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On offsets and curvatures for discrete and semidiscrete surfaces

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Abstract This paper studies semidiscrete surfaces from the viewpoint of parallelity, offsets, and curvatures. We show how various relevant classes of surfaces are defined by means of an appropriate notion of infinitesimal quadrilateral, how offset surfaces behave in the semidiscrete case, and how to extend and apply the mixed-area based curvature theory which has been developed for polyhedral surfaces.

Keywords Semidiscrete surfaces · Curvatures

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A semidiscrete surface $x(i, u)$ is a mapping from $\mathbb{Z} \times \mathbb{R}$ to some vector space, i.e., a bivariate function of one discrete and one continuous variable. Such mixed continuous-discrete objects classically occur in the transformation theory of surfaces. For instance, a pair $x(u, v)$ and $x^+(u, v)$ of surfaces is seen as a semidiscrete mapping defined in $\{0, 1\} \times \mathbb{R}^2$. The viewpoint of smooth parameterized surfaces as limits of discrete nets — systematically exploited by [2] — directly leads to semidiscrete objects, if limits do not apply to all variables, only to some of them. In this way the theory of smooth surfaces, their transformations, and the permutability of their transformations appear as limit cases of a discrete master theory of discrete nets and integrable systems.

This paper is concerned with semidiscrete objects of very simple type. They fit the larger theory if they are considered as a transformation sequence of smooth curves, but they are interesting in their own right and in fact they have turned up in geometry processing applications [8].

Even if in many senses semidiscrete surfaces are limit cases of discrete ones and their properties are similar to both the discrete and continuous cases, they

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nevertheless deserve separare study [14]. Classes of surfaces already treated are the asymptotic surfaces of constant Gaussian curvature [15], the isothermic surfaces [6], and the conjugate surfaces and their circular and conical reductions, which are also relevant for applications [8].

This paper demonstrates how the concept of parallel surfaces (i.e., Combescure transforms) and offsets (i.e., parallel surfaces at constant distance) lead to a theory of curvatures. For smooth surfaces this topic is classical: Steiner's formula

$$dA(x^\tau) = (1 - 2H\tau + K\tau^2)\, dA(x),$$

on the surface area element of an offset at distance τ belongs here. A discrete theory has been given by [7] and [1]. Besides applying this idea to the semidiscrete case, this paper also extends our knowledge of discrete curvatures, for instance by the formula

$$H(f) = -\frac{1}{2\,\mathrm{area}(f)} \sum_{e \in \partial f} \tan\frac{\alpha_e}{2} \times \mathrm{len}(e),$$

for the mean curvature of a face of a *conical* polyhedral surface, more details of which are found in the text below.

1 Smooth, Semidiscrete, and Discrete Surfaces

We define a *net* as a mapping $x : \mathbb{Z}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^n$, which depends on discrete parameters $i_1, \dots, i_k$ and continuous parameters $u_{k+1}, \dots, u_m$. We do not insist on entire $\mathbb{Z}^k \times \mathbb{R}^{m-k}$ as the domain where x is defined, since our study concerns local properties. For the discrete parameter i_r we use the notation x_r for an index shift:

$$x_r(\dots, i_r, \dots) = x(\dots, i_r + 1, \dots),$$

and x_{-r} denotes the inverse shift. Differentiation with respect to i_r is done by the forward difference operator: $\delta_r x = x_r - r$. For the continuous parameters we use partial derivatives $\delta_j x = \frac{\partial x}{\partial u_j}$. For us the most important case is $k = m - k = 1$, i.e.,

$$x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3, \quad x = x(i, u). \tag{1}$$

The derivatives $\delta_1 x, \delta_2 x$ with respect to the one discrete and the one continuous parameter are written as

$$\Delta x(i, u) = x(i + 1, u) - x(i, u), \quad \dot{x}(i, u) = \frac{\partial x}{\partial u}(i, u).$$

A semidiscrete net $x(i, u)$ is visualized as piecewise-smooth surface, namely as the union of line segments

$$(1 - v)x(i, u) + v x(i + 1, u), \quad \text{where } v \in [0, 1]. \tag{2}$$

Recall that this strip is a developable surface, if and only if

$$\{\Delta x, \dot{x}, \dot{x}_1\} \quad \text{linearly dependent.}$$

1.1 **Infinitesimal quadrilaterals**

While the definition of certain classes of smooth surfaces (conjugate, circular, etc.) requires 2nd order derivatives, the analogous definition in the discrete category frequently involves only geometric properties of single faces. It turns out that for semidiscrete surfaces, there is a natural notion of *infinitesimal quadrilateral* which allows a similar approach. Motivated by the decomposition

$$\boxed{\begin{matrix} x & x_1 \\ x_2 & x_{12} \end{matrix}} = \boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}} + \boxed{\begin{matrix} 0 & 0 \\ \delta_2 x & \delta_2 x_1 \end{matrix}}$$

of an elementary quadrilateral of a discrete surface $x : \mathbb{Z}^2 \to U$ we define:

Definition 1.1. *An infinitesimal n-gon is a tangent vector in the affine space of n-gons; we use the notation $(P;V)$ for a tangent vector representing any smooth path $P(\tau)$ of n-gons with $P(0) = P$, $\frac{d}{d\tau}P(0) = V$. The elementary quadrilaterals $(P(i,u);V(i,u))$ of the semidiscrete surface $x(i,u)$ are represented by*

$$P = \boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}}, \quad V = \boxed{\begin{matrix} 0 & 0 \\ \dot{x} & \dot{x}_1 \end{matrix}}. \tag{3}$$

Recall the notions of conjugate and circular discrete surfaces which are characterized by elementary quads being planar or possessing a circumcircle. Conical surfaces have the property that faces adjacent to a vertex always touch some common right circular cone [2,4]. The following definitions for semidiscrete surfaces are natural extensions and have in fact already been given by [8,6]:

Definition 1.2. *A semidiscrete surface $x(i,u)$ is regular/conjugate/circular/ conical $\iff$ the respective condition listed below is fulfilled for all i, u.*

1. *Regularity is linear independence of both $\{\dot{x}, \Delta x\}$ and $\{\dot{x}_{-1}, \Delta x\}$.*
2. *Conjugacy means regularity and linear dependence of $\{\Delta x, \dot{x}, \dot{x}_1\}$.*
3. *Circularity means that if in addition there is a circle passing through x and x_1 such that derivatives $\dot{x}$, $\dot{x}_1$ are tangent there.*
4. *The conical property is conjugacy and existence of a right circular cone with axis through x which touches the adjacent ruled strips along the rulings $x \vee x_1$ and $x \vee x_{-1}$.*

Note that conjugacy and circularity are properties of the infinitesimal quadrilateral $(P;V)$ of Equation (3).

Corollary 1.3 (see [8])**.** *Circularity and conicality are equivalently expressed by the condition that the configuration of straight lines*

$$\begin{aligned} &x \vee x_1, \quad x + \mathrm{span}(\dot{x}), \quad x_1 + \mathrm{span}(\dot{x}_1) && \textit{(circ.)} \\ &x + \mathrm{span}(\dot{x}), \quad x \vee x_1, \quad x \vee x_{-1} && \textit{(con.)} \end{aligned}$$

respectively, is invariant w.r.t. a Euclidean symmetry.

Remark 1.4. Conjugacy of surfaces is uniformly expressed by

$$\{\delta_1 x, \delta_2 x, \delta_{12} x\} \quad \text{linearly dependent},$$

where x may be discrete, semidiscrete, or smooth. This is in accordance with the definition that a conjugate discrete net is defined as a quad mesh with planar faces, i.e., a quad mesh which is a polyhedral surface.

Remark 1.5. Recall that smooth surfaces x, x^+ constitute a Jonas pair if we have linear dependence of $\{x^+ - x, \partial_k x, \partial_k x^+\}$ for all parameters i_k, throughout the domain of definition. They form a Darboux pair if similarly there is a circle passing through x^+, x such that the partial derivatives $\partial_k x, \partial_k x^+$ are tangent there (see e.g. [2]). It follows that for a conjugate semidiscrete surface, curves $x(i, \cdot)$ and $x(i+1, \cdot)$ are a Jonas pair. If x is cirular, they constitute a Darboux pair.

1.2 Parallelity of surfaces

Combescure pairs x, x^+ of surfaces (i.e., surfaces where $\delta_i x$, $\delta_i x^+$ are parallel) turn up frequently in discrete differential geometry. We are particularly interested in surfaces at constant distance which are discussed later. We prefer to speak of *parallel* surfaces, which is written as

$$x \parallel x^+.$$

For semidiscrete surfaces, parallelity means $\Delta x \parallel \Delta x^+$ and $\dot{x} \parallel \dot{x}^+$.

Proposition 1.6. *Assuming regular surfaces, the properties of x being conjugate, circular, or conical are invariant under parallelity.*

This follows directly from Cor. 1.3. It turns out that parallelity is interesting only for conjugate surfaces:

Proposition 1.7. *Assume that x is regular, but not conjugate, i.e., we have linear dependence of $\{\dot{x}, \Delta x, \Delta \dot{x}\}$ only in a set of measure zero w.r.t. 1-dimensional Lebesgue measure in $\mathbb{R} \times \mathbb{Z}$. Then*

$$x^+ \parallel x \iff x^+, x \text{ homothetic}.$$

Proof. Let U be the ambient space which contains all surfaces. The statement is void unless $\dim U \geq 3$. We compute in the exterior algebra $\Lambda^2 U$ and show the statement by proving that the ratios $\lambda = \dot{x} : \dot{x}^+$ and $\gamma = \Delta x : \Delta x^+$ (well defined by parallelity and regularity) are constant and equal. Firstly,

$$\begin{aligned} 0 &= \partial_u(\Delta x \wedge \Delta x^+) = (\dot{x}_1 - \dot{x}) \wedge (\gamma \Delta x) + \Delta x \wedge (\lambda_1 \dot{x}_1 - \lambda \dot{x}) \\ &= (\gamma - \lambda) \Delta x \wedge \dot{x} - (\gamma - \lambda_1) \Delta x \wedge \dot{x}_1. \end{aligned}$$

Linear independence of $\Delta x, \dot{x}, \dot{x}_1$ implies linear independece of $\Delta x \wedge \dot{x}$, $\Delta x \wedge \dot{x}_1$, so we have $\gamma = \lambda$ and $\lambda = \lambda_1$. Secondly, differentiation of γ's defining equation together with $\lambda = \gamma$ yields

$$0 = \partial_u(\Delta x^+ - \gamma \Delta x) = \gamma \dot{x}_1 - \gamma \dot{x} - \dot{\gamma} \Delta x - \gamma \Delta \dot{x} = \dot{\gamma} \Delta x.$$

It follows that $\dot\gamma = 0$. The identities $\gamma = \lambda$, $\dot\gamma = 0$ and $\Delta\gamma = 0$ shown here hold everywhere except in a zero set, so we conclude $\gamma = \lambda = \text{const}$. $\square$

1.3 **Parallel Infinitesimal Polygons.**

The curvature theory of discrete surfaces presented by [7, 1] depends on the notion of parallel polygons, which are required to be contained in parallel planes.

$$\begin{aligned}
&\text{For } P = (p_0, \dots, p_{n-1}), Q = (q_0, \dots, q_{n-1}) \in U^n, \\
&P \parallel Q \iff (p_{i+1} - p_i) \wedge (q_{i+1} - q_i) = 0 \quad (\text{indices modulo } n).
\end{aligned}$$

We extend the definition of parallelity to infinitesimal polygons in a way which serves our purposes when applied to the elementary quads of a semidiscrete surface. We first consider infinitesimal polygons $(P; V)$ and $(Q; W)$ being represented by n-gon paths

$$P + \tau V, \quad Q + \tau W.$$

For parallelity we require $P \parallel Q$; in addition each $\Lambda^2 U$-valued polynomial

$$(p_{i+1}(\tau) - p_i(\tau)) \wedge (q_{i+1}(\tau) - q_i(\tau)) \quad (\text{indices modulo } n)$$

(in the indeterminate τ) shall have a zero at $\tau = 0$ whose order is higher than would be the case for a generic element $V \times W \in T_P U^n \times T_Q U^n$. The next lemma illustrates this definition by means of the elementary quadrilaterals of semidiscrete surfaces.

Lemma 1.8. *Parallelity of conjugate surfaces x, x^+ is equivalent to parallelity of elementary qadrilaterals*

$$\boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}} + \tau \boxed{\begin{matrix} 0 & 0 \\ \dot x & \dot x_1 \end{matrix}}, \qquad \boxed{\begin{matrix} x^+ & x_1^+ \\ x^+ & x_1^+ \end{matrix}} + \tau \boxed{\begin{matrix} 0 & 0 \\ \dot x^+ & \dot x_1^+ \end{matrix}}.$$

The latter is expressed by the conditions

$$\begin{aligned}
&(x_1 - x) \parallel (x_1^+ - x^+), \quad \dot x_1 \parallel \dot x_1^+, \quad \dot x \parallel \dot x^+, \\
&(\dot x - \dot x_1) \wedge (x^+ - x_1^+) + (x - x_1) \wedge (\dot x^+ - \dot x_1^+) = 0.
\end{aligned}$$

Proof. Parallelity of elementary quads firstly means that $(x_1 - x) \parallel (x_1^+ - x^+)$, and secondly that $\Lambda^2 U$-valued polynomials

$$\begin{aligned}
&(\tau \dot x) \wedge (\tau \dot x^+), \quad (\tau \dot x_1) \wedge (\tau \dot x_1^+), \\
&(x - x_1 + \tau(\dot x - \dot x_1)) \wedge (x^+ - x_1^+ + \tau(\dot x^+ - \dot x_1^+))
\end{aligned}$$

have zeros of multiplicities 3, 3, 2, resp., for $\tau = 0$. This verifies the definition of parallelity of infinitesimal quadrilaterals.

Clearly parallelity of elementary quads implies parallelity of surfaces. For the converse we observe that the first three conditions express the definition of $x \parallel x^+$ while the last one is found by differentiating $\Delta x \wedge \Delta x^+ = 0$. $\square$

1.4 **Offsets in three dimensions.**

In the elementary differential geometry of surfaces, an *offset* means a surface at constant distance to a given one. This requirement defines the offset uniquely, but for discrete surfaces there are different ways of defining offsets [4,7,2]. In the semidiscrete category the situation is similar, as has been observed by [8]. We say that a parallel pair x, x^+ of surfaces is an offset pair at distance d, with *Gauss image*

$$s = \frac{1}{d}(x^+ - x),$$

if x, x^+ are at constant distance d from each other. This distance can be measured in different ways:

1. Case (V), *vertex offsets*: Distance of vertices is constant:
$$\mathrm{dist}(x, x^+) = d.$$
2. Case (E_1) *edge offsets, 1st kind*: Distance of rulings is constant:
$$\mathrm{dist}(x \vee x_1, x^+ \vee x_1^+) = d.$$
3. Case (E_2) *edge offsets, 2nd kind*: Distance of tangents is constant:
$$\mathrm{dist}(x + \mathrm{span}(\dot{x}),\ x^+ + \mathrm{span}(\dot{x}^+)) = d.$$
4. Case (F) *face offsets*: Distance of tangent planes is constant:
$$\mathrm{dist}(x + \mathrm{span}(\Delta x, \dot{x}),\ x^+ + \mathrm{span}(\Delta x^+, \dot{x}^+) = d.$$

In the following discussion we restrict ourselves to $U = \mathbb{R}^3$ as ambient space. We will see that there are essentially only two cases of offsets, not four. Before we prove that, we state an obvious but important relation:

Proposition 1.9. *The surface x^+ is an offset of x at distance d $\iff$ $s = \frac{1}{d}(x^+ - x)$ is an offset of the same type of the zero surface.*

Lemma 1.10. *Offset types* (E_2)*,* (F) *are generically equivalent.*

Proof. We translate statements (E_2), (F) into equivalent statements (E_2'), (F') worded in terms of the Gauss image surface s, which is conjugate and parallel to both x, x^+:

1. case (E_2'): Rulings $s + \mathrm{span}(\Delta s) = s \vee s_1$ are tangent to the unit sphere.
2. case (F'): Tangent planes $s + \mathrm{span}(\dot{s}, \Delta s)$ are tangent to the unit sphere.

Assuming (F'), there is a normal vector field $n : \mathbb{Z} \times \mathbb{R} \to S^2$ such that

$$\langle s, n\rangle = \langle s_1, n\rangle = 1 \quad \text{and} \quad \langle \dot{s}, n\rangle = \langle \dot{s}_1, n\rangle = \langle \Delta s, n\rangle = 0$$

(we find n by normalizing $\dot{x} \times \Delta x$). We compute

$$0 = \partial_u \langle s, n\rangle = \langle \dot{s}, n\rangle + \langle s, \dot{n}\rangle = \langle s, \dot{n}\rangle \text{ and similarly } \langle s_1, \dot{n}\rangle = 0,$$
$$0 = \partial_u \langle n, n\rangle = 2\langle \dot{n}, n\rangle = 0.$$

It follows that each of s, s_1, n fulfills the two linear equations $\langle n, \cdot\,\rangle = 1$ and $\langle \dot{n}, \cdot\,\rangle = 0$. If $\dot{n} \neq 0$ (this is the genericity assumption) this implies that they are collinear and $n \in s \vee s_1$. Since the tangent plane touches the unit sphere in the point n, the line $s \vee s_1$ touches the unit sphere in the point n. This shows (E_2').

Assume now (E_2'). Consider the point of contact n of the ruling $s \vee s_1$ with the unit sphere. If $\dot{n}$, Δs are linearely independent (this is the genericity assumption), these two vectors, both orthogonal to n, span the tangent plane which obviously touches the unit sphere. This shows (F'). □

Lemma 1.11. *Offset cases (V) and (E_1) are generically equivalent.*

Proof. Polarity w.r.t. S^2 maps vertices "x", tangents "$x + \mathrm{span}(\dot{x})$", rulings "$x \vee x_1$" and tangent planes of a conjugate surface to the tangent planes, rulings, tangents, and vertices, respectively, of another conjugate surface. It follows that this polarity maps Gauss images of types (V) and (E_1) to Gauss images of types (F) and (E_2), resp., and vice versa. The equivalence (F) $\iff$ (E_2) shown above thus implies (V) $\iff$ (E_1). □

The following statement, which is analogous to the discrete case shown in [9], is stated already in [8].

Theorem 1.12. *Consider a semidiscrete conjugate surface x. A nontrivial vertex offset exists $\iff$ x is circular.*

Sketch of Proof. If x has a vertex offset, there exists a Gauss image s inscribed in S^2. By Prop. 1.6 it is sufficient to show that s is circular. This follows directly from conjugacy of s: The plane through s and s_1 which is spanned by vectors $\dot{s}, \Delta s, \dot{s}_1$ intersects S^2 in the desired circle.

For the converse we assume that x is circular. We construct a Gauss image surface $s : \mathbb{R} \times \mathbb{Z} \to S^2$ parallel to x.

Choose $s(i_0, u_0)$ arbitrarily on $S^2 \cap \dot{x}^\perp$. Since the straight line $s + \mathrm{span}(\Delta x)$, evaluated at (i_0, u_0) has exactly one other intersection point with the unit sphere, $s(i_0 + 1, u_0)$ is uniquely determined by parallelity. The circular condition ensures that also there, $s \in \dot{x}^\perp$. By induction we construct all values $s(i, u_0)$.

As to the continuous variable, $s(i_0, u)$ shall be the integral of the linear ODE $\dot{s} = -\frac{\langle s, \ddot{x}\rangle}{\langle \dot{x}, \dot{x}\rangle}\dot{x}$. By construction, $\partial_{uu}\langle s, s\rangle = 0$, and initial conditions at $u = u_0$ are such that $\langle s, s\rangle = \text{const.} = 1$.

To compute arbitrary values $s(i, u)$ we can either apply the discrete construction to $s(i_0, u)$ or the continuous construction to $s(i, u_0)$. Consistency (i.e., integrability of this difference-differential equation) follows from the circular condition. We omit this calculation. □

1.5 **Support Functions.**

Our discussion of offsets in $\mathbb{R}^3$ leads us to consider support functions of surfaces, which is motivated by the well known concept of the same name in convex geometry. We start by defining a unit normal vector field n of a conjugate surface x, by the requirements

$$\langle \delta_1 x, n\rangle = \langle \delta_2 x, n\rangle = 0, \quad \|n\| = 1.$$

We can locally make the normal vector field unique by requiring

$$\det(\delta_1 x, \delta_2 x, n) > 0, \tag{4}$$

but this does not always serve our purposes; when we speak about a parallel pair x, x^+ with $\frac{1}{d}(x^+ - x) = s$, then we always consider a common normal vector field n for all three surfaces x, x^+, s, even if the handedness condition (4) is fulfilled only for, say, x.

Definition 1.13. *If n is a unit normal vector field of the conjugate surface s, then the associated support function is given by*

$$\sigma_{n,s} : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \quad \sigma_{n,s} = \langle n, s\rangle.$$

Remark 1.14. Obviously, if $x^+ = x + \lambda s$, then $\sigma_{n,x^+} = \sigma_{n,x} + \lambda\sigma_{n,s}$. If s is tangentially circumscribed to the unit sphere, then $\sigma_{n,s} = \text{const.} = 1$.

Theorem 1.15. *Assume a vector field n with* $\det(n,\ \delta_1 n,\ \delta_2 n) \neq 0$. *Then there is a conjugate surface x whose support function w.r.t. n equals the given function σ, if and only if*

$$\det\begin{pmatrix} \sigma & \delta_1\sigma & \delta_2\sigma & \delta_2\delta_1\sigma \\ n & \delta_1 n & \delta_2 n & \delta_2\delta_1 n \end{pmatrix} = 0.$$

This statement applies to discrete, semidiscrete and smooth surfaces, with the appropriate meanings of δ_1, δ_2.

Proof. Define x by solving $\langle n, x\rangle = \sigma$, $\langle \delta_1 n, x\rangle = \delta_1\sigma$, $\langle \delta_2 n, x\rangle = \delta_2\sigma$. The vanishing determinant means that x also solves $\langle \delta_2\delta_1 n, x\rangle = \delta_2\delta_1\sigma$.

(i) In the discrete case a linear combination of equations gives the equivalent condition that the solution of $\langle n, x\rangle = \sigma$, $\langle n_1, x\rangle = \sigma_1$, $\langle n_2, x\rangle = \sigma_2$ also solves $\langle n_{12}, x\rangle = \sigma_{12}$. By an index shift, this is conjugacy.

(ii) In the smooth case, the definition of x means that the surface $x(u_1, u_2)$ is the envelope of tangent planes $\langle n, \cdot\,\rangle = \sigma$. Differentation $0 = \delta_1(\langle \delta_2 n, x\rangle - \delta_2\sigma) = \langle \delta_2\delta_1 n, x\rangle + \langle \delta_2 n, \delta_1 x\rangle - \delta_1\delta_2\sigma$ shows that the determinant condition is equivalent to $\langle \delta_1 n, \delta_2 x\rangle = 0$, which is well known to express conjugacy.

(iii) The semidiscrete case is a mixture of (i) and (ii). A linear combination of equations defines x equivalently as solution of $\langle n, \cdot\rangle = \sigma$, $\langle \dot n, \cdot\rangle = \dot\sigma$, $\langle n_1, \cdot\rangle = \sigma_1$, and the determinant condition means x also solves $\langle \dot n_1, \cdot\rangle = \dot\sigma_1$. The first two equations define the ruling through x of the developable enveloped by planes $\langle n(i,t), \cdot\rangle = \sigma(i,t)$ as t is running and i fixed, while the last two equations define the corresponding ruling through x_1. Conjugacy means that x is contained in the latter ruling, which is just the determinant condition. □

Corollary 1.16. *Assume that conjugate surfaces x, x^+ have the same unit normal vector field n. If $\delta_1 n \neq 0$, $\delta_2 n \neq 0$, then x, x^+ are parallel.*

Thus there is a linear space of surfaces with given unit normal vector field which are conjugate apart from the condition of regularity.

Proof. The previous proof states that $\mathrm{span}(\delta_1 x)$, $\mathrm{span}(\delta_2 x)$ is determined by the normal vector field alone, e.g. in the smooth case by $\langle \delta_1 x, \delta_2 n\rangle = 0$. □

Corollary 1.17. *The conjugate surface x with unit normal vector field n has an offset at constant face-face distance $\iff$ $\det(\delta_1 n, \delta_2 n, \delta_2\delta_1 n) = 0$*

Proof. In view of Prop. 1.9, existence of such offsets means that $\sigma = 1 = \text{const.}$ is an admissible support function. Now Theorem 1.15 immediately gives the result. □

Definition 1.18. *Parallel surfaces x, x^+ with normal vector fields $n = n^+$ and support functions σ, σ^+ are assigned the distance function $\sigma^+ - \sigma$.*

The next result was mentioned by [8]:

Theorem 1.19. *Consider a semidiscrete conjugate surface x and its unit normal vector field n. Provided we are in the generic case $\det(\dot{x}, \Delta x_{-1}, \Delta x) \neq 0$, we have the following equivalences:*

1. *x has a face offset $\iff$*
2. *n is circular $\iff$*
3. *x is conical.*

Proof. The proof uses the 'symmetry' characterizations of circular and conical surfaces mentioned in Cor. 1.3.

1. $\Longrightarrow$ 2. follows from Cor. 1.17: we have $\det(\Delta n, \dot{n}, \Delta\dot{n}) = 0$, i.e., n is conjugate. Being inscribed in S^2, n is circular.

2. $\Longrightarrow$ 3.: The ruled surface strips associated with x are the envelopes of planes with normal vector n. Thus vectors $\Delta x, \Delta x_{-1}, \dot{x}$ are parallel to $n \times \dot{n}$, $n_{-1} \times \dot{n}_{-1}$, $n_{-1} \times n$, respectively, and the symmetry required for n being circular immediately yields a symmetry revealing x as conical.

3. $\Longrightarrow$ 1.: Consider the oriented right circular cone with vertex x which touches the (oriented) adjacent ruled strips along rulings, and parallel translate it to become tangentially circumscribed to S^2. The vertex x thereby moves to a new position s; tangent planes are now tangent to S^2. It follows that the semidiscrete surface s, which by construction is parallel to x, is tangentially circumscribed to S^2. This implies that $\sigma = \text{const.} = 1$ is an admissible support function for x. □

2 Curvatures

We recapitulate how [7] and [1] introduce curvatures associated with the faces of a polyhedral surface. This is done via the classical Steiner formula, which relates the area element of an offset surface x^τ at distance τ from an original surface x via

$$x^\tau = x + \tau s \implies \frac{dA(x^\tau)}{dA(x)} = 1 - 2H\tau + K\tau^2, \tag{5}$$

locally around $\tau = 0$. Here s is the unit normal vector field (the Gauss map), and H, K are mean and Gaussian curvatures, respectively. In the framework of *relative* differential geometry this definition was generalized to a surface x and a Gauss map s which is not necessarily contained in the unit sphere, but such that x, s have the same unit normal vector in corresponding points [13].

For discrete polyhedral surfaces (and in particular for discrete conjugate nets), there is a similar construction which used to define curvatures of circular surfaces in [10,11]. Suppose x, s are parallel, and s is regarded as the Gauss image of x. Then the variation of surface area of faces as we travel through a 1-parameter family of offsets reads

$$x^\tau = x + \tau s \implies \frac{A(f^\tau)}{A(f)} = 1 - 2H(f)\tau + K(f)\tau^2,$$

where f and f^τ are corresponding faces of the surface x and its offset x^τ. The quantities $H(f)$ and $K(f)$ — mean and Gaussian curvatures of the face f — have been introcuded by [7] and are studied by [1]. Their relation to mixed areas is explained in the next section.

2.1 The oriented mixed area of polygons

The above-mentioned curvature theory is based on the oriented area and oriented mixed area of polygons. We therefore first collect some definitions before we proceed to semidiscrete surfaces. The oriented area of an n-gon $P = (p_0, \ldots, p_{n-1})$ in a two-dimensional vector space U, is given by Leibniz' sector formula:

$$A(P) = \frac{1}{2} \sum\nolimits_{0 \le i < n} [p_i, p_{i+1}].$$

Here and in the following indices in such sums are taken modulo n. We use the notation $[a, b]$ for a determinant form in U which defines the area, i.e., $[a, b] = \det(a, b, n)$, for some normal vector n of this plane, the choice of which is irrelvant for curvatures. The sector formula is invariant w.r.t. translation by a vector $t \in \mathbb{R}^3$, not necessarily contained in U;

$$\begin{aligned} \sum \det(p_i + t, p_{i+1} + t, n) &= \sum \det(p_i, p_{i+1}, n) + \det(\sum p_i, t, n) \\ &- \det(\sum p_{i+1}, t, n) = \sum \det(p_i, p_{i+1}, n). \end{aligned}$$

This means that $A(P)$ can be extended to polygons lying in any affine subspace $t+U$. Apparently A is a quadratic form, whose associated symmetric bilinear form is denoted by the symbol $A(P,Q)$:

$$\begin{aligned} A(\lambda P + \mu Q) &= \lambda^2 A(P) + 2\lambda\mu A(P,Q) + \mu^2 A(Q), \quad \text{where} \\ A(P,Q) &= \frac{1}{4}\sum\nolimits_{0\le i<n} [q_i, p_{i+1}] + [p_i, q_{i+1}]. \end{aligned} \tag{6}$$

Note that in Equation (6) the sum of polygons is defined vertex-wise, and that $A(P,Q)$ does not, in general, equal the well known mixed area [12]. For boundaries of convex polygons which happen to be parallel, however, this is the case (as discussed by [1]). Thus $A(P,Q)$ is called the (oriented) mixed area of P,Q, provided $P \parallel Q$.

2.2 **Area and mixed area of infinitesimal quadrilaterals.**

For the differentials of area and mixed area we observe that $A(P+\varepsilon V) = A(P) + 2\varepsilon A(P,V) + \varepsilon^2 A(V)$. A similar relation holds for the mixed area of $P+\varepsilon V$ and $Q+\varepsilon W$. Consequently

$$d_P A(V) = 2A(P,V), \quad d_{P,Q} A(V,W) = A(P,W) + A(V,Q).$$

For the special infinitesimal quadrilaterals according to Def. 1.1, we get

$$P = \boxed{\begin{matrix} x \; x_1 \\ x \; x_1 \end{matrix}}, \; V = \boxed{\begin{matrix} 0 \; 0 \\ \dot{x} \; \dot{x}_1 \end{matrix}} \implies d_P A(V) = \frac{1}{2}[x_1 - x, \dot{x}_1 + \dot{x}].$$

The *area* of a semidiscrete surface is naturally defined as the surface area of the corresponding ruled strips (2) associated with it. In case x is conjugate, a normal vector field of x yields an orientation in the ruled strips and it makes sense to consider *signed* surface area:

$$\operatorname{area}(x(D)) = \int_{(i,u)\in D} dA = \int_{(i,u)\in D} \frac{1}{2}[\Delta x, \dot{x} + \dot{x}_1]\, du. \tag{7}$$

2.3 **Curvatures in the semidiscrete case**

Regarding the surface area of their offsets, semidiscrete surfaces behave in a way similar to their discrete and continuous counterparts.

Proposition 2.1. *Assume x is a conjugate semidiscrete surface, and s (considered to be the Gauss image of x) is parallel to x. Then the surface area of the offset family $x^\tau = x + \tau s$ reads*

$$\operatorname{area}(x^\tau(D)) = \int_{(i,u)\in D} (1 - 2H\tau + K^2\tau)\, dA,$$

where $dA(i,u) = \frac{1}{2}[\Delta x, \dot{x} + \dot{x}_1]\,du$. *With the infinitesimal quadrilaterals*

$$P + V d\varepsilon = \boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}} + \boxed{\begin{matrix} 0 & 0 \\ \dot{x} & \dot{x}_1 \end{matrix}}\,d\varepsilon, \quad Q + W d\varepsilon = \boxed{\begin{matrix} s & s_1 \\ s & s_1 \end{matrix}} + \boxed{\begin{matrix} 0 & 0 \\ \dot{s} & \dot{s}_1 \end{matrix}}\,d\varepsilon,$$

the coefficient functions H *(mean curvature) and* K *(Gauss curvature) can be expressed as*

$$H = -\frac{A(P,W) + A(V,Q)}{2A(P,V)}, \quad K = \frac{A(Q,W)}{A(P,V)}. \tag{8}$$

Proof. Equation (7) implies that the area element of $x + \tau s$ has the form $dA_{P+\tau Q}(V + \tau W)\,du = 2A(P + \tau Q, V + \tau W)\,du$. The rest is using bilinearity of the mixed area. □

Example 2.2. Expanding the previous definition of Gaussian curvature K and mean curvature H leads to the expressions

$$H = -\frac{[\Delta x, \dot{s} + \dot{s}_1] + [\Delta s, \dot{x} + \dot{x}_1]}{2[\Delta x, \dot{x} + \dot{x}_1]}, \quad K = \frac{[\Delta s, \dot{s} + \dot{s}_1]}{[\Delta x, \dot{x} + \dot{x}_1]}. \tag{9}$$

In terms of the infinitesimal area and mixed area of infinitesimal quadrilaterals given above, we also have the expressions

$$H = -\frac{d_{P,Q}A(V,W)}{d_P(V)}, \quad K = \frac{d_Q A(W)}{d_P(V)}.$$

This is a direct analogy to the discrete case.

2.4 **Curvatures from offset distances: discrete case**

Here we derive a formula which expresses the mean curvature of a polyhedral offset pair in terms of edge lengths and dihedral angles. This is interesting because it can directly be compared with other notions of mean curvature derived via the Steiner formula: For a convex polyhedral surface (V, E, F) which bounds a convex set K, the area of an outer parallel body is given by

$$\operatorname{area}(\partial(K + \varepsilon B)) = \sum_{\text{faces } f} \operatorname{area}(f) + \varepsilon \sum_{\text{edges } e} \alpha_e \operatorname{len}(e) + 4\pi\varepsilon^2.$$

Here α_e is the dihedral angle of the edge e. It is therefore natural to consider

$$H(e) = -\frac{1}{2}\alpha_e, \tag{10}$$

as the mean curvature density in the edge e.

Remark 2.3. The values $H(e)$ of (10) and $H(f)$ of Th. 2.4 are not immediately comparable. Total mean curvature "$\int_G H$" of a domain G in the surface under consideration would have to be defined as $\sum_{e \in E} H(e) \operatorname{len}(e \cap G)$, or $\sum_{f \in F} H(f) \operatorname{area}(f \cap G)$ (see e.g. [3] for Geometry Processing applications).

An expression in terms of angles similar to (10) is if we consider polyhedral surfaces which admit offsets at constant face-face distance:

Theorem 2.4. *For a conical mesh, i.e., a polyhedral surface which admits a face-face offset at constant distance, the mean curvature of a face is expressed in terms of the dihedral angles of edges by*

$$H(f) = -\frac{1}{2\,\mathrm{area}(f)} \sum_{edges\ e\subset f} \tan\frac{\alpha_e}{2}\,\mathrm{len}(e).$$

The proof of Theorem 2.4 depends on the following more general result:

Lemma 2.5. *Assume that n-gons $P = (p_0, \dots, p_{n-1})$ and $Q = (q_0, \dots, q_{n-1})$ are corresponding faces in an offset pair of discrete surfaces, and that translating the plane of P by the vector $d \cdot n$, with n a unit normal vector, yields the plane of Q. Similarly corresponding edges $p_i p_{i+1}$ and $q_i q_{i+1}$ lie in parallel planes, carrying faces adjacent to P, Q, resp., whose relative position is given by normal vectors n_i and distances d_i. Then the mean curvature of the face P is expressed as*

$$H(P) = -\frac{1}{2A(P)} \sum_i \frac{d_i - d\cos\alpha_i}{\sin\alpha_i}\|\Delta p_i\|.$$

Here dihedral angles are computed by $\cos\alpha_i = \langle n, n_i\rangle$, $\sin\alpha_i = \langle n \times n_i, \frac{\Delta p_i}{\|\Delta p_i\|}\rangle$.

Proof. We consider the orthogonal projection of Q on the plane of P and measure the oriented distance ϕ_i of corresponding edges $p_i p_{i+1}$ and $q_i q_{i+1}$:

$$\phi_i = \frac{d_i - d\cos\alpha_i}{\sin\alpha_i}.$$

Here ϕ_i is positive, if (after projection) the edge of Q lies to the left of the corresponding edge of P — the plane being oriented by the normal vector n. The mixed area needed for mean curvature is then derived as

$$A(P, Q-P) = \frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A(P + \varepsilon(Q-P)) = \frac{1}{2}\sum \|\Delta p_i\|\phi_i. \qquad \square$$

Remark 2.6. Obviously the mean curvature can also be expressed as

$$H(P) = -\frac{1}{2A(P)} \sum_i \frac{d_i - d\langle n, n_i\rangle}{\det(\Delta p_i, n, n_i)}\|\Delta p_i\|^2.$$

Proof of Theorem 2.4. Let $d = d_i = 1$ and observe the relation

$$\frac{1-\cos x}{\sin x} = \tan\frac{x}{2}. \qquad \square$$

2.5 Curves from offset distances: semidiscrete case

The following result attempts to carry over Lemma 2.5 to the semidiscrete case. Unfortunately the limit process which is involved does not lead to a pretty result, whose proof we omit.[1] Only the semidiscrete version of Theorem 2.4 can be expressed in a reasonably short way:

Corollary 2.8. *Assume that the semidiscrete surface x has face offsets. Consider the angles β between successive normal vectors n, n_1 and the angular velocity ω of the normal vector field:*

$$\sin\beta = \det\Big(\frac{\dot{x}}{\|\dot{x}\|}, n, n_1\Big), \quad \omega = \det\Big(\frac{\Delta x}{\|\Delta x\|}, n, \dot{n}\Big),$$

Then $\langle n, n_1\rangle = \cos\beta$ and $\|\dot{n}\| = |\omega|$, and the mean curvature is expressed as

$$H = -\frac{1}{\det(\Delta x, \dot{x} + \dot{x}_1, n)}\left(\|\dot{x}\| \tan\frac{\beta_{-1}}{2} - \|\dot{x}_1\| \tan\frac{\beta}{2} + \omega\|\Delta x\|\right).$$

Proof. Letting $d = 1$ in Lemma 2.7 yields the result, if we observe the various expressions for β, ω given above, as well as $\langle \dot{n}, \dot{n}\rangle + \langle n, \ddot{n}\rangle = \partial_{uu}\langle n, n\rangle = 0$. Alternatively we may compute a limit of Theorem 2.4. □

3 Examples

3.1 Surfaces of discrete rotational symmetry

Here the surface $x(i, u)$ is generated by rotating the planar meridian $x(0, u) = (\xi(u), \eta(u), 0)$ about the first coordinate axis, by the angle $i\alpha$. Assume that a second surface s (the Gauss image of x) is generated in the same way from a planar meridian curve $s(0, u) = (\sigma(u), \tau(u), 0)$. Obviously,

$$x \parallel s \iff \frac{\Delta s}{\Delta x} = \frac{\tau}{\eta} = \frac{\sigma}{\xi} \text{ and } \frac{\dot{s}}{\dot{x}} = \frac{\dot{\tau}}{\dot{\eta}} = \frac{\dot{\sigma}}{\dot{\xi}}$$

We use Ex. 2.2 to compute the mean and Gaussian curvatures (if $\eta, \dot{\eta} \neq 0$):

$$H = -\frac{\dot{\eta}\tau + \eta\dot{\tau}}{2\eta\dot{\eta}}, \quad K = \frac{\tau\dot{\tau}}{\eta\dot{\eta}}.$$

We discuss various special cases and the relation to smooth surfaces.

- An immediate conseqence of these formulas is $K = 0 \iff \tau = \text{const.}$ and $H = 0 \iff \tau\eta = \text{const.}$

[1] **Lemma 2.7.** *Assume a pair x, x^+ of parallel conjugate surfaces, such that the non-degeneracy condition $\det(\ddot{x}, \dot{x}, \Delta x) \neq 0$ holds. Then the mean curvature associated with the offset pair x, x^+ is expressed via the normal vector field n and the distance function d (cf. Def. 1.18) by the formula $H = -\frac{\delta+\gamma}{\det(\Delta x, \dot{x}+\dot{x}_1, n)}$, where $\delta = \frac{d_{-1} - d\langle n, n_{-1}\rangle}{\det(\dot{x}, n, n_{-1})}\|\dot{x}\|^2 - \frac{d_1 - d\langle n, n_1\rangle}{\det(\dot{x}_1, n, n_1)}\|\dot{x}_1\|^2$ and $\gamma = \frac{2[\Delta\dot{x}, \dot{n}] + \det(\Delta x, \ddot{n}, n)}{[\Delta x, \dot{n}]^2}\dot{d}\|\Delta x\|^2 - \frac{(\ddot{d} + d\|\dot{n}\|^2)\|\Delta x\|^2 + 4\dot{d}\langle\Delta x, \Delta\dot{x}\rangle}{[\Delta x, \dot{n}]}$.*

- The case $K = 0$ is not so interesting, because for non-degenerate Gauss image we get only x, s as co-axial cylinders.
- The given meridian curves, by continuous rotation, can also be used to define *smooth* surfaces of revolution. If s is contained in the unit sphere (so x is circular), then $-\dot{s} : \dot{x} = -\dot{\tau} : \dot{\eta}$ is the normal curvature of the meridian curves, while $\tau : \eta$ is the normal curvature of the parallels. Thus the expressions for H, K above also compute the mean and Gaussian curvatures for smooth surfaces.
- If $s(0, u)$ is contained in the unit sphere, it is determined uniquely by x up to multiplication with -1 by the parallelity condition. This uniqueness is also present in the smooth case. So for circular surfaces, curvatures coincide with the curvatures of associated smooth surface with the same meridian. For instance, vanishing mean curvature implies that x is generated by $x(0, u) = (t, C \cosh(u/C), 0)$.
- Any semidiscrete surface x with the discrete rotational symmetry as described above has a face offset, since Theorem 1.19 applies. The Gauss image $s(i, u)$ is tangentially circumscribed to S^2, so

  $$\tau(u) = \sqrt{1 - \sigma(u)^2} / \cos \frac{\alpha}{2}.$$

  We conclude that the affine mapping

  $$(\xi, \eta, \zeta) \mapsto (\cos \tfrac{\alpha}{2} \cdot \xi, \eta, \cos \tfrac{\alpha}{2} \cdot \zeta)$$

  maps s to the Gauss image of a circular semidiscrete surface, such as considered by the previous paragraph. Since curvatures are defined in an affine-invariant way, we have shown that curvatures of x with respect to s are equal to curvatures of the smooth surface generated by the meridian $(\xi, (u), \eta(u)/\cos \frac{\alpha}{2}, 0)$.
  For instance, minimal surfaces in the face-offset category are generated by meridians of the form $(u, \frac{C}{\cos \frac{\alpha}{2}} \cosh \frac{u}{C}, 0)$.

3.2 **Semidiscrete surfaces of continuous rotational symmetry**

We generate a surface $x(i, u)$ and its associated Gauss image $s(i, u)$ by rotating the respective "meridian" polylines

$$x(i, 0) = (\xi_i, \eta_i, 0), \quad s(i, 0) = (\sigma_i, \tau_i, 0).$$

about the first coordinate axis; the angle of rotation being equal to u. The condition of parallelity $x \parallel s$ is equivalent to $\Delta\eta_i : \Delta\xi_i = \Delta\tau_i : \Delta\sigma_i$. We discuss various special cases:

- Unsurprisingly, the formulas for the mean and Gaussian curvatures coincide with their discrete counterparts given by [1]:

  $$H(i, u) = \frac{\eta_i \tau_i - \eta_{i+1} \tau_{i+1}}{\eta_{i+1}^2 - \eta_i^2}, \quad K(i, u) = \frac{\tau_i^2 - \tau_{i+1}^2}{\eta_{i+1}^2 - \eta_i^2}.$$

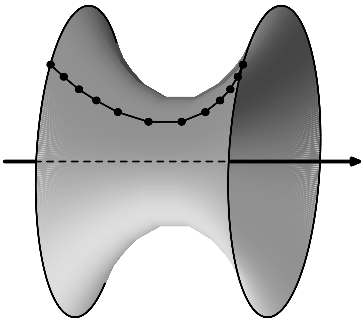

Fig. 1 A semidiscrete minimal surface with the conical property, defined by the angle sequence $\ldots, \frac{\pi}{8}, \frac{\pi}{7}, \frac{\pi}{6}, \frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{5}, \frac{5\pi}{6}, \frac{6\pi}{7}, \frac{7\pi}{8}, \ldots$

– If $H = 0$ then $x(i,u)$ is a minimal surface, and we get the recursion

$$\begin{pmatrix}\xi_{i+1}\\ \eta_{i+1}\end{pmatrix} - \begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix} = \frac{1}{\tau_{i+1}}\begin{pmatrix}0 & \sigma_i - \sigma_{i+1}\\ 0 & \tau_i - \tau_{i+1}\end{pmatrix}\begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix},$$

Thus $x(i,u)$ is uniquely determined by the Gauss image up to scaling. The case of Gauss images inscribed in the unit sphere is analyzed by [5], especially the convergence of meridian polylines to the graph of the *cosh* function.

– If x is to have a face offset we consider a Gauss image s which is tangentially circumscribed to the unit sphere. It follows that s is generated by a polyline of the form

$$(\sigma_i, \tau_i) = \frac{1}{\cos\frac{\alpha_i - \alpha_{i-1}}{2}}\Big(\cos\frac{\alpha_i + \alpha_{i-1}}{2}, \sin\frac{\alpha_i + \alpha_{i-1}}{2}\Big),$$

for some sequence α_i of angles. See Figure 1 for an example.

3.3 Semidiscrete trapezoidal surfaces of the horizontal type

The class of semidiscrete surfaces considered here contains those with discrete rotational symmetry, and indeed we will see that some formulas carry over unchanged.

We require that the infinitesimal quadrilaterals are trapezoids in the sense that the direction of rulings $x \vee x_1$ depends only on the discrete parameter, but not on the continuous parameter (in fact, assuming conjugacy, it is sufficient to require that for each i, the rulings $x(i,u) \vee x(i+1,u)$ are parallel to a fixed plane). We define a *trapezoidal surface of the horizontal type* by a scalar function λ and vectors e by

$$\Delta x(i,u) = \lambda(i,u)e(i), \quad \text{where } \lambda : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ e : \mathbb{Z} \to S^2.$$

Parallel surfaces x, s can without loss of generality be defined by the same vectors $e(i)$, so we now assume that parallel surfaces x, s are defined by vectors $e(i)$ and scalars λ (for x) and μ (for s). Obviously,

$$\Delta s : \Delta x = \mu : \lambda.$$

It turns out that

$$x \parallel s \iff \Delta(\dot\mu : \dot\lambda) = 0 \text{ for all } i, \text{ and } \dot s : \dot x = \dot\mu : \dot\lambda \text{ for all } i. \tag{11}$$

$$\iff \Delta(\dot\mu : \dot\lambda) = 0 \text{ for all } i, \text{ and } \dot s : \dot x = \dot\mu : \dot\lambda \text{ for some } i. \tag{12}$$

Proof. To see this, we assume that for $i = i_0$ we have parallelity of derivatives: $\nu = \dot s : \dot x$. Parallelity of derivatives for $i = i_0 + 1$ means $[\dot s_1, \dot x_1] = [\dot s + \dot\mu e, \dot x + \dot\lambda e] = 0$, which expands to $\nu\dot\lambda = \dot\mu$, i.e., the strips under consideration are parallel if and only if the ratio ν obeys $\nu = \dot\mu : \dot\lambda$. In this case obviously also the ratio $\dot s_1 : \dot x_1$ equals ν. This argument shows the "$\Longrightarrow$" part of (11). "(11) $\Longrightarrow$ (12)" is trivial. Assume now (12). The previous argument shows that $\dot s_1, \dot x_1$ are parallel and that $\dot x_1 : \dot s_1 = \dot x : \dot s$. Induction shows the left hand side of (11). $\square$

If s is considered the Gauss image of x, then similar to Section 3.1 it is straightforward to compute the Gaussian and mean curvatures. We get

$$H = -\frac{1}{2}\Big(\frac{\mu}{\lambda} + \frac{\dot\mu}{\dot\lambda}\Big) = -\frac{\partial_u(\lambda\mu)}{\partial_u(\lambda^2)}, \quad K = \frac{\mu\dot\mu}{\lambda\dot\lambda}. \tag{13}$$

We consider some interesting special cases:

- $H = 0 \iff \lambda\mu$ does not depend on the continuous variable.
- For existence of a face offset we look to the answer given by Theorem 1.19. This leads to the following equivalences:
  1. x has a face-offset.
  2. Throughout the surface, the vector $\dot x(i,u)$ is contained in the bisector plane $P(i,u)$ of normalized vectors $\Delta x(i,u)$ and $-\Delta x(i-1,u)$.
  3. Each curve $x(i,\cdot)$ is planar, lying in the bisector plane $P(i) = P(i,u)$ of rulings $x(i,u)x(i+1,u)$ and $x(i,u)x(i-1,u)$, which does not depend on the continuous variable.

  We focus on a single strip bounded by curves $x(i,\cdot)$ and $x(i+1,\cdot)$ and assume a coordinate system with $0 \in P(i) \cap P(i+1)$. The Gauss image s enjoys each of the properties (a)–(c) mentioned above, but in addition it is tangentially circumscribed to the unit sphere. Therefore, all planes carrying $s(i,\cdot)$ pass through the origin anyway, and we have $s(i,u) \in P(i)$, $s(i+1,u) \in P(i+1)$.
  The Gauss image is uniquely determined by the surface x. It is not difficult to check that

$$s = \frac{\dot x \times e}{\|\dot x \times e\|} - \frac{\det(\dot x \times e, x, \dot x)}{\|\dot x \times e\| \det(e, x, \dot x)} e.$$

  If $x(i,u) = (\xi(u), \eta(u), 0)$, $s(i,u) = (\sigma(u), \tau(u), 0)$, and $e(i) = (\alpha, \beta, \gamma)$, we get $\tau = \frac{\alpha\beta\dot\eta - (\beta^2+\gamma^2)\dot\xi}{\gamma\sqrt{\gamma^2((\dot\xi)^2+(\dot\eta)^2)+(\beta\dot\xi - a\dot\eta)^2}}$ (this is valid for the value i under consideration).

– Recall the formula for mean curvature given in (13). A somewhat lengthy computation shows that for face-offset surfaces, we may characterize the minimality condition $H = 0$ using the local coordinate system described above, by an implicit equation which must be satisfied by $\xi(u)$ and $\eta(u)$:

$$H = 0 \iff \eta = \frac{C_1}{\gamma}\cosh\Big(\frac{(\beta^2+\gamma^2)\xi - \alpha\beta\eta - C_2}{C_1}\Big) \tag{14}$$

for some constants C_1 and C_2. This equation is valid for a previously chosen fixed value i, with $e(i) = (\alpha, \beta, \gamma)$. Any parametric representation $\xi(u), \eta(u)$ of the implicit curve given by this equation generates a minimal surface.

Proof. With the substitution $\xi = \phi + \frac{\alpha\beta}{\gamma}\psi$ and $\eta = \frac{\beta^2+\gamma^2}{\gamma}\psi$ we rewrite the coordinate function $\tau(u)$ as $\tau = -\dot\phi(\beta^2+\gamma^2)^{1/2}\gamma^{-1}(\dot\phi^2+\dot\psi^2)^{-1/2}$ and consequently the mean curvature is expressed as $H = (\beta^2+\gamma^2)^{-1/2}$ $\Big(\frac{\dot\phi}{\psi(\dot\phi^2+\dot\psi^2)^{1/2}} + \frac{\ddot\phi\dot\psi-\dot\phi\ddot\psi}{(\dot\phi^2+\dot\psi^2)^{3/2}}\Big)$ It follows that $H = 0 \iff (\dot\psi^2+\dot\phi^2)\dot\phi + \psi(\ddot\phi\dot\psi - \dot\phi\ddot\psi) = 0$. This equation has the trivial solution $\phi = \text{const.}$, but otherwise it transforms to the equation $(\frac{d\psi}{d\phi})^2 + 1 - \psi\frac{d^2\psi}{d\phi^2} = 0$, whose solutions are $\psi = C\cosh((\phi - C^*)/C))$. Backsubstitution yields the result. □

– It is interesting that we have just found semidiscrete minimal surfaces generated by a sequence of affinely-distorted catenary curves which do not have rotational symmetry. It is tempting to try to achieve smooth minimal surfaces as limit shapes of a sequence of semidiscrete surfaces $x^{(j)}$ of the type studied here, with $e^{(j)} \to 0$. Such a limit unfortunately is always an ordinary catenoid, so we do not find cases of new smooth minimal surfaces in this way.
  This can be seen as follows: Curves $x^{(j)}(i, \cdot)$ converge to planar principal curvature lines $x^\infty(u, \cdot)$ (because of the conical property [4]), which evolve with evolution velocity $\partial_v x^\infty$ orthogonal to the plane they are contained in (being conjugate to a principal curve). It follows that this evolution is isometric, and is actually generated by the rolling of a plane. All principal curves are congruent. In the coordinate system employed in the previous paragraph, the limit of vectors e (which is the direction of evolution) then reads $(0, 0, \pm 1)$, and the principal curve is explicitly given by $\eta = C_1\cosh(\frac{\xi - C_2}{C_1})$. We already know that C_1 is constant (by the congruence property), so we see that the distance of the curve from the axis of rolling is constant. This concludes the argument.

3.4 **Semidiscrete trapezoidal surfaces of the vertical type**

We here consider a class of semidiscrete surfaces which contains the ones with continuous rotational symmetry. Its defining property is that the infinitesimal quadrilaterals are trapezoids in the sense that the infinitesimal edges $\dot{x}\,dt$ and $\dot{x}_1\,dt$ are parallel: We define that x is trapezoidal of the vertical type, if

$$\dot{x}(i,u) = \lambda(i,u)e(u), \quad \text{where } \lambda : \mathbb{Z}\times\mathbb{R} \to \mathbb{R},\ e : \mathbb{R} \to S^2.$$

Assume that such surfaces x, s are defined by coefficient functions λ, μ and unit vectors $e(u), \widetilde{e}(u)$. If x, s are to be parallel, we can without loss of generality assume $\widetilde{e} = e$. We have the following result:

$$x \parallel s \iff \partial_u(\Delta\mu : \Delta\lambda) = 0 \text{ and } \Delta s : \Delta x = \Delta\mu : \Delta\lambda,$$

where the " $\Longrightarrow$ " part is true in the generic case where the directions of $\Delta e, \Delta s$ on the one hand, and e on the other hand are linearely independent.

Proof. Clearly the right hand side implies $x \parallel s$. Conversely assume now $x \parallel s$. Differentiation of $\Delta x \times \Delta s = 0$ yields $(\Delta\lambda\cdot e)\times\Delta s + \Delta x \times (\Delta\mu\cdot e) = (\Delta\lambda\Delta s - \Delta\mu\Delta x)\times e = 0$. The genericity assumption implies $\Delta\lambda\Delta s - \Delta\mu\Delta x = 0$, which is the right hand property we wanted to show. We differentiate again and get $\Delta\lambda\Delta\mu e + \dot{\Delta}\lambda\Delta s - \Delta\lambda\Delta\mu e - \dot{\Delta}\mu\Delta x = \dot{\Delta}\lambda\Delta s - \dot{\Delta}\mu\Delta x = 0$ Therefore, $\dot{\Delta}\lambda : \Delta\lambda - \dot{\Delta}\mu : \Delta\mu = 0$ which can also be written as $\partial_u(\log\Delta\lambda - \log\Delta\mu) = 0$, i.e., $\Delta\lambda : \Delta\mu$ does not depend on the continuous parameter. □

We immediately get the following formulas for mean and Gaussian curvature:

$$H = -\frac{1}{2}\Big(\frac{\mu_1-\mu}{\lambda_1-\lambda} + \frac{\mu_1+\mu}{\lambda_1+\lambda}\Big), \quad K = \frac{\mu_1-\mu}{\lambda_1-\lambda}\cdot\frac{\mu_1+\mu}{\lambda_1+\lambda}.$$

Any condition that either H or K assumes a given value can be converted into a recursion which expresses both μ_1, λ_1 in terms of λ, μ, γ, if we observe that $\Delta\mu = \gamma\Delta\lambda$.

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References

1. Bobenko, A., Pottmann, H., Wallner, J.: A curvature theory for discrete surfaces based on mesh parallelity. Math. Annalen **348**, 1–24 (2010)
2. Bobenko, A., Suris, Yu.: Discrete differential geometry: Integrable Structure. No. 98 in Graduate Studies in Math. American Math. Soc. (2008)
3. Cohen-Steiner, D., Morvan, J.M.: Restricted delaunay triangulations and normal cycle. In: Proc. ACM Symp. Comp. Geometry, pp. 312–321. ACM (2003)
4. Liu, Y., Pottmann, H., Wallner, J., Yang, Y.L., Wang, W.: Geometric modeling with conical meshes and developable surfaces. ACM Trans. Graphics **25**(3), 681–689 (2006)
5. Müller, C.: Discretizations of the hyperbolic cosine. Beitr. Alg. Geometrie (2012). To appear
6. Müller, C., Wallner, J.: Semidiscrete isothermic surfaces. Results Math. (2012). To appear
7. Pottmann, H., Liu, Y., Wallner, J., Bobenko, A., Wang, W.: Geometry of multi-layer freeform structures for architecture. ACM Trans. Graphics **26**(3), #65,1–11 (2007)

8. Pottmann, H., Schiftner, A., Bo, P., Schmiedhofer, H., Wang, W., Baldassini, N., Wallner, J.: Freeform surfaces from single curved panels. ACM Trans. Graphics **27**(3), #76,1–10 (2008)
9. Pottmann, H., Wallner, J.: The focal geometry of circular and conical meshes. Adv. Comp. Math **29**, 249–268 (2008)
10. Schief, W.K.: On the unification of classical and novel integrable surfaces. II. Difference geometry. R. Soc. Lond. Proc. Ser. A **459**, 373–391 (2003)
11. Schief, W.K.: On a maximum principle for minimal surfaces and their integrable discrete counterparts. J. Geom. Physics **56**, 1484–1495 (2006)
12. Schneider, R.: Convex bodies: the Brunn-Minkowski theory, *Encyclopedia of Mathematics and its Applications*, vol. 44. Cambridge Univ. Press (1993)
13. Simon, U., Schwenck-Schellschmidt, A., Viesel, H.: Introduction to the affine differential geometry of hypersurfaces. Lecture Notes. Science Univ. Tokyo (1992)
14. Wallner, J.: Semidiscrete surface representations. In: A. Bobenko, et al. (eds.) Discrete Differential Geometry, Oberwolfach Reports (2009). Abstracts from the workshop held January 12–17, 2009
15. Wallner, J.: On the semidiscrete differential geometry of A-surfaces and K-surfaces. J. Geometry **103**, 161–176 (2012)