ENERGY OF A KNOT: VARIATIONAL PRINCIPLES.

OLEG KARPENKOV

1. Introduction

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and $\tau : S^1 \longrightarrow \mathbb{R}^3$ be a smooth knot. We will assume that $\tau(t)$ is the arc length parametrization. Denote by $D(t_1, t_2)$ the length of the minimal subarc between t_1 and t_2 on the circle. Let $| * |$ denote the absolute value of vectors in $\mathbb{R}^3$.

Following [1], we denote by

$$E(\tau) = E_f(\tau) = \iint\limits_{S^1 \times S^1} f(|\tau(t_1) - \tau(t_2)|, D(t_1, t_2)) dt_1 dt_2$$

the energy of the knot τ, where $f(\rho, \alpha)$ satisfies the following conditions:

1) $f(\rho, \alpha) \in C^{1,1}(U)$, where $U = \{(\rho, \alpha) | 0 < \rho \leq \alpha, \alpha \leq \pi\}$;

2) there exist the following limits:

$$\lim_{\substack{(\rho,\alpha)\in U \\ \rho\to 0, \rho/\alpha \to 1}} f(\rho, \alpha), \quad \lim_{\substack{(\rho,\alpha)\in U \\ \rho\to 0, \rho/\alpha \to 1}} \frac{\partial f(\rho, \alpha)}{\partial \rho}, \quad \lim_{\substack{(\rho,\alpha)\in U \\ \rho\to 0, \rho/\alpha \to 1}} \frac{\partial f(\rho, \alpha)}{\partial \rho}.$$

Almost all energies are not homothety invariant, so we will consider only knots of length 2π.

The energy of a knot is not an invariant of the topological class of this knot. If we make a smooth perturbation of a knot, its energy smoothly changes. We will consider energies with the following important properties. The energy is always positive. When a knot crossing tends to a double point, the energy tends to infinity. So every topological class of knots has a representative with the minimal value of energy. This knot is called a *normal form* of the class. It is unknown whether each class has a unique normal form or not, i.e., whether the normal form for some energy is an invariant of the topological class or not. The normal forms satisfy the variational equations considered below.

Some energies have a physical meaning. For example $f = 1/(|\tau(t_1) - \tau(t_2)|)$ is the energy of a charged knot. Unfortunately, this energy is always infinite. As long as the charged knot does not break there must be some other forces which save the knot. Let us consider a model of such a restriction:

$$f = \frac{D^2(t_1, t_2)}{|\tau(t_1) - \tau(t_2)|}.$$

For this energy we will develop our variational principles.

The study of knot energies began with the work of Moffatt (1969) [5], and was developed by him in [6] following Arnold's work [2]. The first steps in studying properties of the energies of knots were made by O'Hara [7, 8, 9] and the first variational principles for polygons in space were studied by Fukuhara [4].

The aim of this note is to prove that any extremal knot τ satisfies certain variational equations. The paper is organized as follows. We start in Section 2 with the definitions and formulations of the

Date: April 28th, 2002.

Key words and phrases. Energy of a knot, variational principle, perturbation, extremum of a functional.

Partially supported by RBRF-00-15-96084 and RBRF-01-01-00660 projects.

main theorem. In Section 3 we prove this theorem. In Section 4 we prove that the circle unknot always satisfies our extremal conditions. Unfortunately the integrals in the equations do not converge for all possible energies. For example, they do not converge in the case of the most famous energy: Möbius energy. We discuss this also in Section 4. Section 5 seems to be independent from the previous sections. In Section 5 we consider Möbius energy of a knot. We prove some inequality for the energy of a normal form of the connected sum of two knots.

The author is grateful to professor A. B. Sossinsky for constant attention to this work.

2. Notation and definitions

Mostly we will work with knots of fixed length 2π. So let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and let $\tau : S^1 \longrightarrow \mathbb{R}^3$ denote some smooth knot of length 2π. Let $\tau(t)$ be the arc length parametrization.

By $\kappa(t)$ we denote the curvature at t and $R(t) = 1/\kappa(t)$, the radius of curvature at t.

Definition 2.1. Given a smooth knot $\tau : S^1 \longrightarrow \mathbb{R}^3$ and a point $t_0 \in S^1$, a *locally perturbed knot* is a knot (denoted by $\tau_{t_0,\varepsilon}$) such that

a) $|\tau(t) - \tau_{t_0,\varepsilon}(t)| < \varepsilon^2$ if $D(t_0,t) \leq \varepsilon$ and $\tau(t) = \tau_{t_0,\varepsilon}(t)$ if $D(t_0,t) > \varepsilon$;

b) $|\kappa(t) - \kappa_{t_0,\varepsilon}(t)| < \varepsilon$ for $D(t_0,t) < \varepsilon$;

c) $\tau_{t_0,\varepsilon}(t_0+\lambda) = \tau_{t_0,\varepsilon}(t_0) + \lambda\dot{\tau}_{t_0,\varepsilon}(t_0) + (\lambda^2/2)\ddot{\tau}_{t_0,\varepsilon}(t_0) + o(\varepsilon^2)$ if $D(t_0, t_0+\lambda) \leq \varepsilon$.

Note that at the points $t_0 - \varepsilon$ and $t_0 + \varepsilon$ the curvature is not restricted.

The length of the knot $\tau_{t_0,\varepsilon}$ can change, but we regard knots of length 2π only. One of the ways to solve this problem is to consider the restriction of the set of locally perturbed knots to the set of knots of constant length 2π, but this definition is unsatisfactory. Indeed, let a knot τ in some neighborhood of the point t_0 be a piece of a straight line. Then the set of locally perturbed knots at the point t_0 of length 2π consists of the knot τ only.

We will extend this set in the following way.

Definition 2.2. Let the length of $\tau_{t_0,\varepsilon}$ be $(1+\delta)2\pi$. The *locally perturbed length 2π knot* $\tilde{\tau}_{t_0,\varepsilon}$ is the knot obtained from $\tau_{t_0,\varepsilon}$ by homothety with coefficient $1/(1+\delta)$ and center at the origin. We also say that the knot $\tilde{\tau}$ is *associated* with the knot τ.

Consider any $\tau_{t_0,\varepsilon}$. We will show later that $\delta = c_1\varepsilon^3 + \text{o}(e^3)$. Thus by Definition 2.1 we have

$$|\tau_{t_0,\varepsilon}(t_1) - \tau_{t_0,\varepsilon}(t_2)| = |\tau(t_1) - \tau(t_2)| + c_2(t_1,t_2)\varepsilon^2 + o(e^2)$$

if $D(t_0,t_1) < \varepsilon$ or $D(t_0,t_2) < \varepsilon$. Then we may conclude that

$$E(\tau_{t_0,\varepsilon}) = E(\tau) + c_3\varepsilon^3 + o(\varepsilon^3) \quad \text{and} \quad E(\tilde{\tau}_{t_0,\varepsilon}) = E(\tau) + c_4\varepsilon^3 + o(\varepsilon^3).$$

The coefficients c_3 and c_4 of the term ε^3 will be called the *variation* and denoted by $Var(\tau_{t_0,\varepsilon})$ and $Var(\tilde{\tau}_{t_0,\varepsilon})$ respectively.

Now all is prepared for the definition of a locally extremal point of a knot.

Definition 2.3. Any $t_0 \in S^1$ is called *locally extremal point* of τ if $Var(\tilde{\tau}_{t_0,\varepsilon}) = 0$ for each locally perturbed knot $\tilde{\tau}_{t_0,\varepsilon}$ of length 2π.

Definition 2.4. The knot τ is said to be *locally extremal* if all its points are locally extremal.

Let us find necessary and sufficient conditions for the point t_0 be locally extremal. We denote the vector product of two vectors a and b by $[a,b]$. By (a,b,c) we denote the mixed product (oriented

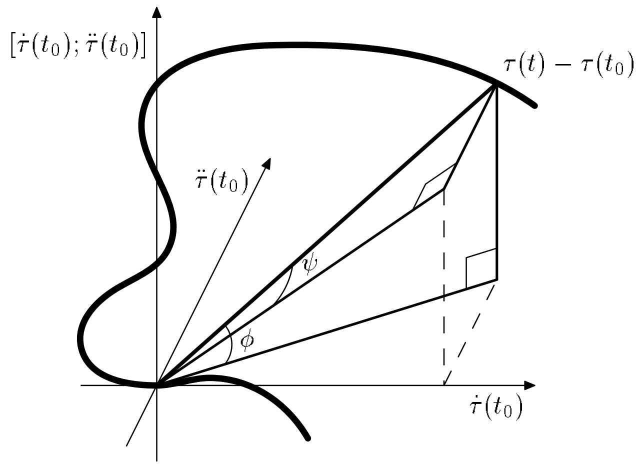

FIGURE 1. The geometric interpretation of $\psi(t_0, t)$ and $\phi(t_0, t)$.

volume) of the vectors a, b and c. Let $\dot\tau(t)$ be the velocity vector and $\ddot\tau(t)$ be the acceleration vector. Now we define the functions $\Psi(t_0, t)$ and $\Phi(t_0, t)$.

$$\Psi(t_0,t)=\begin{cases}\left(\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|},\frac{\ddot\tau(t_0)}{|\ddot\tau(t_0)|},\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|}\right) & \text{, if } \ddot\tau(t_0)\neq 0;\\ \left(\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|},\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|}\right) & \text{, if } \ddot\tau(t_0)=0.\end{cases}$$

$$\Phi(t_0,t)=\begin{cases}\left(\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|},\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|},\left[\frac{\dot\tau(t_0)}{|\dot\tau(t_0)|},\frac{\ddot\tau(t_0)}{|\ddot\tau(t_0)|}\right]\right) & \text{, if } \ddot\tau(t_0)\neq 0;\\ 0 & \text{, if } \ddot\tau(t_0)=0.\end{cases}$$

Note that $|\dot\tau(t_0)| = 1$ and $|\tau(t) - \tau(t_0)| \neq 0$ if $t \neq t_0$. Thus Ψ and Φ are well defined.

We also remark that $\Psi(t_0, t) = \sin\psi(t_0, t)$, where $\psi(t_0, t)$ is the angle between the vector $\tau(t)-\tau(t_0)$ and the oriented plane spanning of $\dot\tau(t_0)$ and $\ddot\tau(t_0)$. The function Φ has a similar representation: $\Phi(t_0, t) = \sin\phi(t_0, t)$, where $\phi(t_0, t)$ is the angle between the vector $\tau(t) - \tau(t_0)$ and the oriented plane spanning of $\dot\tau(t_0)$ and $[\dot\tau(t_0), \ddot\tau(t_0)]$. (See Fig. 1). These angles can be either positive or negative.

Theorem 2.1. *Let τ be a smooth knot. The point t_0 is a locally extremal point of τ if and only if the following conditions hold:*

$$V_1(t_0) := \frac{2}{3R(t_0)}\left(4\int\limits_{S^1}\left(f + R(t_0)\Phi(t_0,t)\frac{\partial f}{\partial\rho}\right)dt - \frac{1}{\pi}\iint\limits_{S^1\times S^1}\left(2f + D(t_1,t_2)\frac{\partial f}{\partial\rho} + |\tau(t_1)-\tau(t_2)|\frac{\partial f}{\partial\alpha}\right)dt_1dt_2 + 2\iint\limits_{A}\frac{\partial f}{\partial\alpha}dt_1dt_2\right) = 0;$$

$$V_2(t_0) := \frac{4}{3R(t_0)}\int\limits_{S}\frac{\partial f}{\partial\rho}\Psi(t_0,t)dt = 0.$$

Here $A \subset S^1 \times S^1$ is the set of points (t_1, t_2) such that $D(t_1, t_2) = D(t_1, t_0) + D(t_0, t_2)$.

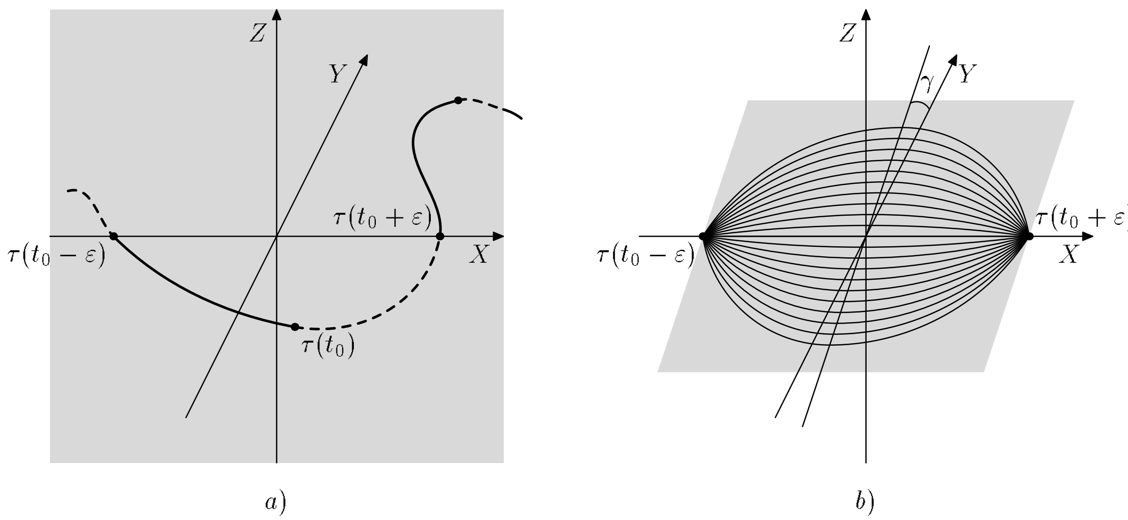

FIGURE 2. a)The choice of X, Y and Z-axes. b)The parabolic arcs (α, γ) where γ is fixed.

Corollary 2.1. *A knot τ is locally extremal if and only if almost all of its points are locally extremal, i.e.,*

$$\int_{S^1} \Big(V_1^2(t) + V_2^2(t)\Big) dt = 0.$$

3. PROOFS

Let t_0 be any point of S^1. We choose orthonormal coordinates in $\mathbb{R}^3$ such that $\tau(t_0)$ is on the (X, Y)-plane, $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ lie symmetrically on the X-axis. If $\tau(t_0 - \varepsilon)$, $\tau(t_0)$ and $\tau(t_0 + \varepsilon)$ are on the same line, then we make any possible choice of the Y-axis. Finally, we choose the Z-axis such that the orientation of the (X, Y, Z)-space is positive (see Fig. 2a)).

Let P_ε be the class of parabolic arcs and one segment such that all the parabolas have their vertex in the (Y, Z)-plane, $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ are the endpoints of the arcs, and the endpoints of the segment are $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$. Each parabola can be specified by two parameters (λ, γ), where 2λ is the "acceleration" and γ is the angle between the (X, Y)-plane and the plane containing the parabola (see Fig. 2b)). Notice also that $(0, \gamma)$ is some segment.

Denote by $M_{P,t_0,\varepsilon}$ the 2-dimensional set of knots $\tau_{t_0,\varepsilon,\lambda,\gamma}$, where the curve connecting $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ belongs to the class P_ε with the following property: the knot $(\tau_{t_0,\varepsilon,\lambda,\gamma} + \tau)/2$ is a locally perturbed knot. Denote by $\tilde{M}_{P,t_0,\varepsilon}$ the set of knots associated with the knots in the class P_ε.

Theorem 3.1. *Let τ be a smooth knot. The point t_0 is a locally extremal point if and only if $Var(\tilde{\tau}_{t_0,\varepsilon}) = 0$ for each locally perturbed (at t_0) knot $\tilde{\tau}_{t_0,\varepsilon} \in \tilde{M}_{P,t_0,\varepsilon}$.*

Proof of Theorem 3.1.
We begin the proof with the following lemma.

Lemma 3.1. *Let $C = \{(x, y, z) \in \mathbb{R}^3 | \sqrt{y^2 + z^2} < r, |x| < s\}$ be a cylinder. Suppose a point moves inside C with velocity of constant modulus 1 and so that the absolute value of its acceleration is bounded by K (see Fig.3). Let $x(0) = -s$, $x(T) = s$, $s \gg r$ and $K < 1/(4r)$. Then the length of the*

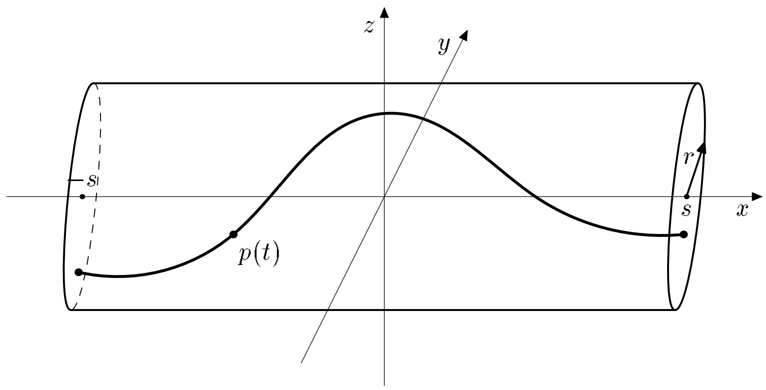

FIGURE 3. The trajectory of the point $p(t)$ inside the cylinder C.

trajectory of a point (i.e. T) is bounded:

$$T < \frac{2s}{\sqrt{1-4Kr}}.$$

First let us prove that $\dot{y}^2(t_0) < 2Kr$. We first consider the case for which $x(t_0) < 0$, $\dot{x}(t_0) > 0$ and $y(t_0) > 0$; then

$$y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}(\xi) d\xi < r.$$

By the assumption, we have

$$\dot{y}(\xi) = \dot{y}(t_0) + \int_{t_0}^{\xi} \ddot{y}(\zeta) d\zeta > \dot{y}(t_0) - \int_{t_0}^{\xi} K d\zeta = \dot{y}(t_0) - (\xi - t_0)K.$$

It follows that

$$y(t) > y(t_0) + \int_{t_0}^{t} \dot{y}(x_0) - (\xi - t_0)K d\xi = y(t_0) + (t-t_0)\dot{y}(t_0) - \frac{(t-t_0)^2}{2}K.$$

But $y(t_0) > -r$ and $y(t) < r$, so

$$(t-t_0)\dot{y}(t_0) - \frac{(t-t_0)^2}{2}K - 2r < 0.$$

By assumption $x < 0$ and $s \gg r$, so the vertex of the parabola is at the point $t - t_0 = \dot{y}(t_0)/K < s$. This yields the inequality $\dot{y}^2(t_0) < 2Kr$.

The proof for the cases in which $\dot{x}(t_0) > 0$ and $y(t_0) < 0$; $\dot{x}(t_0) < 0$ and $y(t_0) > 0$; $\dot{x}(t_0) < 0$ and $y(t_0) < 0$ is similar.

Secondly, we claim that $\dot{z}^2(t_0) < 2Kr$. The proof is similar to the inequality for $\dot{y}^2(t_0)$.

By the previous statements, it follows that

$$\dot{x}^2(t_0) = 1 - \dot{y}^2(t_0) - \dot{z}^2(t_0) > 1 - 4Kr > 0$$

for every $t_0 \in [0, T]$. So we have $T < 2s/(1-4Kr)$

This completes the proof of Lemma 3.1

We continue the proof with a generalization of the previous lemma.

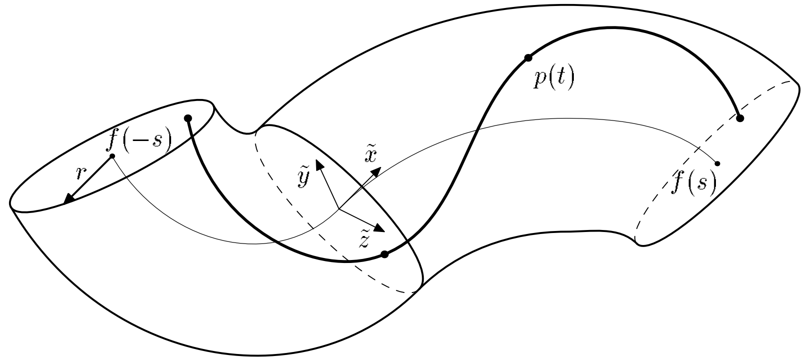

FIGURE 4. The trajectory of the point p(t) inside the cylinder C_f.

Lemma 3.2. *Let $f : [-s,s] \longmapsto \mathbb{R}^3$ be a unit-length smooth map, let the curvature of f be bounded ($|\ddot{f}(t)| < K_1$) and $sK_1 < 1$. Let $D^2(t) \in \mathbb{R}^3$, where $t \in [-s,s]$ is the disk of radius r centered at $f(t)$ with the plane of the disc orthogonal to $\dot{f}$. Let also $rK_1 < 1$. Denote by $C_f = \bigcup_{[-s,s]} D^2(t)$ the tubular neighborhood of the curve f. Suppose a point $p(t) = (x(t), y(t), z(t))$ moves inside C_f (see Fig. 4) with velocity of constant absolute value 1 and let the absolute value of its acceleration be bounded by K_2. Let $p(0) \in D^2(-s)$, $p(T) \in D^2(s)$. Let $s \gg r$ and*

$$K_2 + \frac{1}{1 - rK_1} K_1 < \frac{1}{4r}.$$

Then the length of the trajectory of the point (i.e., T) is bounded and

$$2s(1 - rK_1) < T < \frac{2s(1 + rK_1)}{\sqrt{1 - 4K_2 r}}.$$

Let us define $\tilde{x} = t$.

Now we describe some map π from C_f to the standard cylinder C (see Fig. 3). Let

$$\pi(D^2(\tilde{x})) = \{(\tilde{x}, y, z) \in \mathbb{R} | \sqrt{y^2 + z^2}\}$$

be isometric images of the disk $D^2(\tilde{x})$ for each $\tilde{x} \in [-s,s]$. If we fix a preimage $\tilde{y}$-axis of the y-axis and a preimage $\tilde{z}$-axis of the z-axis in the disc $D^2(\tilde{x})$ for each $\tilde{x} \in [-s,s]$, then the map will be completely described. As long as $sK_1 < 1$ and $rK_1 < 1$, this map is well defined and the manifold $N_f = \bigcup_{[-s,s]} \partial D^2(t)$ with boundary $\partial D^2(-s) \cup \partial D^2(s)$ is smooth.

Let $\pi(\tilde{y}_{-s}) = (-s, r, 0)$ for some $\tilde{y}_{-s} \in \partial D^2(-s)$. Consider the vector field on N_f with the following property: if the point q lies on the circle $\partial D^2(\tilde{x})$, then the vector v_q equals $\dot{f}(\tilde{x})$; this means that v_q is the unit-length vector orthogonal to the disc $D^2(\tilde{x})$ with the corresponding direction. Denote the integral trajectory of this field passing through the point $\tilde{y}_{-s}$ by $\tilde{y} = \{\tilde{y}(\tilde{x}) | \tilde{x} \in [-s,s]\}$. This trajectory defines the $\tilde{y}$ coordinate in each disc $D^2(\tilde{x})$. Finally we define the unit-length $\tilde{z}$-vector as the vector product of the unit-length $\tilde{x}$-vector and unit-length $\tilde{y}$-vector (in each $D^2(\tilde{x})$).

The image $\pi(p)$ of the point p moves inside C. We denote $\pi(p)$ by $\hat{p}$. Notice that

$$\frac{|\dot{p}(t)|}{|\dot{\hat{p}}(t)|} = \frac{1}{|\dot{\hat{p}}(t)|} \in [1 - rK_1, 1 + rK_1].$$

Note also that if the curvature of the trajectory is K at some point $p(t)$, then the curvature of the image of this trajectory will be

$$\hat{K} < K + \frac{1}{\frac{1}{K_1} - r}$$

at the point $\hat{p}(t)$, as can be easily shown.

Now Lemma 3.2 follows from Lemma 3.1.

We continue the proof of Theorem 3.1. Let $\tau_{t_0,\varepsilon}$ be any locally perturbed knot at the point t_0 and let $t \in S^1$ such that $D(t_0, t) < \varepsilon$. Consider

$$P_{\tau_{t_0,\varepsilon}}(t) = \tau_{t_0,\varepsilon}(t_0) + (t - t_0)\big(\dot{\tau}_{t_0,\varepsilon}(t_0) + c_1\big) + \frac{(t - t_0)^2}{2}\big(\ddot{\tau}(t_0) + c_2\big)$$

We choose the constants $c_1 = o(\varepsilon^2)$ and $c_2 = o(\varepsilon^2)$ so that

$$P_{\tau_{t_0,\varepsilon}}(t + \varepsilon) = \tau_{t_0,\varepsilon}(t + \varepsilon), \quad P_{\tau_{t_0,\varepsilon}}(t - \varepsilon) = \tau_{t_0,\varepsilon}(t - \varepsilon).$$

Here we take the unit-length parametrization and denote the length of curves by $l(*)$. Then $P_{\tau_{t_0,\varepsilon}}(t)$ is a parabolic arc in the ε-neighborhood of the point t_0. From Lemma 3.2 it follows that $|P_{\tau_{t_0,\varepsilon}}(t) - \tau_{t_0,\varepsilon}(t)| = o(\varepsilon^2)$ and also $l(P_{\tau_{t_0,\varepsilon}}(t)) = l(\tau_{t_0,\varepsilon}) + o(\varepsilon^3)$. So $E_{\tau_{t_0,\varepsilon}} - E_{P_{\tau_{t_0,\varepsilon}}(t)} = o(\varepsilon^3)$.

Consider the perturbed curve $\tau_{t_0,\varepsilon,\lambda,\gamma}$ passing through the point $\tau_{t_0,\varepsilon}(t_0)$. We have

$$|\tau_{t_0,\varepsilon,\lambda,\gamma}(t) - \tau_{t_0,\varepsilon}(t)| < \varepsilon.$$

We also have $E_{\tau_{t_0,\varepsilon,\lambda,\gamma}} - P_{E_{\tau_{t_0,\varepsilon}}} = o(\varepsilon^3)$.

Finally we conclude that $E_{\tau_{t_0,\varepsilon,\lambda,\gamma}} - E_{\tau_{t_0,\varepsilon}} = o(\varepsilon^3)$.

One can see that the knot $\tau_{t_0,\varepsilon,\lambda,\gamma}$ belongs $M_{P,t_0,\varepsilon}$. We note again that $l(P_{\tau_{t_0,\varepsilon}}(t)) = l(\tau_{t_0,\varepsilon}) + o(\varepsilon^3)$. Hence

$$E_{\tilde{\tau}_{t_0,\varepsilon,\lambda,\gamma}} - E_{\tilde{\tau}_{t_0,\varepsilon}} = o(\varepsilon^3).$$

By definition, the knot $\tilde{\tau}_{t_0,\varepsilon,\lambda,\gamma}$ belongs $\tilde{M}_{P,t_0,\varepsilon}$. This completes the proof of Theorem 3.1.

Proof of Theorem 2.1 Without loss of generality, we put

$$t_0 = 0, \qquad \gamma = o(1), \quad \text{and} \quad \lambda = 1/(2R(0)) + o(1),$$

where R(0) is the radius of curvature at the point 0. According to Theorem 3.1, we can consider only the class $\tilde{M}_P$ of knots. Let $\tilde{\tau}_{0,\varepsilon,\lambda,\gamma}$ be a knot in $\tilde{K}_P$. Denote

$$\Delta := \left[\frac{\varepsilon}{1+\delta}, \frac{\varepsilon}{1+\delta}\right] \subset S^1.$$

Now note that for any τ we have

$$E(\tau) = \iint\limits_{S^1\times S^1} f dxdy = 2\iint\limits_{\Delta\times S^1} f dxdy - \iint\limits_{\Delta\times\Delta} f dxdy + \iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} f dxdy + \iint\limits_{S^1\times S^1\setminus A} f dxdy =: 2E_1(\tau) - E_2(\tau) + E_3(\tau) + E_4(\tau).$$

Here $f = f(\rho(\tau(x), \tau(y)), \alpha(\tau(x), \tau(y)))$. Further note that

$$Var(\tau) = 2Var_1(\tau) - Var_2(\tau) + Var_3(\tau) + Var_4(\tau),$$

where Var_i is the variation of E_i.

First we calculate Var_1. We recall that $\sin\phi = \Phi$ and $\sin\psi = \Psi$.

Lemma 3.3.

$$Var_1(\tilde\tau_{0,\varepsilon,\lambda,\gamma}) = \frac{4}{3}\Big(\int\limits_{S^1} \frac{f}{R(0)} + \sin\phi\frac{\partial f}{\partial\rho}dy\Big)\big(\lambda - \frac{1}{2R(0)}\big) + \frac{2}{3R(0)}\Big(\int\limits_{S^1}\sin\psi\frac{\partial f}{\partial\rho}dy\Big)\gamma.$$

The length of the arc of the parabola is $2\varepsilon + \frac{2}{3}\lambda^2\varepsilon^3 + o(e^3)$. So $\delta = \frac{2}{3}\lambda^2\varepsilon^3$. Note also that the coefficient of homothety is $o(e^2)$ and thus $Var_1(\tau_{0,\varepsilon,\lambda,\gamma}) = Var_1(\tilde\tau_{0,\varepsilon,\lambda,\gamma})$. Let

$$(a,b,c) = (a(t),b(t),c(t)) = \tau_{0,\varepsilon,\lambda,\gamma}(t), \quad \ell = \ell(t) = \sqrt{a(t)^2+b(t)^2+c(t)^2}, \quad f = f(\rho,\alpha).$$

Thus we have

$$E_1(\tau_{0,\varepsilon,\lambda}) = \int\limits_{S^1}\int\limits_{-\varepsilon}^{\varepsilon}\Big(\Big[1+(2\lambda t_1)^2\Big]^{1/2}$$

$$f\Big(\big[((t_1-a(t_2))^2 + ((\lambda {t_1}^2 - \lambda\varepsilon^2)\cos\gamma - b(t_1))^2 + ((\lambda {t_1}^2-\lambda\varepsilon^2)\sin\gamma - c(t_1))^2)\big]^{1/2},$$

$$D(t_1,t_2)\Big)\Big)dt_1dt_2 + o(\varepsilon^3) =$$

$$\int\limits_{S^1}\int\limits_{-\varepsilon}^{+\varepsilon}\Big(\Big(1+2\lambda^2{t_1}^2+o({t_1}^2)\Big)\Big(f + \Big(\frac{{t_1}^2 - at_1 + \lambda(\varepsilon^2-{t_1}^2)(b\cos\gamma + c\sin\gamma)}{\ell} - \frac{a^2{t_1}^2}{2\ell^2}\Big)\frac{\partial f}{\partial\rho} +$$

$$\frac{a^2{t_1}^2\frac{\partial^2 f}{\partial\rho^2}}{2\ell} + D(0,t_1)\frac{\partial f}{\partial\alpha}\Big)\Big)dt_1dt_2 + o(\varepsilon^3) =$$

$$= \int\limits_{S^1}\int\limits_{-\varepsilon}^{+\varepsilon}\Big(2\lambda^2{t_1}^2f + f + \Big(\frac{{t_1}^2 - at_1 + \lambda(\varepsilon^2-{t_1}^2)(b\cos\gamma+c\sin\gamma)}{\ell} -$$

$$\frac{a^2{t_1}^2}{2\ell^2}\Big)\frac{\partial f}{\partial\rho} + \frac{a^2{t_1}^2\frac{\partial^2 f}{\partial\rho^2}}{2\ell} + D(t_1,0)\frac{\partial f}{\partial\alpha}\Big)dt_1dt_2 + o(\varepsilon^3) =$$

$$= \int\limits_{S^1}\Big(\frac{2\lambda^2\varepsilon^3}{3}f + \varepsilon f + \Big(\frac{2\varepsilon^3 + 4\lambda\varepsilon^3(b\cos\gamma + c\sin\gamma)}{3\ell} - \frac{a^2\varepsilon^3}{2\ell^2}\Big)\frac{\partial f}{\partial\rho} + \frac{a^2\varepsilon^3\frac{\partial^2 f}{\partial\rho^2}}{2\ell}\Big)dt_2 +$$

$$\int\limits_{S^1}\int\limits_{-\varepsilon}^{+\varepsilon} D(t_1,0)\frac{\partial f}{\partial\alpha}dt_1dt_2 + o(\varepsilon^3)$$

This yields

$$dE_1(\lambda,\gamma) = d\Big(\int\limits_{S^1}\Big(\frac{2\lambda^2\varepsilon^3}{3}f + \Big(\frac{4\lambda\varepsilon^3(b\cos\gamma+c\sin\gamma)}{3\ell}\Big)\frac{\partial f}{\partial\rho}\Big)dt_2 + o(\varepsilon^3)\Big) =$$

$$\Big(\int\limits_{S^1}\Big(\frac{4\lambda\varepsilon^3}{3}f + \Big(\frac{4\varepsilon^3(b\cos\gamma + c\sin\gamma)}{3\ell}\Big)\frac{\partial f}{\partial\rho}\Big)dt_2 + o(\varepsilon^3)\Big)d\lambda +$$

$$\Big(\int\limits_{S^1}\Big(\frac{4\varepsilon^3\lambda(-b\sin\gamma + c\cos\gamma)}{3\ell}\Big)\frac{\partial f}{\partial\rho}dt_2 + o(\varepsilon^3)\Big)d\gamma.$$

Finally we substitute

$$\frac{b}{\ell} = \sin\phi, \quad \frac{c}{\ell} = \sin\psi, \quad \gamma = o(1), \quad \lambda = \frac{1}{2R(0)} + o(1),$$

where $R(0)$ is the radius of curvature at the point 0, obtaining

$$Var_1(\tilde{\tau}_{0,\varepsilon,\lambda,\gamma}) = \frac{4}{3}\Big(\int\limits_{S^1} \frac{f}{R(0)} + \sin\phi\frac{\partial f}{\partial\rho}dt_2\Big)\Big(\lambda - \frac{1}{2R(0)}\Big) + \frac{2}{3R(0)}\Big(\int\limits_{S^1} \sin\psi\frac{\partial f}{\partial\rho}dt_2\Big)\gamma.$$

The proof of Lemma 3.3 is complete.

Lemma 3.4. $Var_2 = 0$.

Since $E_2(\tau) - E_2(\tilde{\tau}_{0,\varepsilon}) = o(\varepsilon^3)$, we immediately have $Var_2 = 0$.

Lemma 3.5.

$$Var_3 = \Big(\frac{2}{3\pi R(0)} \iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} -2f - \ell\frac{\partial f}{\partial\rho} + (2\pi - D(t_1,t_2))f_\lambda dt_1 dt_2\Big)\big(\lambda - \frac{1}{2R(0)}\big).$$

The following calculations prove this lemma.

$$E_3(\tau_{0,\varepsilon,\lambda}) = \iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} f d\tilde{t_1} d\tilde{t_2} =$$

$$\iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} f\Big(\ell\Big(1 - \frac{2\lambda^2\varepsilon^3}{3\pi}\Big), D\Big(\Big(|t_2 - t_1| + \frac{4\lambda^2\varepsilon^3}{3}\Big)\Big(1 - \frac{2\lambda^2\varepsilon^3}{3\pi}\Big), 0\Big)\Big)$$

$$d\Big((1 - \frac{2\lambda^2\varepsilon^3}{3\pi})t_1\Big) d\Big((1 - \frac{2\lambda^2\varepsilon^3}{3\pi})t_2\Big) + o(\varepsilon^3) =$$

$$\iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} \Big(f + \Big(-\frac{2}{\pi}f - \frac{\ell}{\pi}\frac{\partial f}{\partial\rho} + 2f_\lambda - \frac{D(t_1,t_2)f_\lambda}{\pi}\Big)\frac{2\lambda^2\varepsilon^3}{3}\Big) dt_1 dt_2 + o(\varepsilon^3).$$

Let us remark that

$$\iint\limits_{\Delta\times S^1 \bigcup S^1\times\Delta} \Big(-\frac{2f}{\pi} - \frac{\ell\frac{\partial f}{\partial\rho}}{\pi} + 2f_\lambda - \frac{D(t_1,t_2)f_\lambda}{\pi}\Big)\frac{2\lambda^2\varepsilon^3}{3} dt_1 dt_2 = o(\varepsilon^3).$$

Therefore

$$Var_3 = \Big(\frac{2}{3\pi R(0)} \iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} -2f - \ell\frac{\partial f}{\partial\rho} + \big(2\pi - D(t_1,t_2)\big)f_\lambda dt_1 dt_2\Big)\Big(\lambda - \frac{1}{2R(0)}\Big).$$

Lemma 3.6.

$$Var_4 = \Big(\frac{2}{3\pi R(0)} \iint\limits_{S^1\times S^1\setminus A} -2f - \ell\frac{\partial f}{\partial\rho} - D(t_1,t_2)\frac{\partial f}{\partial\alpha} dt_1 dt_2\Big)\Big(\lambda - \frac{1}{2R(0)}\Big)$$

The proof of this lemma is similar to the previous one.

Lemmas 3.3-3.6 complete the proof of Theorem 2.1.

4. Corollaries

In [1] it is shown that the circle is not always the global maximum, or the global minimum for the energy considered. Let us show that circle is a locally extremal knot for any energy E satisfying the conditions 1), 2) of the Introduction.

Corollary 4.1. *The circle is always a locally extremal knot.*

If τ is a circle, then

$$\ell(t_1,t_2)=2\sin\frac{t_2-t_1}{2},\quad R(t_1)=1,\quad \psi(t_1,t_2)=0,\quad \phi=\frac{t_2-t_1}{2}.$$

So $V_2(t_1)=0$ for any $t_1\in S^1$. Further

$$V_1(t_1)=\frac{1}{3}\Big(8\int\limits_{S^1} f+\sin\Big(\frac{|t_1|}{2}\Big)f_\rho dt_1-\frac{2}{\pi}\iint\limits_{S^1\times S^1} 2f+2\sin\Big(\frac{|t_2-t_1|}{2}\Big)\frac{\partial f}{\partial\rho}$$
$$+D(t_1,t_2)\frac{\partial f}{\partial\alpha}dt_1dt_2+4\iint\limits_{A}\frac{\partial f}{\partial\alpha}dt_1dt_2\Big)=$$
$$\frac{1}{3}\Big(8\int\limits_{S^1} f+\sin\Big(\frac{|t_1|}{2}\Big)\frac{\partial f}{\partial\rho}dt_1-4\int\limits_{S^1}2f+2\sin\Big(\frac{|t_1|}{2}\Big)\frac{\partial f}{\partial\rho}+D(0,t_1)\frac{\partial f}{\partial\alpha}dt_1+$$
$$4\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2D(t_1,t_2)\frac{\partial f}{\partial\alpha}dt_1dt_2\Big)\overset{f(\rho,\alpha)=f(\rho,2\pi-\alpha)}{=}-4\int\limits_{S^1}D(0,t_1)\frac{\partial f}{\partial\alpha}dt_1+4\int\limits_{S^1}D(0,t_1)\frac{\partial f}{\partial\alpha}dt_1=0.$$

Therefore any point of the circle is a locally extremal point. Hence the circle is locally extremal. The corollary is proved.

Now let us say a few worlds about Möbius energy which is (in the version from [3])

$$f_M=\frac{1}{|\tau(t_1)-\tau(t_2)|^2}-\frac{1}{D^2(t_1,t_2)}.$$

It has many remarkable properties (see [7] and [3]). Möbius energies of homothetic knots are equal. This energy is invariant for Möbius transformations (see also Section 5). The variational equations and the gradient flow equation of Möbius energy was studied in [3].

Unfortunately, for Möbius energy, the variation Var is always infinite, and this mean that we can not perturb the knot in the way considered above.

The main property of Möbius energy is as follows. When a knot crossing tends to a double point, the energy tends to infinity. The energy is always positive. So every topological type of knot has a representative with minimal value of energy, some normal form.

Notice that the main part of Möbius energy is $1/|\tau(t_1)-\tau(t_2)|^2$. The other part $1/D^2(t_1,t_2)$ is only a normalization that makes the integral convergent. So let us make another normalization of the "main part" of Möbius energy. In this case we often lose the invariance for Möbius transformations. Let us consider the following energy:

$$\tilde f=\frac{D^3(x,y)}{|\tau(x),\tau(y)|^2}.$$

It is easily seen that this energy on one hand has the above property and on the other we can use our variational principles. Note also that such an energy is the same for homothetic knots.

Corollary 4.2. *We present V_1 and V_2 for this energy:*

$$V_1(t_0) = \frac{2}{3R(t_0)}\Bigg(4\int\limits_{S^1}\Big(\frac{|\tau(t)-\tau(t_0)|^3}{D(t,t_0)^2}\Big(1-2\frac{R(t_0)}{D(t,t_0)}\Phi(t_0,t)\Big)\Big)dt -$$
$$\frac{3}{\pi}\iint\limits_{S^1\times S^1}\frac{|\tau(t_2)-\tau(t_1)|^3}{D(t_2,t_1)^2}dt_1dt_2 + 6\iint\limits_{A}\frac{|\tau(t_2)-\tau(t_1)|^2}{D(t_2,t_1)^2}dt_1dt_2\Bigg);$$
$$V_2(t_0) = -\frac{8}{3R(t_0)}\int\limits_{S}\frac{|\tau(t_1)-\tau(t_2)|^3}{D(t_0,t)^3}\Psi(t_0,t)dt.$$

5. Möbius energy of the connected sum of knots.

In this section we consider only the standard Möbius energy

$$E(\tau) = \iint\limits_{S^1\times S^1} f_M dt_1dt_2 = \iint\limits_{S^1\times S^1}\Big(\frac{1}{|\tau(t_1)-\tau(t_2)|^2} - \frac{1}{D^2(t_1,t_2)}\Big)dt_1dt_2.$$

Denote the topological class of the knot τ by $[\tau]$ and the minimal energy for the class $[\tau]$ (the energy of the normal form of this class) by $E_{[\tau]}$. Let also $[\tau_1+\tau_2]_i$ denotes any possible class of the connected sums of the classes $[\tau_1]$ and $[\tau_2]$. From now we fix the orientations of the summands τ_1 and τ_2. This mean that we choose some class of the connected sum i.

We give some restriction for the energy of the normal form of the connected sum.

Theorem 5.1. *Let $[\tau_1]$ and $[\tau_2]$ be classes of knots. Then the following inequality holds:*

$$E_{[\tau_1+\tau_2]_i} \le E_{[\tau_1]} + E_{[\tau_2]} - 4.$$

In the proof of the Theorem 5.1, we use a nice property of Möbius energy. Möbius energy is invariant for Möbius transformations. Here we recall the theorem from [3].

Theorem 5.2 (Freedman, He, Wang). *Let τ be a knot in $\mathbb{R}^3$ and let T be a Möbius transformation of $\mathbb{R}^3\cup\{\infty\}$. The following statements hold:*
(i) if $T\circ\tau\subseteq\mathbb{R}^3$, then $E(T\circ\tau)=E(\tau)$;
(ii) if $T\circ\tau$ passes through ∞, then $E(T\circ\tau)=E(\tau)-4$.

Let $\varepsilon<\pi$; then we define the function $\chi_\varepsilon : [-\pi,\pi)\to\mathbb{R}$ as follows:

$$\chi_\varepsilon = \begin{cases} 1 & ,|t|<\frac{\varepsilon}{2} \\ 1-\frac{1}{2}\exp\Big(\big(\frac{4|t|}{\varepsilon}-2\big)^{-2}\Big) & ,\frac{\varepsilon}{2}\le|t|<\frac{3\varepsilon}{4} \\ \frac{1}{2}\exp\Big(\big(4-\frac{4|t|}{\varepsilon}\big)^{-2}\Big) & ,\frac{3\varepsilon}{4}\le|t|<\varepsilon \\ 0 & ,|t|\ge\varepsilon \end{cases}.$$

Further we will consider a function χ_ε as the function defined on the circle.

Let τ be a C_1^1 knot (i.e., there exist the derivative of τ and this derivative belongs to Lipschitz class), t_0 a point of this knot and r the radius of curvature at the point t_0. Then in a small neighborhood of t_0 in some orthogonal coordinates τ can be expressed

$$\tau(t) = \Big(r\sin\Big(\frac{t-t_0}{r}\Big),\quad r\cos\Big(\frac{t-t_0}{r}\Big)+f(t-t_0),\quad g(t-t_0)\Big),$$

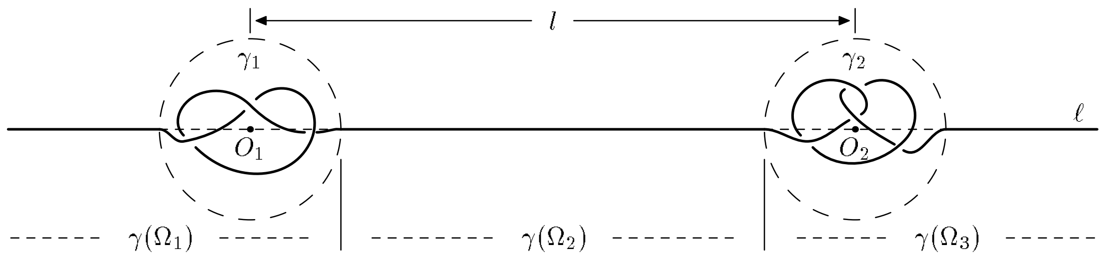

FIGURE 5. The long double knot γ.

where $f(t-t_0)=o((t-t_0)^2)$ and $g(t-t_0)=o((t-t_0)^2)$. Now we are ready to define τ_ε.

$$\tau_\varepsilon(t)=\tau(t)-\Big(0,\quad f(t-t_0)\chi_\varepsilon(t-t_0),\quad g(t-t_0)\chi_\varepsilon(t-t_0)\Big).$$

Lemma 5.1. *For any $\delta>0$ there exists some small $\varepsilon>0$ such that $|E(\tau)-E(\tau_\varepsilon)|<\delta$.*

Direct calculations show that $\chi'_\varepsilon(t)\le O(\varepsilon^{-1})$ and $\chi''_\varepsilon(t)\le O(\varepsilon^{-2})$. Thus we can obtain

$$\begin{aligned}
&f\chi_\varepsilon \le O(\varepsilon^3)O(1)=O(\varepsilon^3);\\
&g\chi_\varepsilon \le O(\varepsilon^3)O(1)=O(\varepsilon^3);\\
&(f\chi_\varepsilon)'=f'\chi_\varepsilon+f\chi'_\varepsilon\le O(\varepsilon^2)O(1)+O(\varepsilon^3)O(\varepsilon^{-1})\le O(\varepsilon^2);\\
&(g\chi_\varepsilon)'=g'\chi_\varepsilon+g\chi'_\varepsilon\le O(\varepsilon^2)O(1)+O(\varepsilon^3)O(\varepsilon^{-1})\le O(\varepsilon^2);
\end{aligned}$$

Therefore the knot τ is the limit in the C^1_1-topology of the knots τ_ε as ε tends to 0. Möbius energy is a smooth functional from the set of C^1_1 knots in the C^1_1-topology (see [3] for the proof of this fact). Hence we can find an ε satisfying the condition of the lemma.

Lemma 5.1 is proven.

Now we consider some class of smooth maps $\gamma:\mathbb{R}\to U\subset\mathbb{R}^3$ without self-intersections, where U is described below. Consider some straight line ℓ and two point O_1 and O_2 on it. We denote the distance between O_1 and O_2 by l. Let r_1 and r_2 be two positive real numbers such that $r_1+r_1<l$. We define U as the union of two open balls B_1 and B_2 of radii r_1 and r_2 centered at O_1 and O_2 and of the straight line ℓ. The map γ sends bijectively some segment and two rays to the set $\ell\setminus(B_1\cup B_2)$. Inside the balls the map γ is smooth and has no any self-intersections. We denote the map restricted to B_1 and B_2 by γ_1 and γ_2 (see Fig. 5). Let also $\gamma(t)$ be a unit length parametrization.

Definition 5.1. We call a map from the class described above a *long double knot.*

Consider some one-parametric family of long double knots $\gamma(l)$ with fixed radii of the balls r_1 and r_2, and the fixed functions γ_1 and γ_2. The parameter of this family is $l=|O_2-O_1|>r_1+r_2$. Denote by q_1 and q_2 the length of the curves γ_1 and γ_2. Let also γ^- be the long double knot with the function γ_1 in the first ball and the straight segment in the second. Similarly, let γ^+ be the knot with the function γ_2 in the second ball and the straight segment in the first. We denote by Ω_2 the preimage of the central segment, and by Ω_1 and Ω_3 the connected components of $\mathbb{R}\setminus\Omega_2$ (see Fig. 5).

Lemma 5.2. *For any $\varepsilon>0$ there exists an $l>r_1+r_2$ such that*

$$E\big(\gamma(l)\big)<E(\gamma^-)+E(\gamma^+)+\varepsilon.$$

Note that

$$E(\gamma(l)) = \Big(\iint\limits_{(\Omega_1\cup\Omega_2)\times(\Omega_1\cup\Omega_2)} + \iint\limits_{(\Omega_1\cup\Omega_2)\times\Omega_3} + \iint\limits_{\Omega_3\times(\Omega_1\cup\Omega_2)} + \iint\limits_{\Omega_3\times\Omega_3} \Big) f_M dt_1 dt_2 =$$
$$\iint\limits_{(\Omega_1\cup\Omega_2)\times(\Omega_1\cup\Omega_2)} f_M dt_1 dt_2 + \iint\limits_{(\Omega_2\cup\Omega_3)\times(\Omega_2\cup\Omega_3)} f_M dt_1 dt_2 - \iint\limits_{\Omega_2\times\Omega_2} f_M dt_1 dt_2 +$$
$$\iint\limits_{\Omega_1\times\Omega_3} f_M dt_1 dt_2 + \iint\limits_{\Omega_3\times\Omega_1} f_M dt_1 dt_2 <$$
$$E(\gamma^-) + E(\gamma^+) - 0 + \iint\limits_{\Omega_1\times\Omega_3} f_M dt_1 dt_2 + \iint\limits_{\Omega_3\times\Omega_1} f_M dt_1 dt_2.$$

Let us estimate the last two integrals.

$$\iint\limits_{\Omega_1\times\Omega_3} f_M dt_1 dt_2 = \iint\limits_{\Omega_3\times\Omega_1} f_M dt_1 dt_2 M < \int\limits_{-\infty}^{r_1-l/2} \int\limits_{l/2-r_2}^{+\infty} \Big(\frac{1}{(t_1-t_2-q_1-q_2)^2} - \frac{1}{(t_1-t_2)^2}\Big) dt_1 dt_2 =$$
$$\ln\Big(\frac{l-r_1-r_2}{l-r_1-r_2-q_1-q_2}\Big) = \ln\Big(1 + \frac{q_1+q_2}{l-r_1-r_2-q_1-q_2}\Big) \overset{l\to+\infty}{\longrightarrow} 0.$$

Therefore for any $\varepsilon > 0$ the desired l exists. Lemma 5.2 is proven.

Now we prove Theorem 5.1.

Let τ_1 and τ_2 be the normal forms in the classes $[\tau_1]$ and $[\tau_2]$. Take any $\delta > 0$. We fix some t_1 and t_2. By Lemma 5.1 there exist two knots τ_{1_ε} and τ_{2_ε} with the small arcs in some neighborhood of this two points, such that

$$|E(\tau_{1_\varepsilon}) - E(\tau)| < \delta \quad \text{and} \quad |E(\tau_{2_\varepsilon}) - E(\tau)| < \delta.$$

Consider the Möbius transformations T_1 and T_2 sending the points t_1 and t_2 of the knots τ_{1_ε} a τ_{2_ε} to infinity. The arcs in the neighborhood of t_1 and t_2 map to the rays of the same straight line. Therefore we can combine $T_1 \circ \tau_{1_\varepsilon}$ and $T_1 \circ \tau_{2_\varepsilon}$ to obtain the long double knot.

By Theorem 5.2 we have:

$$E(T_1 \circ \tau_{1_\varepsilon}) = E(\tau_{1_\varepsilon}) - 4 \quad \text{and} \quad E(T_2 \circ \tau_{2_\varepsilon}) = E(\tau_{2_\varepsilon}) - 4.$$

Further, by Lemma 5.2, using the long knots $E(T_1 \circ \tau_{1_\varepsilon})$ and $E(T_2 \circ \tau_{2_\varepsilon})$ we construct the long double knot γ so that

$$E(\gamma) < E(T_1 \circ \tau_{1_\varepsilon}) + E(T_2 \circ \tau_{2_\varepsilon}) + \delta.$$

Finally, consider a Möbius transformation T which maps the long double knot γ to the knot $T \circ \gamma$. This knot belongs to the class $[\tau_1 + \tau_2]_i$. We use Theorem 5.2 again to obtain the following:

$$E_{[\tau_1+\tau_2]_i} < E(T \circ \gamma) = E(\gamma) + 4 < E(T_1 \circ \tau_{1_\varepsilon}) + E(T_2 \circ \tau_{2_\varepsilon}) - 4 + \delta =$$
$$E(\tau_{1_\varepsilon}) + E(\tau_{2_\varepsilon}) - 4 + \delta = E(\tau_1) + E(\tau_2) - 4 + 3\delta.$$

The inequality

$$E_{[\tau_1+\tau_2]_i} < E(T \circ \gamma) < E(\tau_1) + E(\tau_2) - 4 + 3\delta$$

holds for any $\delta < 0$. This proves Theorem 5.1.

References

[1] A. Abrams, J. Cantarella, J. H. G. Fu, M. Ghomi, and R. Howard, *Circles minimize most knot energies*, http://arXiv.org/abs/math/0105138.

[2] V. I. Arnold, *The asymptotic Hopf invariant and its applications*, in Proc. Summer School in Differential Equations, Erevan, Armenian SSR Academy of Science (1974); English translation in Selecta Math. Sov. 5(1986), 327-345.

[3] M. H. Freedman, Z. -H. He, and Z. Wang, *Mëbius energy of knots and unknots*, Ann. of Math. (2)139 (1994), no. 1, 1-50.

[4] W. Fukuhara, *Energy of a knot*, The fête of topology, Academic Press, (1988), 443-451.

[5] H. K. Moffatt, *The degree of knottedness of tangled vortex lines*, J. Fluid Mech. 35(1969), 117-129.

[6] H. K. Moffatt, *Magnetostatic equilibria and analogous Euler flows of arbitrary complex topology. Part 1. Fundamentals*, J. Fluid Mech. 35(1985), 359-378.

[7] J. O'Hara, *Energy of a knot*, Topology 30(1991), no. 2, 241-247.

[8] J. O'Hara, *Family of energy functionals of knots*, Topology Appl. 48(1992), no. 2, 147-161.

[9] J. O'Hara, *Energy functionals of knots II*, Topology Appl. 56(1994), no. 1, 45-61.

E-mail address, Oleg Karpenkov: `karpenk@mccme.ru`