilon}) = E(\tau_{2_\varepsilon}) - 4.$$

Further, by Lemma 4.2, using the long knots $E(T_1 \circ \tau_{1_\varepsilon})$ and $E(T_2 \circ \tau_{2_\varepsilon})$ we construct the long double knot γ so that

$$E(\gamma) < E(T_1 \circ \tau_{1_\varepsilon}) + E(T_2 \circ \tau_{2_\varepsilon}) + \delta.$$

Finally, consider a Möbius transformation T which maps the long double knot γ to the knot $T \circ \gamma$. This knot belongs to the class $[\tau_1 + \tau_2]_i$. We use Theorem 4.2 again to obtain the following:

$$E_{[\tau_1+\tau_2]_i} < E(T \circ \gamma) = E(\gamma) + 4 < E(T_1 \circ \tau_{1_\varepsilon}) + E(T_2 \circ \tau_{2_\varepsilon}) - 4 + \delta =$$
$$E(\tau_{1_\varepsilon}) + E(\tau_{2_\varepsilon}) - 4 + \delta = E(\tau_1) + E(\tau_2) - 4 + 3\delta.$$

The inequality

$$E_{[\tau_1+\tau_2]_i} < E(T \circ \gamma) < E(\tau_1) + E(\tau_2) - 4 + 3\delta$$

holds for any $\delta < 0$. This proves Theorem 4.1.

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