arXiv:1309.5040v1 [math.AP] 19 Sep 2013

MEAN VALUE PROPERTY FOR NONHARMONIC FUNCTIONS

TETIANA BOIKO, OLEG KARPENKOV

Abstract. In this article we extend the mean value property for harmonic functions to the nonharmonic case. In order to get the value of the function at the center of a sphere one should integrate a certain Laplace operator power series over the sphere. We write explicitly such series in the Euclidean case and in the case of infinite homogeneous trees.

Contents

Introduction

In this article we extend the mean value property for harmonic functions to the case of nonharmonic functions. Our goals are to study this problem in Euclidean case and in the case of infinite homogeneous trees.

Mean value property for harmonic function. In what follows we denote by $S^{d-1}(r)$ the $(d-1)$-dimensional sphere in the Euclidean space $\mathbb{R}^d$ with radius r and center at the origin. Let $\mathrm{Vol}(S^{d-1}(x))$ be its volume and let $d\mu$ be the standard surface volume measure on each of the spheres.

Recall the classical *mean value property* for a harmonic function f:

$$f(0) = \frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} f d\mu.$$

Date: 18 September 2013.
Key words and phrases. Mean value property, Laplacian, discrete Laplacian, homogeneous trees.

See [4] for general reference to potential theory.

Mean value property for nonharmonic function in Euclidean space $\mathbb{R}^d$.

In Section 1 we prove the following general formula for analytic functions f under some natural convergency conditions (see Theorem A):

$$f(0)=\frac{1}{\mathrm{Vol}(S^{d-1}(r))}\int\limits_{S^{d-1}(r)}\sum_{i=0}^{\infty}\alpha_{i,d}r^{2i}\triangle^i f d\mu.$$

The coefficients $\alpha_{i,d}$ are generated as follows

$$\sum_{i=0}^{\infty}\alpha_{i,d}x^{2i}=\frac{(Ix/2)^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}\right)J_{\frac{d-2}{2}}(Ix)},$$

where J_q denotes the Bessel function of the first kind and $I=\sqrt{-1}$.

We would like to mention that for harmonic functions all $\triangle^i f=0$ (for $i\geq 1$) and hence we get a classical mean value property.

Mean value property for nonharmonic function for homogeneous trees. Harmonic functions on trees for the first time were introduced in 1972 by P. Cartier in [1]. In Section 2 of this article we show the generalized version of Poisson-Martin integral representation for holomorphic functions to the case of non-harmonic functions (under certain natural convergency conditions). For a general theory of harmonic functions on graphs and, in particular, trees we refer to [2, 3, 5].

Consider a homogeneous tree of degree $q+1$ which we denote by T_q. We prove the following formula (see Theorem B and Corollary 2.3):

$$f(v)=\frac{q}{q+1}\int\limits_{\partial T_q}\left[\sum_{i=0}^{\infty}\left((q+1)^i\Big(\gamma_i(\infty)+q^{\infty}\gamma_i(-\infty)\Big)\triangle^i f(t)\right)\right]_v dt,$$

where

$$\gamma_i(n)=c_{i,i}n^i+\ldots+c_{i,1}n+c_{i,0}, \tag{1}$$

whose collection of coefficients $c_{i,j}$ (for a fixed i) is the solution of the following linear system

$$A\begin{pmatrix}c_{i,i}\\ \vdots\\ c_{i,1}\\ c_{i,0}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} \tag{2}$$

where

$$A = \begin{pmatrix} 0 & \dots & 0 & 2 \\ 1^i(1+(-1)^i q^1) & \dots & 1^1(1+(-1)^1 q^1) & 1^0(1+(-1)^0 q^1) \\ 2^i(1+(-1)^i q^2) & \dots & 2^1(1+(-1)^1 q^2) & 2^0(1+(-1)^0 q^2) \\ \vdots & \ddots & \vdots & \vdots \\ (i-1)^i(1+(-1)^i q^{i-1}) & \dots & (i-1)^1(1+(-1)^1 q^{i-1}) & (i-1)^0(1+(-1)^0 q^{i-1}) \\ i^i(1+(-1)^i q^i) & \dots & i^1(1+(-1)^1 q^i) & i^0(1+(-1)^0 q^i) \end{pmatrix}.$$

Here we consider the integration in the following sense

$$\int\limits_{\partial T_q} \left[\sum_{i=0}^{\infty} \lambda_i(\infty) \triangle^i f(t) \right]_v dt = \lim_{n\to\infty} \left(\sum_{i=0}^{n} \lambda_i(n) \sum_{w\in S_n(v)} \triangle^i f(w) \right),$$

where $S_n(v)$ is the set of all vertices at distance n to the vertex v.

1. Generalized mean value property in $\mathbb{R}^n$

In this section we we show how to generalize the mean value property in $\mathbb{R}^n$ to the case of nonharmonic functions. Without loss of generality we study the value at the origin and take the integrals over the spheres centered at the origin. In Subsections 1.1 and 1.2 we introduce some preliminary general notions and definitions. Further in Subsection 1.3 we formulate and prove the main results concerning the mean value property in $\mathbb{R}^n$.

1.1. **Operator on $\mathbb{R}^1$ associated to the d-dimensional Laplace operator.** Consider the Laplace operator $\triangle$ on $\mathbb{R}^d$. In polar coordinates one can write

$$\triangle(f) = \triangle_r(f) + \frac{1}{r^2}\triangle_{S^{d-1}} f, \quad \text{where} \quad \triangle_r(f) = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\Big(r^{d-1}\frac{\partial f}{\partial r}\Big),$$

the radial part, and $\triangle_{S^{d-1}}$ is the Laplace–Beltrami operator on the $(d-1)$-sphere. Let us associate to the Laplace operator $\triangle$ the following operator on a real line:

$$\tilde{\triangle}_d(g) = \frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x}\Big(\frac{g(x)}{x^{d-1}}\Big)\Big).$$

Proposition 1.1. *For an analytic function f it holds*

$$\int\limits_{S^{d-1}(x)} \triangle f(v) d\mu = \tilde{\triangle}_d \left(\int_{S^{d-1}(x)} f(v) d\mu \right).$$

Proof. First, notice that for Laplace–Beltrami operator $\triangle_{S^{d-1}}$ it holds

$$\int\limits_{S^{d-1}(x)} h(v)\triangle_{S^{d-1}} f(v) d\mu = -\int\limits_{S^{d-1}(x)} \langle \operatorname{grad} h(v), \operatorname{grad} f(v)\rangle d\mu,$$

where the function grad is the gradient operator on the tangent space to the sphere $S^{d-1}(x)$, and $\langle v, w\rangle$ is the scalar product of v and w. Therefore, substituting $h = 1$ we have

$$\int\limits_{S^{d-1}(x)} \triangle_{S^{d-1}} f(v)d\mu = 0.$$

Second, we make the following transformations.

$$\begin{aligned}\tilde{\triangle}_d\Big(\int_{S^{d-1}(x)} f(v)d\mu\Big) &= \tilde{\triangle}_d\Big(x^{d-1}\int_{S^{d-1}(1)} f(xv)d\mu\Big) = \frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial}{\partial x}\Big(\int_{S^{d-1}(1)} f(xv)d\mu\Big)\right)\\ &= \int_{S^{d-1}(1)} \frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x} f(xv)\Big)d\mu = \int_{S^{d-1}(x)} \frac{1}{x^d}\frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x} f(xv)\Big)d\mu\\ &= \int_{S^{d-1}(x)} \triangle_r f(v)d\mu = \int_{S^{d-1}(x)} \triangle f(v)d\mu.\end{aligned}$$

This concludes the proof of Proposition 1.1. □

Iteratively applying Proposition 1.1 we get the following corollary.

Corollary 1.2. *For an analytic function f on $\mathbb{R}^d$ and a nonnegative integer n it holds*

$$\int\limits_{S^{d-1}(x)} \triangle^n f(v)d\mu = \tilde{\triangle}_d^n\left(\int_{S^{d-1}(x)} f(v)d\mu\right).$$

□

1.2. **Bessel functions and some important generating functions.** Let J_p denote Bessel functions of the first kind. Recall that the power series decomposition of J_p at $x = 0$ is written as

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{p+2k}}{k!\Gamma(p+k+1)}.$$

Let us define two collections of coefficients $\alpha_{i,d}$ and $\beta_{i,d}$. Recall that

$$\sum_{i=0}^{\infty} \alpha_{i,d} x^{2i} = \frac{(Ix/2)^{\frac{d-2}{2}}}{\Gamma\big(\frac{d}{2}\big) J_{\frac{d-2}{2}}(Ix)}.$$

Remark 1.3. In case if $d = 1$ and $d = 3$ we have the following

$$\operatorname{sech} x = \sum_{i=0}^{\infty} \alpha_{i,1} x^{2i} \qquad \text{and} \qquad x \operatorname{csch} x = \sum_{i=0}^{\infty} \alpha_{i,3} x^{2i}.$$

Set the coefficients $\beta_{i,d}$ as follows

$$\sum_{i=0}^{\infty} \beta_{i,d} x^{2i} = \frac{J_{\frac{d-2}{2}}(Ix)}{(Ix/2)^{\frac{d-2}{2}}},$$

Proposition 1.4. *Let k be a nonnegative integer and d be a positive integer. Then it holds*

(i) $\beta_{k,d} = \dfrac{1}{4^k k! \Gamma(p+k+1)}$;

(ii) $\sum\limits_{i=0}^{k} \alpha_{i,d}\beta_{k-i,d} = \begin{cases} \frac{1}{\Gamma(d/2)}, & \text{if } k=0; \\ 0, & \text{if } k \geq 1. \end{cases}$

Proof. The first statement follows directly from the power series decomposition for the function $J_{\frac{d-2}{2}}(Ix)$. The second statement holds, since by the definition of generating functions

$$\sum_{i=0}^{\infty} \alpha_{i,1} x^{2i} \sum_{i=0}^{\infty} \beta_{i,1} x^{2i} = \frac{(Ix/2)^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}\right) J_{\frac{d-2}{2}}(Ix)} \cdot \frac{J_{\frac{d-2}{2}}(Ix)}{(Ix/2)^{\frac{d-2}{2}}} = \frac{1}{\Gamma(d/2)}.$$

□

1.3. **Generalized mean value property.** We start with several definitions.

Definition 1. For an arbitrary nonnegative integer d a smooth functions f on $\mathbb{R}^n$, and a smooth function g on $\mathbb{R}^1$ set

$$T_d(f,r)(v) = \sum_{i=0}^{\infty} \alpha_{i,d} r^{2i} \triangle^i f(v),$$

$$\tilde{T}_d g(x) = \sum_{i=0}^{\infty} \alpha_{i,d} x^{2i} \tilde{\triangle}_d^i g(x),$$

where the generating function for the coefficients $\alpha_{i,d}$ is as above.

For an arbitrary function $f : \mathbb{R}^d \to \mathbb{R}$ we denote by $\tilde{f} : \mathbb{R} \to \mathbb{R}$ the function defined as follows. For positive x we set

$$\tilde{f}(x) = \frac{1}{\mathrm{Vol}(S^{d-1}(x))} \int_{S^{d-1}(x)} f(v) d\mu.$$

For negative x we put $f(x) = f(-x)$. Finally we define

$$\tilde{f}(0) = \lim_{x \to 0} \left(\frac{1}{\mathrm{Vol}(S^{d-1}(x))} \int_{S^{d-1}(x)} f(v) d\mu \right) = f(0).$$

Definition 2. We say that a function f is *spherically a-analytic at* 0 for some $a > 0$ if the Taylor series for $\tilde{f}$ at the origin converges to $\tilde{f}$ on the segment $[-a, a]$.

Theorem A. *Consider $0 < r < a$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function that is spherically a-analytic at* 0. *Then we have*

$$f(0) = \frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} T_d(f) d\mu.$$

Example 1.5. Let a function φ on $\mathbb{R}^3$ satisfy the Poisson's equation

$$\triangle\varphi = f$$

for some harmonic function f. Then it holds

$$\varphi(0) = \frac{1}{4\pi}\int\limits_{S^2(1)}\Big(\varphi(x) - \frac{1}{6}\triangle\varphi(x)\Big)d\mu.$$

We start the proof of Theorem A with the following lemma.

Lemma 1.6. *Let k be a nonnegative integer. Then*

$$\tilde{T}_d(x^{2k+d-1}) = \begin{cases} x^{d-1}, & \text{if } k=0; \\ 0, & \text{if } n \ge d. \end{cases}$$

Proof. First, observe the following

$$\tilde{\triangle}_d x^n = (n-d+1)(n-1)x^{n-2}.$$

Therefore,

$$\tilde{\triangle}_d^i x^{2k+d-1} = 4^i \frac{k!}{(k-i)!}\frac{\Gamma(k+\frac{d}{2})}{\Gamma(k-i+\frac{d}{2})}x^{n-2i}.$$

In particular, this means that for $i > k$ we have $\tilde{\triangle}_d^i(x^{2k+d-1}) = 0$. Hence we get

$$\begin{aligned}
\tilde{T}_d(x^{2k+d-1}) &= \sum_{i=0}^{\infty}\alpha_{i,d}x^{2i}4^i\frac{k!}{(k-i)!}\frac{\Gamma(k+\frac{d}{2})}{\Gamma(k-i+\frac{d}{2})}x^{n-2i} \\
&= 4^k k!\Gamma\Big(k+\frac{d}{2}\Big)x^{2k+d-1}\sum_{i=0}^{k}\alpha_{i,d}\frac{1}{4^{k-i}(k-i)!\Gamma(k-i+\frac{d}{2})} \\
&= 4^k k!\Gamma\Big(k+\frac{d}{2}\Big)x^{2k+d-1}\sum_{i=0}^{k}\alpha_{i,d}\beta_{k-i,d} \\
&= \begin{cases} x^{d-1}, & \text{if } k=0; \\ 0, & \text{if } k \ge 0. \end{cases}
\end{aligned}$$

The last two equalities follows from Proposition 1.4(i) and Proposition 1.4(ii) respectively. □

Corollary 1.7. *Consider an even analytic function f whose Taylor series taken at 0 converges on the segment $[-a, a]$. Let also x satisfy $0 < x < a$. Then*

$$\frac{\tilde{T}_d\big(x^{d-1}f(x)\big)}{x^{d-1}} = f(0).$$

Remark. In fact, if f is not even then a more general statement holds

$$f(0) = \frac{\tilde{T}_d\big(x^{d-1}f(x)\big) + \tilde{T}_d\big(x^{d-1}f(-x)\big)}{2x^{d-1}}.$$

Proof. Let f be an even function, i.e.,

$$f(x) = \sum_{i=0}^{\infty} c_i x^{2i}.$$

Then from Lemma 1.6 we have

$$\frac{\tilde{T}_d\big(x^{d-1}f(x)\big)}{x^{d-1}} = \frac{\tilde{T}_d\big(\sum_{i=0}^{\infty} c_i x^{2i+d-1}\big)}{x^{d-1}} = \frac{\big(\sum_{i=0}^{\infty} c_i \tilde{T}_d(x^{2i+d-1})\big)}{x^{d-1}} = \frac{c_0 x^{d-1}}{x^{d-1}} = c_0 = f(0).$$

We demand the convergence of Taylor series in order to exchange the sum operation with $\tilde{T}_d$ in the second equality. □

Proof of Theorem A. By Corollary 1.2 and by the definition of $\tilde{f}$ we have

$$\int\limits_{S^{d-1}(x)} \triangle^n f(v) d\mu = \tilde{\triangle}_d^n \left(\int_{S^{d-1}(x)} f(v) d\mu \right) = \mathrm{Vol}(S^{d-1}(1)) \tilde{\triangle}_d^n \big(x^{d-1} \tilde{f}(x)\big).$$

Since f is spherically a-analytic, the function $\tilde{f}$ satisfies all the conditions of Corollary 1.7. Applying Corollary 1.7 we get

$$f(0) = \tilde{f}(0) = \frac{\tilde{T}_d\big(x^{d-1}\tilde{f}(x)\big)}{x^{d-1}} = \frac{1}{x^{d-1}\,\mathrm{Vol}(S^{d-1}(1))} \int\limits_{S^{d-1}(x)} T_d(f) d\mu$$

$$= \frac{1}{\mathrm{Vol}(S^{d-1}(x))} \int\limits_{S^{d-1}(x)} T_d(f) d\mu.$$

This concludes the proof of Theorem A. □

2. Horocyclic formula for homogeneous trees

In this section we study the situation in the discrete case of homogeneous trees. We start in Subsection 2.1 with necessary notions and definitions. Further in Subsection 2.2 we formulate the statements regarding the generalization of the Poisson-Martin integral representation theorem. In Subsection 2.3 we study some necessary tools that are further used in the proofs of the main result. We conclude the proofs in Subsection 2.4.

2.1. **Notions and definitions.**

Consider a homogeneous tree T_q (i.e., every vertex of such tree has $q+1$ neighbors) and denote its Martin boundary by ∂T_q. If v and w are connected by an edge we write $v \sim w$.

2.1.1. *Laplace operator.* In this section we consider the standard Laplace operator on the space of all functions on T_q, which is defined as

$$\triangle f(v) = \frac{\sum\limits_{w \sim v} f(w)}{q+1} - f(v).$$

The composition of $i \geq 1$ Laplace operators we denote by $\triangle^i$. Set also $\triangle^0$ the identity operator.

Remark 2.1. Similarly one might consider majority of weighted Laplace operators. The statements of this section has a straightforward generalization to arbitrary locally finite graphs. For simplicity reasons we restrict ourselves entirely to homogeneous trees.

2.1.2. *Maximal cones and horocycles.* We start with the definition of maximal proper cones.

Definition 3. Consider two vertices $v, w \in T_q$ connected by an edge e. The maximal connected component of $T_q \setminus e$ containing v is called the *maximal proper cone with vertex at* v (*with respect to* w). We denote it by C^{v-w}.

The *distance* between two vertices $v, w \in T_q$ is the minimal number of edges needed to reach the vertex w starting from the vertex v. For an arbitrary nonnegative integer r and an arbitrary vertex v we denote by $S_r(v)$ the set of all vertices at distance r to v, we call such set the *circle of radius* r *with center* v. Note that $S_r(v)$ contains exactly $(q+1)q^{r-1}$ points.

Definition 4. Let C^{v-w} be a maximal proper cone of T_q and let n be a nonnegative integer. The set

$$C_n^{v-w} = C_{v-w} \cap S_n(v)$$

is called the *horocycle of radius* n *with center at* v (*with respect to* w).

2.1.3. *Integral series.* For an arbitrary function $f : T_q \to \mathbb{R}$ we write

$$f(C_n^{v-w}) = \frac{1}{q^n} \sum_{u \in C_n^{v-w}} f(u).$$

Definition 5. In what follows we consider the *horocyclic integrals* defined by the following expression:

$$\int\limits_{\partial C^{v-w}} \left[\sum_{i=0}^{\infty} \lambda_i(\infty) \triangle^i f(t) \right] dt = \lim_{n \to \infty} \left(\sum_{i=0}^{n} \lambda_i(n) \triangle^i f(C_n^{v-w}) \right),$$

where f is a function on the tree, λ_i are arbitrary functions on the set of positive integers. Respectively we write

$$\int\limits_{\partial T_q} \left[\sum_{i=0}^{\infty} \lambda_i(\infty) \triangle^i f(t) \right]_v dt = \lim_{n \to \infty} \left(\sum_{i=0}^{n} \lambda_i(n) \sum_{u \in S^n(v)} \frac{\triangle^i f(u)}{q^n} \right).$$

Here we specify by an index v that the series are taken with respect to the vertex v, since now it is not reconstructed from the integration domain.

For instance,

$$\int\limits_{\partial C^{v-w}} [2^{\infty}(1+(-\infty)^3)f(t)]dt = \lim_{n\to\infty}\Big(2^n(1-n^3)f(C^{v-w}_{n+1})\Big) = \lim_{n\to\infty}\Big(2^n(1-n^3)\sum_{u\in C^{v-w}_n}\frac{f(u)}{q^n}\Big).$$

Remark. Notice that the limit operation is not always commute with the sum operation. To illustrate this we mention, that the expression from the limit exists for every holomorphic function even if the integral at Martin boundary diverges (see Theorem 2.4). So the notion of integral series extends the notion of integration of functions at Martin boundary.

2.2. **Horocyclic formula.** In this subsection we formulate the mean value property for certain nonharmonic functions.

2.2.1. *Horocyclic integrals for horosummable functions.* We start with the following definition.

Definition 6. We say that a functions f is C^{v-w}*-horosummable* if

$$\lim_{n\to\infty}\big(q^n f(C^{v-w}_{2n})\big) = \lim_{n\to\infty}\Big(\sum_{u\in C^{v-w}_{2n}}\frac{f(u)}{q^n}\Big) = 0.$$

Theorem B. *Consider two vertices $v, w \in T_q$ connected by an edge, and let f be a C^{v-w}-horosumable function. Then*

$$f(v) = \int\limits_{\partial C^{v-w}} \left[\sum_{i=0}^{\infty}\left((q+1)^i\Big(\gamma_i(\infty)+q^{\infty}\gamma_i(-\infty)\Big)\triangle^i f(t)\right)\right] dt,$$

where γ_i are polynomials as below (see (1)) whose coefficients are the solutions of System (2). In addition, the condition that the horocyclic integral in the right part of the equation converges to $f(v)$ is equivalent to the condition that f is C^{v-w}-horosumable.

We prove this theorem later in Subsection 2.4.

Note that it would be interesting to relate the coefficients at terms $\triangle^i(f)$ with discretizations of Bessel functions.

Example 2.2. Let us check Theorem B for the function χ_v that is zero everywhere except for the point v and $\chi_v(v) = 1$. We have

$$\triangle^i\chi_v(C^{v-w}_n) = \frac{1}{q^n}\sum_{u\in C^{v-w}_n} f(u) = \begin{cases} 0, & \text{if } i<n;\\ \frac{1}{(q+1)^n}, & \text{if } i=n.\end{cases}$$

(notice that C^{v-w}_n contains exactly q^n vertices). Therefore,

$$\int\limits_{\partial C^{v-w}} \left[\sum_{i=0}^{\infty}\left((q+1)^i\Big(\gamma_i(\infty)+q^{\infty}\gamma_i(-\infty)\Big)\triangle^i \chi_v(t)\right)\right] dt$$
$$= \lim_{n\to\infty} a_{n,n}\triangle^n\chi_v(C^{v-w}_n) = \lim_{n\to\infty}(q+1)^n\frac{1}{(q+1)^n} = 1 = \chi_v(v).$$

It is clear from this example that it is not always possible to exchange the sum operator and the limit operator. For the function χ_v we have

$$\sum_{i=0}^{\infty} \lim_{n\to\infty} \left((q+1)^i \Big(\gamma_i(n) + q^n \gamma_i(-n)\Big) \triangle^i \chi_v(C_n^{v-w}) \right) = \sum_{i=0}^{\infty} 0 = 0 \neq 1 = \chi_v(v).$$

Let us write a weaker version of Theorem B for the integration over all the Martin boundary.

Corollary 2.3. *Consider a vertex $v \in T_q$, and let f be a C^{v-w}-horosumable function for all vertices w adjacent to v. Then*

$$f(v) = \frac{q}{q+1} \int\limits_{\partial T_q} \left[\sum_{i=0}^{\infty} \left((q+1)^i \Big(\gamma_i(\infty) + q^{\infty} \gamma_i(-\infty)\Big) \triangle^i f(t) \right) \right]_v dt.$$

Proof. Let us sum up the expression obtained in Theorem B for all maximal proper cones with vertex at v. From one hand there are exactly $q+1$ such horocycles so the sum equals to $(q+1)f(v)$. From the other hand each point of the Martin boundary was integrated q times. Therefore, we get the constant $\frac{q}{q+1}$ in the statement of the corollary. □

Remark. Note that it is possible to write similar series for arbitrary locally-finite trees, although the formulas for the coefficients would be more complicated.

2.2.2. *Horocycle formula for harmonic functions.* We conclude this subsection with the following more general statement for harmonic functions.

Corollary 2.4. *Consider an arbitrary harmonic function h on a homogeneous tree T_q. Let v be a vertex of T_q and G_v be one of the corresponding horocyclic parts. Then the following holds:*

$$h(v) = \int\limits_{\partial C^{v-w}} [h(t)] dt + \int\limits_{\partial C^{v-w}} \left[q^{\infty} \Big(h(t) - \int_{\partial C^{v-w}} [h(t)] dt \Big) \right] dt.$$

□

Remark. Suppose that h is integrable on ∂C^{v-w} with respect to the probability measure on the Martin boundary. Then this integral coincides with

$$\int\limits_{\partial C^{v-w}} [h(t)] dt.$$

In case if h is not integrable with respect to probability measure, the horocyclic integral nevertheless exists. In some sense horocyclic integrability is a conditional integrability with respect to integration over probability measure. Horocyclic integral exists for every harmonic function h and for every cone C^{v-w}.

2.3. **Relations on special Laurent polynomial.** In this subsection we prove some supplementary statements. For every integer n we denote

$$D_n(x) = x^n + \frac{q^n}{x^n}.$$

Note that $D_0(x) = x^0 + \frac{q^0}{x^0}$.

For every nonnegative integer we set

$$S_n(x) = \frac{(x-1)^n(x-q)^n}{(q+1)^n x^n}.$$

We have the following recurrent relation for the defined above Laurent polynomials.

Proposition 2.5. *For every integer n we have*

$$S_1 D_n = \frac{D_{n+1} - (q+1)D_n + qD_{n-1}}{q+1}.$$

Proof. For every integer n (including $n = -1, 0, 1$) it holds

$$\begin{aligned} S_1 D_n &= \Big(\frac{(x-1)(x-q)}{(q+1)x}\Big)\Big(x^n + \frac{q^n}{x^n}\Big) \\ &= \frac{x^{n+1}}{q+1} - x^n + \frac{q}{q+1}x^{n-1} + \frac{q^n}{(q+1)x^{n-1}} - \frac{q^n}{x^n} + \frac{q^{n+1}}{(q+1)x^{n+1}} \\ &= \frac{1}{q+1}\Big(x^{n+1} + \frac{q^{n+1}}{x^{n+1}}\Big) - \Big(x^n + \frac{q^n}{x^n}\Big) + \frac{q}{q+1}\Big(x^{n-1} + \frac{q^{n-1}}{x^{n-1}}\Big) \\ &= \frac{D_{n+1} - (q+1)D_n + qD_{n-1}}{q+1}. \end{aligned}$$

□

The following proposition is straightforward.

Proposition 2.6. *For every integer n there exists a unique decomposition*

$$D_n = \sum_{i=0}^{n} a_{n,i} S_i.$$

□

Now we are interested in the coefficients $a_{n,i}$. The next statement follows directly from Proposition 2.5.

Corollary 2.7. *For every positive integer i and every integer n it holds*

$$a_{n,i-1} = \frac{a_{n+1,i} - (q+1)a_{n,i} + qa_{n-1,i}}{q+1}.$$

Additionally in the case $i = 0$ it holds

$$0 = a_{n+1,0} - (q+1)a_{n,0} + qa_{n-1,0}.$$

Proof. By the definition we have

$$S_1 S_k = S_{k+1}.$$

Propositions 2.5 and 2.6 imply

$$\sum_{i=1}^{n+1} a_{n,i-1} S_i = S_1 D_n = \frac{D_{n+1} - (q+1)D_n + qD_{n-1}}{q+1}$$
$$= \frac{1}{q+1}\Big(\sum_{i=0}^{n+1} a_{n+1,i} S_i - (q+1)\sum_{i=0}^{n} a_{n,i} S_i + q\sum_{i=0}^{n-1} a_{n-1,i} S_i\Big).$$

Collecting the coefficients at S_i we get the recurrence relations of the corollary. □

Definition 7. For a positive integer k we define the linear form L_k in $2k+1$ variables as follows

$$L_k(y_1, \ldots, y_{2k+1}) = \sum_{i=-n}^{n} c_{i,n} y_i,$$

where $c_{i,n}$ are defined as the coefficients of S_n, i.e., from the expression

$$S_n(x) = \frac{(x-1)^n (x-q)^n}{(q+1)^n x^n} = \sum_{i=-n}^{n} c_{i,n} x^i.$$

Proposition 2.8. *For every nonnegative integer i and every integer n we have*

$$L_i(a_{n-i,i}, a_{n-i+1,i}, \ldots, a_{n+i,i}) = 0.$$

Proof. We prove the proposition by induction in i.

Base of induction. For the case $i = 0$ the statement holds by Corollary 2.7.

Step of induction. Suppose that the statement holds for $i-1$. Let us prove it for i. We have

$$L_i(a_{n-i,i}, \ldots, a_{n+i,i}) = 0.$$

By Corollary 2.7 and linearity of L_i we have

$$\begin{aligned} &L_i(a_{n-i,i}, \ldots, a_{n+i,i}) \\ &\quad = L_i\big(\tfrac{a_{n-i+1,i+1}-(q+1)a_{n-i,i+1}+qa_{n-i-1,i+1}}{q+1}, \ldots, \tfrac{a_{n+i+1,i+1}-(q+1)a_{n-i,i+1}+qa_{n-i-1,i+1}}{q+1}\big) \\ &\quad = \frac{1}{q+1}\Big(L_i(a_{n-i+1,i+1}, \ldots, a_{n+i+1,i+1}) - (q+1)L_i(a_{n-i,i+1}, \ldots, a_{n+i,i+1}) \\ &\qquad + qL_i(a_{n-i-1,i+1}, \ldots, a_{n+i-1,i+1})\Big) \\ &\quad = L_{i+1}(a_{n-i-1,i+1}, a_{n-i,i+1}, \ldots, a_{n+i,i+1}, a_{n+i+1,i+1}). \end{aligned}$$

Therefore, by induction assumption we have

$$L_{i+1}(a_{n-i-1,i+1}, a_{n-i,i+1}, \ldots, a_{n+i,i+1}, a_{n+i+1,i+1}) = L_i(a_{n-i,i}, \ldots, a_{n+i,i}) = 0.$$

This concludes the proof of the induction step. □

Corollary 2.9. *For every fixed nonnegative integer k we have the*

$$a_{n,k} = P_k(n) + q^n \hat{P}_k(n),$$

where $P_k(n)$ and $\hat{P}_k(n)$ are polynomials of degree at most k. □

We skip the proof here. This is a general statement about linear recursive sequences whose characteristic polynomial has roots 1 and q both of multiplicity n.

Example 2.10. Direct calculations show that in case $q = 2$ we have

$$\begin{aligned}
&a_{n,0} = 1 + 2^n,\\
&a_{n,1} = \tfrac{3^1}{1!}(-n + 2^n n),\\
&a_{n,2} = \tfrac{3^2}{2!}\big(n^2 + 3n + 2^n(n^2 - 3n)\big),\\
&a_{n,3} = \tfrac{3^3}{3!}\big(-n^3 - 9n^2 - 26n + 2^n(n^3 - 9n^2 + 26n)\big),\\
&\dots
\end{aligned}$$

Let us prove a general theorem on numbers $a_{n,i}$.

Theorem 2.11. *For every admissible k and n it holds*

$$a_{n,k} = (q+1)^k \big(\gamma_k(n) + q^n \gamma_k(-n)\big),$$

where the coefficients of γ_k are defined by System (2).

We start the proof of Theorem 2.11 with the following two lemmas.

Lemma 2.12. *For every nonnegative integer k and every n we have*

$$\hat{P}_k(-n) = P_k(n).$$

Proof. For every integer x we have

$$D_{-n} = x^{-n} + \frac{q^{-n}}{x^{-n}} = \frac{1}{q^n}\left(\frac{q^n}{x^n} + x^n\right) = \frac{D_n}{q^n}.$$

By Proposition 2.6 the coefficients $a_{n,i}$ and $a_{-n,i}$ are uniquely defined, therefore,

$$a_{n,k} = q^n a_{-n,k}.$$

Let us rewrite this equality in terms of polynomials P_k and $\hat{P}_k$:

$$P_k(n) + q^n \hat{P}_k(n) = q^n (P_k(-n) + q^{-n} \hat{P}_k(-n)),$$

and hence

$$P_k(n) + q^n \hat{P}_k(n) = \hat{P}_k(-n) + q^n P_k(-n).$$

Since this equality is fulfilled for every n we have $\hat{P}_k(-n) = P_k(n)$. This concludes the proof. □

Lemma 2.13. *For every nonnegative k it holds*

$$P_k(k) + q^k P_k(-k) = (q+1)^k$$

Proof. We prove the proposition by induction in k.

Base of induction. For the case $k=0,1$ we have

$$P_0(0)+q^0P_0(0)=a_{0,0}=1 \qquad \text{and} \qquad P_1(1)+qP_1(-1)=a_{1,1}=q+1.$$

Step of induction. Let $P_k(k)+q^kP_k(-k)=(q+1)^k$. Then

$$(q+1)^k=P_k(k)+q^kP_k(-k)=a_{k,k}=\frac{a_{k+1,k+1}-(q+1)a_{k,k+1}+qa_{k-1,k+1}}{q+1}=\frac{a_{k+1,k+1}}{q+1}$$
$$=\frac{P_{k+1}(k{+}1)+q^{k+1}P_{k+1}(-k{-}1)}{q+1}.$$

The third equality follows from the recursive formula of Corollary 2.7. Hence

$$P_{k+1}(k{+}1)+q^{k+1}P_{k+1}(-k{-}1)=(q+1)^{k+1}.$$

This concludes the step of induction. □

Proof of Theorem 2.11. From Lemma 2.12 we know that $\hat{P}_k(-n)=P_k(n)$. In addition, by Corollary 2.9 the degree of P_k equals to k, and hence it has $k{+}1$ coefficient. The coefficients of the polynomial P_k are uniquely defined by the conditions for $a_{j,k}$ for $j=0,\ldots,k$:

$$P_k(j)+q^jP_k(-n)=0 \text{ for } j=0,\ldots,k-1, \quad \text{and} \quad P_k(k)+q^kP_k(-k)=(q+1)^k.$$

The expression for k follows from Lemma 2.13. We consider these equalities as linear conditions on the coefficients of the polynomial $\frac{P_k}{(q+1)^k}$. These conditions form a linear system, which coincides with System (2) (substituting k to i).

We should also show that the determinant of the matrix in System (2) is nonzero. We prove this by reductio ad absurdum. Suppose the determinant of the matrix is zero. Thus, it has a nonzero kernel. Therefore, there exists an expression

$$R(n)=r(n)+r(-n)q^n,$$

where $r(n)$ is a polynomial of degree k having at least one nonzero coefficient, satisfying

$$R(-k)=R(-k+1)=\ldots=R(k)=0.$$

Let $R(k{+}1)=a$. Let us find the value $R(-k{-}1)$. From one hand, our sequence satisfy the linear recursion condition determined by the coefficients of the polynomial $(x{-}1)^k(x{-}q)^k$, and hence

$$R(-k-1)=-\frac{a}{q^{k+1}}.$$

From another hand,

$$R(-k-1)=r(-k-1)+r(k+1)q^{-k-1}=\frac{r(k+1)+r(-k-1)q^{k+1}}{q^{k+1}}=\frac{a}{q^{k+1}}.$$

This implies that $a=0$, and hence $R(k+1)=R(-k-1)=0$.

Therefore, the linear recursive sequence $R(n)$ determined by the coefficients of the polynomial of degree $2k+3$ has $2k+3$ consequent elements equal zero. Hence $R(n) = 0$ for any integer n, which implies that all the coefficients of $r(n)$ equal zero. We come to the contradiction. Hence the determinant of the matrix in System (2) is nonzero.

So both the coefficients of $\frac{P_k}{(q+1)^k}$ and the coefficients of γ_k are solutions of System (2). Since System (2) has a unique solution, the polynomials P_k and $(q+1)^k\gamma_k$ coincide. Therefore, by Lemma 2.12 it holds

$$a_{n,k} = P_k(n) + q^n\hat{P}_k(n) = P_k(n) + q^n P_k(-n) = (q+1)^k\big(\gamma_k(n) + q^n\gamma_k(-n)\big).$$

This concludes the proof of Theorem 2.11. □

Observe the following corollary.

Corollary 2.14. *For every integer $k > 0$ we have $P_k(0) = 0$, and $P_0(1) = 1$.* □

2.4. **Proof of Theorem B.** Finally we have all necessary tools to prove of Theorem B. We start with the following lemma.

Lemma 2.15. *Let f be a function on T_q and v, w be two vertices of T_q connected by an edge. Then for every nonnegative n it holds*

$$f(v) + q^n f(C_{2n}^{v-w}) = \sum_{k=0}^{n}\Big((q+1)^k\big(\gamma_k(n) + q^n\gamma_k(-n)\big)\triangle^k f(C_n^{v-w})\Big).$$

Proof. For $0 < k \le n$ set

$$\hat{D}_{k,n} = f(C_{n-k}^{v-w}) + q^k f(C_{n+k}^{v-w}),$$
$$\hat{S}_{k,n} = \sum_{i=-k}^{k} c_{i,k} f(C_{n+i}^{v-w}),$$

where the coefficients $c_{i,k}$ are generated by

$$S_k = \frac{\big((x-1)(x-q)\big)^k}{(q+1)^k x^k} = \sum_{i=-k}^{k} c_{i,k}x^i.$$

Notice that all linear expressions over S_k and D_k are identically translated to the linear expressions over $\hat{S}_{k,n}$ and $\hat{D}_{k,n}$. Then from Proposition 2.6 it follows

$$f(v) + q^n f(C_{2n}^{v-w}) = \hat{D}_{n,n} = \sum_{k=0}^{n} a_{n,k}\hat{S}_{k,n},$$

where the coefficients $a_{n,k}$ as in Theorem 2.11, i.e.,

$$a_{n,k} = (q+1)^k(\gamma_k(n) + q^n\gamma_k(-n)),$$

where the coefficients of γ_k are defined by System (2). In addition note that

$$\hat{S}_{k,n} = \triangle^k(C_n^{v-w}).$$

Therefore, we obtain

$$f(v)+q^n f(C^{v-w}_{2n}) = \sum_{k=0}^{n}\Big((q+1)^k\big(\gamma_k(n)+q^n\gamma_k(-n)\big)\triangle^k f(C^{v-w}_{n})\Big).$$

This concludes the proof. □

Proof of Theorem B. From Lemma 2.15 we have

$$f(v)+q^n f(C^{v-w}_{2n}) = \sum_{i=0}^{n}\Big((q+1)^i\big(\gamma_i(n)+q^n\gamma_i(-n)\big)\triangle^i f(C^{v-w}_{n})\Big).$$

Hence,

$$\begin{aligned}\int\limits_{\partial C^{v-w}} & \left[\sum_{i=0}^{\infty}\left((q+1)^i\Big(\gamma_i(\infty)+q^{\infty}\gamma_i(-\infty)\Big)\triangle^i f(t)\right)\right]dt \\ &= \lim_{n\to\infty}\sum_{i=0}^{n}\left((q+1)^i\Big(\gamma_i(n)+q^n\gamma_i(-n)\Big)\triangle^i f(C^{v-w}_{n})\right)\\ &= \lim_{n\to\infty}\big(f(v)+q^n f(C^{v-w}_{2n})\big) = f(v)+\lim_{n\to\infty}\big(q^n f(C^{v-w}_{2n})\big)\\ &= f(v).\end{aligned}$$

Therefore, the integral converges to $f(v)$ if and only if the sequence $\big(q^n f(C^{v-w}_{2n})\big)$ converges to zero as n tends to infinity. This means that f is C^{v-w}-horosumable. This concludes the proof. □

Acknowledgements. The authors are grateful to Prof. Woess for constant attention to this work. This first author is supported by the Austrian Science Fund (FWF): W1230, Doctoral Program "Discrete Mathematics".

References

[1] P. Cartier. Fonctions harmoniques sur un arbre. In *Symposia Mathematica, Vol. IX (Convegno di Calcolo delle Probabilità, INDAM, Rome, 1971)*, pages 203–270. Academic Press, London, 1972.

[2] J. L. Doob. *Classical potential theory and its probabilistic counterpart*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1984 edition.

[3] E. B. Dynkin. The boundary theory of Markov processes (discrete case). *Uspehi Mat. Nauk*, 24(2 (146)):3–42, 1969.

[4] L. L. Helms. *Introduction to potential theory*. Robert E. Krieger Publishing Co., Huntington, N.Y., 1975. Reprint of the 1969 edition, Pure and Applied Mathematics, Vol. XXII.

[5] W. Woess. *Denumerable Markov chains*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2009. Generating functions, boundary theory, random walks on trees.

Tetiana Boiko, TU Graz, 30/III, Steyrergasse, 8010, Graz, Austria
E-mail address: boiko@math.tugraz.at

Oleg Karpenkov, University of Liverpool, Mathematical Sciences Building, Liverpool L69 7ZL, United Kingdom
E-mail address: karpenk@liv.ac.uk