RATIONAL APPROXIMATION OF THE MAXIMAL COMMUTATIVE SUBGROUPS OF $GL(n,\mathbb{R})$

OLEG N. KARPENKOV[1)], ANATOLY M. VERSHIK[2)]

ABSTRACT. How to find "best rational approximations" of maximal commutative subgroups of $GL(n,\mathbb{R})$? In this paper we pose and make first steps in the study of this problem. It contains both classical problems of Diophantine and simultaneous approximations as a particular subcases but in general is much wider. We prove estimates for $n = 2$ for both totaly real and complex cases and write the algorithm to construct best approximations of a fixed size. In addition we introduce a relation between best approximations and sails of cones and interpret the result for totally real subgroups in geometric terms of sails.

CONTENTS

Introduction: the problem and its relationships

We pose and investigate a problem of approximation of maximal commutative subgroups of $GL(n, \mathbb{R})$ by rational subgroups, or more geometrically in other words a problem of approximation of arbitrary simplicial cones in $\mathbb{R}^n$ by rational simplicial cones. This problem is a natural multidimensional generalization of a problem on rational approximations of real numbers that is contained in the case of $n = 1$. As a particular example it also contains a simultaneous approximation problem and closely related to multidimensional generalizations of continued fractions. The problem of approximation of real spectrum maximal commutative subgroups has much in common with the problem of approximations of nondegenerate simplicial cones. This in particular allows to use methods dealing with multidimensional continued fractions.

Maximal commutative subgroups. We consider a Cartan subgroup of the group $GL(n, \mathbb{R})$ or maximal abelian semisimple subgroups of $GL(n, \mathbb{R})$. Some times it is convenient to consider such subgroup as the set of all matrices, commuting with given semisimple element $A \in GL(n, \mathbb{R})$, i.e., the centralizer $C_{GL(n, \mathbb{R})}(A)$. The centralizer is commutative if and only if A has distinct eigenvalues. So we work with centralizers of "generic" matrices. For the field of real numbers not all Cartan subgroups are mutually conjugate: the general Cartan subgroup in $GL(n, \mathbb{R})$ has k one-dimensional and l two-dimensional minimal eigenspaces (where $k+2l = n$). We will study mainly the Cartan subgroups with only one-dimensional minimal eigenspaces, which we call "real Cartan subgroup", but all the definitions are extended to the general Cartan subgroups of $GL(n, \mathbb{R})$ and can be extended to the case of the Cartan subgroup of $GL(n, \mathbb{C})$ or more general semisimple groups. In that case all elements of the Cartan subgroup has real eigenvalues.

We will use term "maximal commutative subgroup" or shortly MCRF, and denote the space of it as $\mathfrak{C}_n$.

The space of simplicial cones. It is convenient to deal with geometric analog of MCRF-subgroups. Let us describe a relation of real maximal commutative subgroups and nondegenerate simplicial cones.

A *nondegenerate simplicial cone* in $\mathbb{R}^n$ is a conical convex hull of a set of n unordered linearly-independent vectors. Further we omit "nondegenerate", since we work only with nondegenerate cones. Together with any simplicial cone K one may study its symmetric with respect to origin cone $-K$. All further discussions, constructions, notions, and statements are invariant with respect to the map $x \mapsto -x$ of $\mathbb{R}^n$, and hence they all deal with both cones K and its symmetric one $-K$. Therefore, we identify the cones K and $-K$ and define $Simpl_n$ as *a space of pairs of symmetric cones*.

There exists a natural (2^{n-1})-folded covering of the space $\mathfrak{C}_n$ of all maximal commutative subgroups by the space $Simpl_n$:

$$Simpl_n \to \mathfrak{C}_n$$

the cones map to the subalgebras whose eigendirections are the extremal rays of the cones. So for any element of $Simpl_n$ we have a maximal commutative subgroups.

Therefore, approximation problems, which we discuss below and which are local problems, can be studied in terms of the groups as well as in terms of simplicial cones.

A space $Simpl_n$ of all simplicial cones in $\mathbb{R}^n$ can be defined directly with coordinates of cones generators, nevertheless it is very important to understand this space as a *homogeneous space of the group* $GL(n,\mathbb{R})$ in the following way.

Consider a group $GL(n,\mathbb{R}), n > 1$ of all linear invertible transformations in $\mathbb{R}^n$ with a fixed basis. Take D_n — the subgroup of the diagonal matrices in the chosen basis which have positive numbers on the diagonal, i.e. a positive part of the corresponding Cartan subgroup or connected component of the unity of that subgroup. The elements of this subgroup leaves invariant each of the 2^n of coordinate cones. The left homogeneous space $GL(n,\mathbb{R})/D_n$ can be considered as a space of all connected parts of the Cartan subgroups of the group $GL(n,\mathbb{R})$. To get a cone (or actually a pair of symmetric cones K and $-K$) we should add a symmetric group of coordinate permutations S_n (Weil group) which is also contained in the normalizer of D_n. Denote by $\hat{D}_n$ the skew-product $S_n \ltimes D_n$ of the symmetric group and the subgroup of diagonal matrices.

A homogeneous space

$$GL(n,\mathbb{R})/\hat{D}_n$$

of left conjugacy classes in $GL(n,\mathbb{R}), n > 1$ *with respect to the subgroup* $\hat{D}_n$ *is naturally identified with the space of all* (*pairs of*) *nondegenerate simplicial cones* $Simpl_n$.

Indeed, the subgroup of $GL(n,\mathbb{R})$ preserving the positive coordinate cone $\mathbb{R}_+^n$ as well as its reflection coincides with the group $\hat{D}_n$, and $GL(n,\mathbb{R})$ transitively acts on $Simpl_n$.

Notice that it is sometimes convenient to take the group $SL(n,\mathbb{R})$ instead of $GL(n,\mathbb{R})$ (factoring the last by the subgroups of positive scalar matrices and taking $\hat{D}_n$ as the subgroup of positive diagonal matrices with unit determinant in $SL(n,\mathbb{R})$:

$$Simpl_n = SL(n,\mathbb{R})/\{\hat{D}_n \cap SL(n,\mathbb{R})\}$$

A homogeneous space $Simpl_n, n > 1$ is not compact. This space admits a transitive right action of the whole group $GL(n,\mathbb{R})$ and it possess an essential absolutely continuous measure μ_n, that is quasihomogeneous with respect of the action. This measure is called Möbius measure, it was studied in [18]. We are mostly interested in the actions of $SL(n,\mathbb{Z})$ and $SL(n,\mathbb{Q})$ on the space $Simpl_n$ but not in the action of the whole group $GL(n,\mathbb{R}), n > 1$. These actions are ergodic.

Definition 0.1. Consider a simplicial cone $C \in Simpl_n$. The boundary of the convex hall of the integer points in this cone without an origin, i.e.

$$\partial\Big(\operatorname{conv}\big\{C \cap \mathbb{Z}^n \setminus (0,\dots,0)\big\}\Big),$$

is called *the sail of the simplicial cone.*

The space of the simplicial cones could be identified with the space of the sails of simplicial cones.

Remark. Note that one can consider the sail for other convex bodies, for instance of the interiors of conics.

For the simplest case of $n = 2$ a simplicial cone is a convex angle between two rays on the plane, and the space $Simpl_2$ of all cones is a two dimensional torus without a diagonal modulo the involution: $\{S^1 \times S^1 \smallsetminus Diag\}/ \approx$, where $Diag$ is the diagonal in $S^1 \times S^1$ and $\approx$ is a factorization: $(x, y) \approx (y, x)$. Here the points of the circles S^1 are the oriented lines in $\mathbb{R}^2$ that contains critical rays of the angles, and quasiinvariant measure is the Lebesque measure. Actually $Simpl_2$ is a Möbius strip without a boundary or equivalently a punctured projective plane. The geometry of the corresponding cone includes a part of the classical theory of continuous fraction. The sail for $n = 2$ is the boundary of noncompact convex polygon. The two-dimensional case is tightly connected with classical continued fractions (see in Section 2).

The problem of approximations. The described relation between simplicial cones and real spectrum (i.e. having real eigenvalues, see further) maximal commutative subgroups in $GL(n, \mathbb{R})$ preserving the corresponding cones is a covering (up to an identification of the cone and its central symmetrical image). Therefore approximations of such subgroups and approximations of simplicial cones (we speak about this further) are the same up to the lifting. Recall that we have fixed a system of coordinates in $\mathbb{R}^n$, and hence we have a special coordinate simplicial cone $K_0 = \mathbb{R}^n_+$ (a hyperoctant).

Definition 0.2. A *rational simplicial cone* (or respectively a *rational commutative subgroup*) is a cone (a subgroup) whose all extremal rays (eigen-directions) contains points distinct to the origin with all rational coordinates, actually this implies the existence of points with all integer coordinates as well.

A simplicial cone (maximal commutative subgroup) is called *algebraic* if there exists a matrix $g \in SL(n, \mathbb{Z})$ with distinct eigenvalues whose eigen-directions generates this cone (respectively integer matrix whose centralizer in $SL(n, \mathbb{R})$ coincides with this subgroup).

It is clear that the rational cones form the orbit of the coordinate cone K_0 with respect to the group $SL(n, \mathbb{Q})$.

An example of an algebraic simplicial cone is the conical convex hull of the two eigenvectors of the Fibonacci matrix:

$$g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Definition 0.3. Consider some cone $C \in Simpl_n$ and take nonzero linear forms $L_1, \ldots, L_n$ that annulates the hyperfaces of the cone. A *Markoff-Davenport form* is

$$\Phi_C(x) = \frac{\prod\limits_{k=1}^{n} \big(L_k(x_1, \ldots, x_n)\big)}{\Delta(L_1, \ldots, L_n)}$$

where $\Delta(L_1, \ldots, L_n)$ is the volume of the parallelepiped spanned by L_k for $k = 1, \ldots, n$ in the dual space.

This form is defined by a cone uniquely up to a sign. Now having Markoff-Davenport form Φ one can define distances between two cones. For two cones C_1 and C_2 consider two forms

$$\Phi_{C_1}(v) + \Phi_{C_2}(v) \quad \text{and} \quad \Phi_{C_1}(v) - \Phi_{C_2}(v).$$

Take the maximal absolute values of the coefficients of these forms separately, the minimal of them would be the distance between C_1 and C_2. Further in Subsection 1.1 we define Markoff-Davenport form in a more general situation.

Now we are ready to formulate the main problem of approximations:
For a given simplicial cone (or maximal commutative subgroup of $SL(n,\mathbb{R})$) find a rational simplicial cone (rational maximal commutative real subgroup) that for a chosen Markoff-Davenport metric is the closest rational simplicial cone (subgroup) in some fixed class of rational cones (subgroups).
Such classes of rational cones can chosen to be finite classes including only cones having fixed "sizes" of integer points on their rays (for more information see below in Section 1).

First of all the approximations problem by rational simplicial cones (subgroup) must be considered for algebraic cones (subgroups). The most intriguing things are connected with generalization of the beautiful theory of Markoff-Lagrange spectra [31] and Markoff-Davenport n-ary forms [10].

Relations with theory of multidimensional continued fractions. The problem on approximation of commutative subgroups or simplicial cones formulated above and studied in this work is intimately connected with the theory of multidimensional continued fractions but does not reduce to that.

The recent work by V. I. Arnold [2] and the following works by him [4], E. I. Korkina [26], G. Lachaud [29], J.-O. Mussafir [33], Karpenkov [14], etc., revived the interest to one of classical generalizations of continued fractions theory, considered for the first time by F. Klein in [23]. From geometrical point of view the generalization deals with *sails*. The classical theory of ordinary continued fractions i.e. theory of Gauss transformations in algebro-dynamical terms related to the case $n = 2$ was made by R. L. Adler and L. Flatto in [1]. M. L. Kontsevich, Yu. M. Suhov in [24] made an improved version admitting an extension to multidimensional case. In the work [24] the authors considered the following approach to these questions: to study the homogeneous space $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$, i.e. the space of lattices in $SL(n,\mathbb{R})$, and the action of the Cartan subgroup D_n on it. For $n = 2$ this action is reduced to the action of the group $\mathbb{R}^1$ and as it is known from [1] it is a special suspension over the Gauss automorphism that lies in a definition of continued fractions.

One can suppose that the solution of the approximation problem reduced to the geometry of the sails in the following sense: in order to find the best approximation of the cone (equivalently maximal commutative subgroup) one must find the appropriate basis of the vectors which belong to the vertices of the sail of this cone or adjacent cone. Up to now *this is an open question*. The experiments show that it could be not always the case (see for instance in Example 3.11).

Let us show connections of our problem with this geometry. First of all the space $Simpl_n$ as we had mentioned can be interpreted as the *space of sails of simplicial cones*. Let us compare our approach to the geometry of sails with [24].

One can think of dynamical systems as of triples: (a space, a group action, an invariant or quasiinvariant measure). Then in [24] the authors study the dynamical system

$$\{SL(n,\mathbb{R})/SL(n,\mathbb{Z}), \quad D_n, \quad \nu_n\}.$$

i.e. in our terms it is multidimensional suspension (time here is a Cartan subgroup) in a given or an arbitrary cone.

Our approach to theory of sails is in some sense dual to the approach of [24]. We consider another dynamical system, namely, the action of a discrete (noncommutative) group $SL(n,\mathbb{Z})$ (or $SL(n,\mathbb{Q})$) in the space of sails (or equivalently simplicial cones):

$$\{Simpl_n(= SL(n,\mathbb{R})/\hat{D}_n), \quad SL(n,\mathbb{Z}), \quad \mu_n\}.$$

Roughly speaking the "time" and the subgroup defining the homogeneous space has been transposed.

Both approaches have their own advantages and limitations. However the main aim of the current work is not in studying of multidimensional sails, their statistics and other properties, but in their applications to approximations.

More about geometry of sails. The geometry of sails is very interesting by itself. One of the essential subjects here is a statistical analysis of their geometric characteristics with respect to the measure on the space of the sails $Simpl_n$. For instance, *what is the measure of sails with given properties: say with given number of faces of some given combinatorial type* (see [24], [5], [6], [15], [18]). This would generalize Gauss-Kuzmin theorem (see in [28]) and some others for ordinary continued fractions. The work in this direction has just started and it is not much known now, first theorems on this subject can be found in [18].

Faces of different dimensions of a sail were studied in [29], [33], [13], [25], [17]. In algebraic cases all faces are polyhedra. It is also natural to consider the sails in the adjacent hyperoctants. The important problem here is to study the condition for a polygonal surface to be a sail form some cone. This problem was posed by V. I. Arnold and was studied in several papers ([3], [4], [14], [16], [17], [20], [26], [27] [29], [33]). In [37] H. Tsuchihashi showed the relation between sails of cones and cusp singularities, introducing a new application to toric geometry. This relation is studied in detail in [19] for the two-dimensional case.

Actually in the study of $Simpl_n$ the other multidimensional generalizations of continued fractions can be useful. This in particular includes the considered before convex-geometric ([23], [4], [26], [29], [14]) local minima type ([32], [7]), Voronoy ([39], [9]), and algorithmic([34], [35]) generalizations of continued fractions.

Connections with limit shape problems. Another link of the approximation problem is with so called limit shape problems. We want only to emphasize here that the problems

like limit shape problems about Young diagrams or convex lattice polygons (see [38]) can be considered in the simplicial cones (instead of traditional posing in the hyperoctant $\mathbb{Z}_n^+$), and in this case the rational approximation of the cone becomes an important argument. We hope to consider this in the appropriate place.

Description of obtained results. Let us briefly describe the results of this work. Apparently the problem of approximations of arbitrary commutative subgroups in $SL(n, \mathbb{R})$ was never stated in such generality. By the problem of approximation we mean the problem of finding of best approximation of a simplicial cone by rational cones (similar to the classical problem on best approximations of real numbers by rational numbers). This problem is very complicated already in the case of $n = 2$. That is also applied even to the algebraic cones. We give several estimates that suggest an idea that best approximations are not always related to sails or to sails of adjacent cones (see also in Example 3.11).

First, we show that the classical case of approximations of real numbers by rational numbers is really one of particular cases of the proposed new approximation model. In addition we also indicate that simultaneous approximations are also covered by our approach.

Further we work in general case of $n = 2$. We give upper and lower estimates for the discrepancy between best approximations and original simplicial cones in the following important case (Theorem 3.1): *let $\alpha_1, \alpha_2 \in \mathbb{R}$ both have infinite continued fractions with bounded elements, consider a simplicial cone bounded by two lines $y = \alpha_1$ and $y = \alpha_2$, then the growth rate of the best approximation of size N is bounded by C_1/N^2 and C_2/N^2 while N tends to infinity.* Then we translate this statement to the language of sails and their generalizations (Theorem 3.8) and finally show an algorithm to construct best approximations of a fixed size.

Remark. In this paper we work in a slightly extended way including commutative subgroups of $SL(n, \mathbb{R})$ having complex conjugate eigenvectors as well. This is the main reason for our choice to use terminology of commutative subgroups instead of simplicial cones (that are convenient only for the totally real case).

We conclude the paper with several examples of approximations in the three-dimensional case, coming from simultaneous approximations.

The paper is organized as follows. In Section 1 we give basic notions and definitions of maximal subgroup approximation theory. We introduce sizes and discrepancies for the subgroups and define the notion of "best approximations" in our context. In Section 2 we briefly show how the classical theory of Diophantine approximations is embedded into theory of subgroup approximations.

Further we make first steps to study a general two-dimensional case. It is rather complicated since we need to approximate an object defined by four entries of 2×2 matrices that vary. Hence this case is comparable with a general case of simultaneous approximations of vectors in $\mathbb{R}^4$. Nevertheless it is simpler to find the best approximations in the case of subgroups, especially in special algebraic case when a certain periodicity of approximations take place. In Section 3 we write estimates for the quality of best approximations for

both hyperbolic and non-hyperbolic cases of rays whose continued fractions has bounded elements. This in particular includes an algebraic case. We also show geometric origins of the bounds in terms of continued fractions for the hyperbolic algebraic case.

Finally in Section 4 we study in a couple examples the case of simultaneous approximations of vectors in $\mathbb{R}^3$ in the frames of subgroup approximations. We test two algebraic examples coming from totally real and non-totally real cases.

1. RATIONAL APPROXIMATIONS OF MCRF-GROUPS

In this section we give general definitions and formulate basic concepts of maximal commutative subgroups approximations. We recall a definition of a Markoff-Davenport form in Subsection 1.1. Further in Subsection 1.2 we define rational subgroups and choose "size" for them. We define the distance function (discrepancy) between two subgroups in Subsection 1.3.

As we have already mentioned we will continue with terminology of maximal commutative subgroups. In case when we deal with real spectra subgroups the statements can be directly translated to the case of simplicial cones.

1.1. **Regular subgroups and Markoff-Davenport forms.** Consider a real space $\mathbb{R}^n$ and fix some coordinate basis in it. A real operator is called *regular* if all its eigenvalues are distinct (but not necessary real). A maximal commutative subgroup of $GL(n, \mathbb{R})$ is said to be *regular*, or *MCRS-group* for short, if it contains regular operators.

We say that a one-dimensional complex space is an *eigenspace* of an MCRF-group if it is an eigenspace of one of its regular operators. Actually any two regular operators of the same MCRS-group have the same eigenspaces, therefore each MCRF-group has exactly n distinct eigenspaces.

Consider an arbitrary MCRS-group $\mathcal{A}$ and denote its eigenspaces by $l_1, \ldots, l_n$. Denote by L_i a nonzero linear form over $\mathbb{C}^n$ that attains zero values at all vectors of the complex lines l_j for $j \neq i$. Let $\Delta(L_1, \ldots, L_n)$ be the determinant of the matrix having in the k-th column the coefficients of the form L_k for $k = 1, \ldots, n$ in the dual basis.

Definition 1.1. We say that the form

$$\frac{\prod\limits_{k=1}^{n} \big(L_k(x_1, \ldots, x_n)\big)}{\Delta(L_1, \ldots, L_n)}$$

is the *Markoff-Davenport form* for the MCRS-group $\mathcal{A}$ and denote it by $\Phi_{\mathcal{A}}$.

Example 1.2. Consider an MCRS-group containing a Fibonacci operator

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Fibonacci operator has two eigenlines

$$y = -\theta x \quad \text{and} \quad y = \theta^{-1} x,$$

where θ is the *golden ration* $\frac{1+\sqrt{5}}{2}$. So the Markoff-Davenport form of Fibonacci operator is

$$\frac{(y + \theta x)(y - \theta^{-1} x)}{\theta - \theta^{-1}} = \frac{1}{\sqrt{5}}(-x^2 + xy + y^2).$$

A Markoff-Davenport form is uniquely defined by an MCRS-group up to a sign, since the linear forms L_i are uniquely defined by the MCRS-group up to multiplication by a scalar and permutations. By definition any MCRS-group contains a real operator with distinct roots, therefore all the coefficients of the Markoff-Davenport form are real.

Remark 1.3. The minima of the absolute values of such forms on the integer lattice were studied by A. Markoff in [31] for two-dimensional case, and further by H. Davenport in [10], [11], and [12] for three-dimensional totally real case. A few three-dimensional totally real examples were exhoustively studied by A. D. Bryuno, V. I. Parusnikov (see for instance in [8]). The first steps in general multidimensional case were made in paper [21].

1.2. **Rational subgroups and their sizes.** We start with the following definition.

Definition 1.4. An MCRS-group $\mathcal{A}$ is called *rational* if all its eigenspaces contain *Gaussian* vectors, i. e. vectors whose coordinates are of type $a + Ib$ for integers a and b, where $I^2 = -1$. Denote the set of all rational MCRS-groups of dimension n by Rat_n.

Example 1.5. The following two operators

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{with eigenvectors } (I, 1) \text{ and } (-I, 1),$$

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{with eigenvectors } (1, 2) \text{ and } (1, -2)$$

represents rational MCRS-groups (denote them by $\mathcal{A}_i$ and $\mathcal{A}_{ii}$) with real and complex conjugate eigen-directions.

For a complex vector $v = (a_1 + Ib_1, \ldots, a_n + Ib_n)$ denote by $|v|$ the norm

$$\max_{i=1,\ldots,n} \left(\sqrt{a_i^2 + b_i^2} \right).$$

A Gaussian vector is said to be *primitive* if all its coordinates are relatively prime.

Suppose that a complex one-dimensional space has Gaussian vectors, then the minimal value of the norm $|*|$ for the Gaussian vectors is attained at primitive Gaussian vectors.

Definition 1.6. Consider a rational MCRS-group $\mathcal{A}$. Let $l_1, \ldots, l_n$ be the eigenspaces of $\mathcal{A}$. The *size* of $\mathcal{A}$ is a real number

$$\max_{i=1,\ldots,n} \left\{ |v_i| \big| v_i - \text{is a primitive Gaussian vector in } l_i \right\},$$

we denote it by $\nu(\mathcal{A})$.

The sizes of operators in Example 1.5 are 1 and 2 respectively.

1.3. Discrepancy functional and approximation model.

We are focused mostly on the following approximation problem: *how to approximate an MCRS-group by rational MCRS-groups (or even by a certain subset of rational MCRS-groups)?*

Let us first define a natural distance between MCRF-groups. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two MCRS-groups. Consider the following two symmetric bilinear forms

$$\Phi_{\mathcal{A}_1}(v) + \Phi_{\mathcal{A}_2}(v) \quad \text{and} \quad \Phi_{\mathcal{A}_1}(v) - \Phi_{\mathcal{A}_2}(v)$$

for vectors in $\mathbb{R}^n$. Take the maximal absolute values of the coefficients of these forms (separately). The minimal of these two maximal values we consider as a distance between $\mathcal{A}_1$ and $\mathcal{A}_2$, we call it *discrepancy* and denote by $\rho(\mathcal{A}_1, \mathcal{A}_2)$.

Let us calculate the discrepancy between the MCRS-groups of Example 1.5. We have

$$\left| \Phi_{\mathcal{A}_i}(v) \pm \Phi_{\mathcal{A}_{ii}}(v) \right| = \left| I \frac{x^2 + y^2}{2} \pm \frac{y^2 - 4x^2}{4} \right|$$

therefore $\rho(\mathcal{A}_i, \mathcal{A}_{ii}) = \frac{\sqrt{3}}{2}$.

Definition 1.7. Let $\Omega \subset \mathrm{Rat}_n$ for a fixed n. The problem of *best approximations* of an MCRS-group $\mathcal{A}$ by MCRS-groups in Ω is as follows. *For a given positive integer N find a rational MCRS-group $\mathcal{A}_N$ in Ω with size not exceeding N such that*

$$\rho(\mathcal{A}, \mathcal{A}_N) = \min \left\{ \rho(\mathcal{A}, \mathcal{A}') \middle| \mathcal{A}' \in \Omega, \nu(\mathcal{A}') \leq N \right\}.$$

Remark 1.8. There are another important classes of MCRS-groups that contain matrices of $GL(n, \mathbb{Z})$ and $GL(n, \mathbb{Q})$ respectively. The MCRS-group is said to be *algebraic* if it contains regular operators of $GL(n, \mathbb{Z})$. It is natural to consider approximations of MCRS-groups by algebraic MCRS-groups, and approximations of algebraic MCRS-groups by rational MCRS-groups.

2. Diophantine approximations and MCRS-group approximations

A classical problem of approximating real numbers by rational numbers is a particular case of the problem of best approximations of MCRS-groups.

For a real α denote by $\mathcal{A}[\alpha]$ an MCRS-group of $GL(2, \mathbb{R})$ defined by the two spaces $x = 0$ and $y = \alpha x$. Consider any two MCRS-groups $\mathcal{A}[\alpha_1]$ and $\mathcal{A}[\alpha_2]$ with positive α_1 and α_2 and calculate a discrepancy between them.

$$\Phi_{\mathcal{A}[\alpha_1]} - \Phi_{\mathcal{A}[\alpha_2]} = \frac{x(y - \alpha_1 x)}{1} - \frac{x(y - \alpha_2 x)}{1} = (\alpha_2 - \alpha_1)x^2$$

$$\Phi_{\mathcal{A}[\alpha_1]} + \Phi_{\mathcal{A}[\alpha_2]} = \frac{x(y - \alpha_1 x)}{1} + \frac{x(y - \alpha_2 x)}{1} = 2xy - (\alpha_2 + \alpha_1)x^2$$

Since $\alpha_1 > 0$ and $\alpha_2 > 0$ we have

$$\rho(\mathcal{A}[\alpha_1], \mathcal{A}[\alpha_2]) = |\alpha_1 - \alpha_2|.$$

Denote by $\Omega_{[0,1]}^{\mathbb{Q}}$ a subset of all $\mathcal{A}[\alpha]$ for rational α in the segment $[0, 1]$.

For any couple of relatively prime integers (m, n) satisfying $0 \leq \frac{m}{n} \leq 1$ we have

$$\nu\left(\mathcal{A}\left[\frac{m}{n}\right]\right) = n.$$

A classical problem of approximations of real numbers by rational numbers having bounded denominators in our terminology is as follows.

Theorem 2.1. *Consider a real number α, $0 \leq \alpha \leq 1$. Let $[0, a_1, \ldots]$ (or $[0, a_1, \ldots, a_k]$) be an ordinary infinite (finite) continued fraction for α. Then the set of best approximations consists of MCRS-groups $\mathcal{A}[m/n]$ for $m/n = [0, a_1, \ldots, a_{l-1}, a_l]$ where $l = 1, 2, \ldots$ (In case of finite continued fraction we additionally have $\mathcal{A}[m/n]$ for $m/n = [0, a_1, \ldots, a_{k-1}, a_k - 1]$).* $\square$

3. General approximations in two-dimensional case

In this section we prove estimates on the quality of best approximations for MCRS-groups whose eigen-directions are expressed by continued fractions with bounded denominators. We study separately the cases of hyperbolic and non-hyperbolic MCRS-groups. Especially we study geometric interpretation of the bounds in turms of geometric continued fractions for the algebraic hyperbolic MCRS-groups.

3.1. **Hyperbolic case.** An MCRS-group is called *hyperbolic* if it contains a hyperbolic operator (whose all eigenvalues are all real and pairwise distinct).

3.1.1. *Lagrange estimates for a special case.* In this subsection we prove an analog of Lagrange theorem on the approximation rate for an MCRS-groups that has eigenspaces defined by $y = \alpha_1 x$ and $y = \alpha_2 x$ with bounded elements of the continued fractions for α_1 and α_2. In particular this includes all algebraic MCRS-groups. Here we do not consider the case when one of the eigenspaces is $x = 0$, this case was partially studied in Section 2.

Theorem 3.1. *Let α_1 and α_2 be real numbers having infinite continued fractions with bounded elements. Consider an MCRS-group $\mathcal{A}$ with eigenspaces $y = \alpha_1 x$ and $y = \alpha_2 x$. Then there exist positive constants C_1 and C_2 such that for any positive integer N the best approximation $\mathcal{A}_N$ in Ω satisfies*

$$\frac{C_1}{N^2} < \rho(\mathcal{A}, \mathcal{A}_N) < \frac{C_2}{N^2}.$$

We will start the proof with the following two lemmas.

Denote by $\mathcal{A}_{\delta_1, \delta_2}$ the MCRS-group defined by the lines $y = (\alpha_i + \delta_i)x$ for $i = 1, 2$.

Lemma 3.2. *Consider a positive real number ε_1 such than $\varepsilon_1 < 1/|\alpha_1 - \alpha_2|$. Suppose that $\rho(\mathcal{A}, \mathcal{A}_{\delta_1, \delta_2}) < \varepsilon_1$ then*

$$|\delta_1| < \frac{(1+|\alpha_1|)(\alpha_1-\alpha_2)^2}{|\alpha_2|(1-\varepsilon_1|\alpha_1-\alpha_2|)}\varepsilon_1 \qquad and \qquad |\delta_2| < \frac{(1+|\alpha_2|)(\alpha_1-\alpha_2)^2}{|\alpha_1|(1-\varepsilon_1|\alpha_1-\alpha_2|)}\varepsilon_1.$$

Proof. Let us remind that the Markoff-Davenport form of $\mathcal{A}_{\delta_1,\delta_2}$ is

$$\Phi\mathcal{A}_{\delta_1,\delta_2}(x,y) = \frac{\big(y - (\alpha_1 + \delta_1)x\big)\big(y - (\alpha_2 + \delta_2)x\big)}{(\alpha_2 + \delta_2) - (\alpha_1 + \delta_1)}.$$

Consider the absolute values of the coefficients at y^2 and at xy for the difference of Markoff-Davenport forms for the MCRS-groups $\mathcal{A}$ and $\mathcal{A}_{\delta_1,\delta_2}$. By the conditions of the lemma these coefficients are less then ε_1:

$$\left|\frac{\delta_2 - \delta_1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2 + \delta_1 - \delta_2)}\right| < \varepsilon_1 \quad \text{and} \quad \left|\frac{\alpha_1\delta_2 - \alpha_2\delta_1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2 + \delta_1 - \delta_2)}\right| < \varepsilon_1.$$

From the first inequality we have:

$$|\delta_1 - \delta_2| < \frac{(\alpha_1 - \alpha_2)^2}{1 - \varepsilon_1|\alpha_1 - \alpha_2|}\varepsilon_1.$$

The second inequality implies:

$$|\delta_1| < \frac{|(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2 + \delta_1 - \delta_2)|\varepsilon_1 + |\alpha_1(\delta_1 - \delta_2)|}{|\alpha_2|},$$

and therefore

$$|\delta_1| < \frac{|\alpha_1 - \alpha_2|(|\alpha_1 - \alpha_2| + \frac{(\alpha_1-\alpha_2)^2}{1-\varepsilon_1|\alpha_1-\alpha_2|}\varepsilon_1)\varepsilon_1 + |\alpha_1|\frac{(\alpha_1-\alpha_2)^2}{1-\varepsilon_1|\alpha_1-\alpha_2|}\varepsilon_1}{|\alpha_2|} = \frac{(1 + |\alpha_1|)(\alpha_1 - \alpha_2)^2}{|\alpha_2|(1 - \varepsilon_1|\alpha_1 - \alpha_2|)}\varepsilon_1.$$

The inequality for δ_2 is obtained in the same way. $\square$

Lemma 3.3. *Let ε_2 be a positive real number. Suppose $|\delta_1| < \varepsilon_2$ and $|\delta_2| < \varepsilon_2$, then*

$$\rho(\mathcal{A}, \mathcal{A}_{\delta_1,\delta_2}) < \frac{\max\big(2, 2(|\alpha_1| + |\alpha_2|), \alpha_1^2 + \alpha_2^2 + |\alpha_1 - \alpha_2|\varepsilon_2\big)}{(|\alpha_1 - \alpha_2|)(|\alpha_1 - \alpha_2| + 2\varepsilon_2)}\varepsilon_2.$$

Proof. The statement of lemma follows directly form the estimate of the coefficients for the difference of Markoff-Davenport forms for the MCRS-groups $\mathcal{A}$ and $\mathcal{A}_{\delta_1,\delta_2}$. $\square$

Proof of Theorem 3.1. Let us start with the first inequality. Let $\alpha_1 = [a_0, a_1, \ldots]$, and $m_i/n_i = [a_0, a_1, \ldots, a_i]$. Without loss of generality we assume that $N > a_0$. Suppose k is the maximal positive integer for which $m_k \leq N$ and $n_k \leq N$. Then we have

$$\min\left(\left|\alpha_1 - \frac{m}{n}\right|\ \middle|\ |m| \leq N, |n| \leq N\right) \geq \left|\alpha_1 - \frac{m_{k+1}}{n_{k+1}}\right| \geq \frac{1}{n_{k+1}(n_{k+1} + n_{k+2})} \geq$$

$$\frac{1}{(a_{k+1} + 1)n_k\big((a_{k+1} + 1)n_k + (a_{k+1} + 1)(a_{k+2} + 1)n_k\big)} \geq \frac{1}{(a_{k+1} + 1)^2(a_{k+2} + 2)} \cdot \frac{1}{N^2}.$$

For the second and the third inequalities we refer to [22].

The same calculations are valid for α_2. Hence we get C_1 from Lemma 3.2.

Now we prove the second inequality.

$$\left|\alpha_1 - \frac{m_k}{n_k}\right| < \frac{1}{n_k n_{k+1}} < \frac{a_{k+1} + 1}{n_{k+1}^2} < \frac{(a_{k+1} + 1)}{N^2}\max\big(1, (\alpha_1 + 1)^2\big).$$

The first inequality is classical and can be found in [22]. We take maximum in the last inequality for the case of $m_{k+1} > N$ and $n_{k+1} < N$. From conditions of the theorem the set of a_i's is bounded. Therefore, there exists a constant $C'_{2,1}$ such that for any N there exists an approximation of α_1 of quality smaller than $C'_{2,1}/N^2$.

The same holds for α_2. Therefore, we can apply Lemma 3.3 in order to obtain the constant C_2. $\square$

Let us say a few words about the case of unbounded elements of continued fractions for α_i. Take any positive ε. If the elements of a continued fraction (say for α_1) are growing fast enough than there exists a sequence N_i for which the approximations $\mathcal{A}_{N_i}$ are of a quality $\frac{C}{(N_i)^{1+\varepsilon}}$. We show this in the following example.

Example 3.4. Let M be a positive integer. Consider $\alpha_1 = [a_0, a_1, \ldots]$, such that $a_0 = 1$, $a_n = (n_{k-1})^{M-1}$. Denote $\frac{m_k}{n_k} = [a_0, \ldots, a_k]$. Let $\alpha_2 = 0$. Take $N_k = \frac{n_k + n_{k+1}}{2}$. Then there exists a positive constant C such that for any integer i we have

$$\rho(\mathcal{A}, \mathcal{A}_{N_i}) \geq \frac{C}{N_i^{1+1/M}}.$$

Proof. For any i we have

$$n_{i+1} \geq a_i n_i = n_i^{M-1} n_i = n_i^M.$$

Therefore, the best approximation with denominator and numerator less than N_k is not better than

$$\left| \alpha_1 - \frac{m_k}{n_k} \right| \geq \frac{1}{n_k(n_{k+1} + n_k)} \geq \frac{1}{n_{k+1}^M(n_{k+1} + n_k)} \geq \frac{2^{1+1/M}}{N_k^{1+1/M}}.$$

Now we apply Lemma 3.2 to complete the proof. $\square$

We suspect the existence of *badly approximable* MCRS-group $\mathcal{A}$ and a constant C such that there are only finitely many solutions N of the following equation

$$\rho(\mathcal{A}, \mathcal{A}_N) \leq \frac{C}{N},$$

like in the case of simultaneous approximations of vectors in $\mathbb{R}^3$ (see for instance in [30]).

3.1.2. *Periodic sails and best approximations in algebraic case.* Let us show one relation between classical geometry of numbers (for example see in [4]) and best simultaneous approximations.

First we recall the notion of sails. Consider an arbitrary cone C in $\mathbb{R}^2$ with vertex at the origin and boundary rays r_1 and r_2. We also suppose that the angle between r_1 and r_2 is non-zero and less than π. Denote the set of all integer points in the closure of the cone except the origin by I_{r_1,r_2}. The *sail* of this cone is the boundary of the convex hull of I_{r_1,r_2}. It is homeomorphic to a line and contains rays in case of r_i has an integer point distinct to the origin.

Definition 3.5. Define inductively the n-sail for the cone C.
 — let *1-sail* be the sail of C.
 — suppose all k-sails for $k < k_0$ are defined then let k_0-*sail* be

$$\partial\Big(\text{conv}\Big(I_{r_1,r_2} \setminus \bigcup_{k=1}^{k_0-1} k\text{-sail}\Big)\Big),$$

where $\text{conv}(M)$ denote the convex hull of M.

The k-sails have the following interesting property.

Proposition 3.6. *Consider a cone C. The k-sail of C is homothetic to the 1-sail of C and the coefficient of homothety is k.* $\square$

Now consider an arbitrary MCRS-group. Let l_1 and l_2 be the two eigenlines for all the operators of MCRS-group. The union of all four k-sails for the cones defined by the lines l_1 and l_2 is a *k-geometric continued fraction* of the MCRS-group.

Further we proceed with an algebraic case. So a hyperbolic MCRS-group $\mathcal{A}$ contains an $GL(2,\mathbb{Z})$-operator with distinct eigenvalues. In this case the mentioned operator acts on a k-geometric continued fraction (for any k) as a transitive shift. In addition the values of the function

$$\Phi_{\mathcal{A}}(m,n), \quad \text{for } m,n \in \mathbb{Z},$$

are contained in the set $\alpha\mathbb{Z}$ where the value α is attained at some point of the 1-geometric continued fraction. The value $\alpha = \alpha(\mathcal{A})$ is an essential characteristic of $\mathcal{A}$, it is sometimes called *Markoff minima* of the form $\Phi_{\mathcal{A}}$.

Lemma 3.7. *Let an integer point (m,n) be in the k-geometric continued fraction of $\mathcal{A}$. Then*

$$|\Phi_{\mathcal{A}}(m,n)| \geq k\alpha.$$

Proof. We use induction.
    The statement clearly holds for $k = 1$.
    Suppose the statement holds for $k = k_0$ let us prove it for $k = k_0 + 1$. From the step of induction we have the following: for any cone the convex hull of real points $|\Phi_{\mathcal{A}}(a,b)| = k_0\alpha$ contains the k_0-sail of the cone. From the other hand all integer points with $|\Phi_{\mathcal{A}}(m,n)| = k_0\alpha$ (if any) are on the boundary of this convex hull. Hence all of them are in k_0-sail, and thus they are not contained in (k_0+1)-sail. $\square$

Theorem 3.8. *Let $\mathcal{A}$ be an algebraic MCRS-group. Then there exists a positive constants C such that for any positive integer N the following holds. Let the best approximation $\mathcal{A}_N \in \Omega$ be defined by primitive vectors v_1 and v_2 contained in k_1- and k_2-geometric continued fractions respectively, then $k_1, k_2 < C$.*

Proof. By Lemma 3.7 it is sufficient to prove that the set of values of $|\Phi_{\mathcal{A}}(v_i)|$ is bounded.

Let $\mathcal{A}$ has eigenlines $y = \alpha_i x$, $i = 1, 2$. Notice that

$$|\Phi_{\mathcal{A}}(m, n)| = \left| \frac{(m - \alpha_1 n)(m - \alpha_2 n)}{\alpha_1 - \alpha_2} \right| = \left| \frac{m}{n} - \alpha_1 \right| \cdot \left| \frac{m - \alpha_2 n}{\alpha_1 - \alpha_2} n \right|$$

Let $v_1 = (x_1, y_1)$. By Lemma 3.2 (without loss of generality we suppose that v_1 corresponds to δ_1 in the lemma) the first multiplicative is bounded by $\tilde{C}/N^2$ for some constant $\tilde{C}$ that does not depend on N.

Hence,

$$|\Phi_{\mathcal{A}}(x_1, y_1)| \leq \tilde{C} \left| \frac{y_1^2}{N^2} \cdot \frac{\frac{x_1}{y_1} - \alpha_2}{\alpha_1 - \alpha_2} \right| \leq \tilde{C} \left| \frac{\frac{x_1}{y_1} - \alpha_2}{\alpha_1 - \alpha_2} \right|$$

Finally, the last expression is uniformly bounded. The same holds for v_2.

Therefore, the set of values of $|\Phi_{\mathcal{A}}(v_i)|$ is bounded. $\qquad\square$

Conjecture 1. We conjecture that for almost all N the vectors v_1 and v_2 defining $\mathcal{A}_N$ are in 1-geometric continued fraction.

3.1.3. *Technique of calculation of best approximations in the hyperbolic case.* In this subsection we show a general technique of calculation of best approximations for an arbitrary MCRS-group $\mathcal{A}$ with eigenspaces $y = \alpha_1 x$ and $y = \alpha_2 x$ for distinct real numbers α_1 and α_2.

Proposition 3.9. *Let m and n be two integers. Suppose $|\alpha_1 - \frac{m}{n}| < \varepsilon_3$ (or $|\alpha_2 - \frac{m}{n}| < \varepsilon_3$ respectively), then the following holds:*

$$\left| \alpha_1 - \frac{m}{n} \right| > \frac{|\alpha_1 - \alpha_2|}{|\alpha_1 - \alpha_2| + \varepsilon_3} \frac{|\Phi_{\mathcal{A}}(m, n)|}{n^2} \quad \left(\left| \alpha_2 - \frac{m}{n} \right| > \frac{1}{|\alpha_1 - \alpha_2| + \varepsilon_3} \frac{|\Phi_{\mathcal{A}}(m, n)|}{n^2} \right).$$

Proof. We have

$$\left| \alpha_1 - \frac{m}{n} \right| = \frac{1}{n} |m - \alpha_1 n| = \frac{1}{n} \frac{|m - \alpha_1 n|(m - \alpha_2 n)}{m - \alpha_2 n} = \frac{|\Phi_{\mathcal{A}}(m,n)|}{n^2} \frac{|\alpha_1 - \alpha_2|}{|\alpha_1 - \alpha_2 + (\frac{m}{n} - \alpha_1)|} > \frac{|\alpha_1 - \alpha_2|}{|\alpha_1 - \alpha_2| + \varepsilon_3} \frac{|\Phi_{\mathcal{A}}(m,n)|}{n^2}.$$

The same holds for the case of the approximations of α_2. $\qquad\square$

Procedure of best approximation calculation.

1). Find best Diophantine approximations of α_1 and α_2 using continued fractions in the square $N \times N$. Suppose for α_i it is m_i/n_i, and the following best approximation is m_i'/n_i'.

2). Consider now the MCRS-group $\overline{\mathcal{A}}$ with invariant lines $y = \frac{m_i}{n_i} x$. By Lemma 3.3 we get an upper bound for $\rho(\mathcal{A}, \overline{\mathcal{A}})$ (where $\varepsilon_2 = \max(1/(n_1 n_1'), 1/(n_2 n_2'))$).

3). Now having the estimate for discrepancy we use Lemma 3.2 to get estimates C_1 and C_2 for $|\alpha_1 - \frac{p_1}{q_1}|$ and $|\alpha_2 - \frac{p_2}{q_2}|$ for the best approximation of $\mathcal{A}$ with rays $y = \frac{p_1}{q_1} x$ and $y = \frac{p_2}{q_2} x$.

4). By Proposition 3.9 we write an estimate for $\frac{\Phi_{\mathcal{A}}(p_i, q_i)}{q_i^2}$ for $i = 1, 2$.

5). Finally we compare the discrepancies for all MCRS-groups that satisfies the estimates for $\frac{\Phi_{\mathcal{A}}(k_i, l_i)}{l_i^2}$ obtained in 4).

Example 3.10. Consider an MCRS-group containing Fibonacci matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Denote by F_n the n-th Fibonacci number.

Consider any integer $N \geq 100$.

1). Consider a positive integer k such that $F_k \leq N < F_{k+1}$ and choose an approximation $\overline{\mathcal{A}}$ with eigenspaces $F_{k-1}y - F_k x = 0$ and $F_k y + F_{k-1}x = 0$. Then

$$\left| \alpha_1 - \frac{F_k}{F_{k-1}} \right| \leq 1/(F_{k-1}F_k), \quad \left| \alpha_1 + \frac{F_{k-1}}{F_k} \right| \leq 1/(F_k F_{k+1})$$

2). So, $\varepsilon_2 = 1/(F_{k-1}F_k) < 1/(55 \cdot 89)$. Therefore,

$$\rho(\mathcal{A}, \mathcal{A}_{\delta_1, \delta_2}) < \frac{\max\left(2, 2\sqrt{5}, 3 + \sqrt{5}/4895\right)}{5 + \frac{2\sqrt{5}}{4895}} \frac{1}{F_{k-1}F_k} < \frac{2\sqrt{5}}{5 + \frac{2\sqrt{5}}{4895}} \frac{(89/55)^3}{N^2} < \frac{3.79}{N^2}.$$

3). Hence, by Lemma 3.2 we get ($\varepsilon_1 < 3.79/100^2$):

$$|\delta_1| < \tfrac{80.35}{N^2} \qquad \text{and} \qquad |\delta_2| < \tfrac{18.97}{N^2}.$$

4). The estimates for $\frac{\Phi_{\mathcal{A}}(p_1, q_1)}{q_1^2}$ and $\frac{\Phi_{\mathcal{A}}(p_2, q_2)}{q_2^2}$ for the corresponding rays of best approximation are as follows.

$$\frac{|\Phi_{\mathcal{A}}(m_1, n_1)|}{n_1^2} < \frac{80.65}{N^2}, \quad \frac{|\Phi_{\mathcal{A}}(m_2, n_2)|}{n_2^2} < \frac{18.99}{N^2}.$$

5). Notice that the number of approximations whose discrepancies we compare in this step is bounded by some constant not depending on N. We have completed the computations for $N = 10^6$, the answer in this case is the matrix with eigenspaces: $F_{29}y - F_{30}x = 0$ and $F_{30}y + F_{29}x = 0$.

We conjecture that for the Fibonacci matrix we always get the best approximation with eigenspaces $F_{k-1}y - F_k x = 0$ and $F_k y + F_{k-1}x = 0$.

We conclude this subsection with an example showing that the continued fractions do not always give best approximations.

Example 3.11. Consider an operator A with eigenvectors:

$$v_1 = (1, 2) \qquad \text{and} \qquad v_2 = (2, 3),$$

and the corresponding maximal subgroup $\mathcal{A}$. Then there are four different best approximations of size 1, they have invariant lines defined by the following couples of vectors:

$$\left(w_1 = (1, 0), w_2 = (1, 1) \right), \quad \left(w_1 = (1, 0), w_2 = (1, -1) \right),$$
$$\left(w_1 = (1, 0), w_2 = (0, 1) \right), \quad \text{and} \quad \left(w_1 = (0, 1), w_2 = (1, 1) \right).$$

(the discrepancy between $\mathcal{A}$ and any of them equals 6). The continued fraction (or the union of sails) of A contains only four integer points

$$(1, 2), \quad (2, 3), \quad (-1, -2), \quad \text{and} \quad (-2, -3).$$

Therefore the invariant lines of all four best approximations do not contain vectors of the sail of A.

Remark 3.12. Actually, for a generic MCRS-group the best approximation of any size $N > 0$ is unique. In the previous example we have four best approximations since we are approximating MCRS-group defined by vectors with integer coefficients.

3.2. **Non-hyperbolic case.** Now we prove similar statements for the complex case.

3.2.1. *Lagrange estimates for a special case.* In this subsection we prove an analog of Lagrange theorem on the approximation rate for an MCRS-groups that has complex conjugate eigenspaces defined by $y = (\alpha + I\beta)x$ and $y = (\alpha - I\beta)x$ with bounded elements of the continued fractions for α and β. In particular this includes all complex algebraic MCRS-groups.

Theorem 3.13. *Let α and β be real numbers having infinite continued fractions with bounded elements. Consider an MCRS-group $\mathcal{A}$ with eigenspaces $y = (\alpha + I\beta)x$ and $y = (\alpha - I\beta)x$. Then there exist positive constants C_1 and C_2 such that for any positive integer N the best approximation $\mathcal{A}_N$ in Ω satisfies*

$$\frac{C_1}{N^2} < \rho(\mathcal{A}, \mathcal{A}_N) < \frac{C_2}{N^2}.$$

We will start the proof with the following two lemmas.

Denote by $\mathcal{A}_{\delta_1, \delta_2}$ the MCRS-group defined by the lines $y = \big((\alpha + \delta_1) \pm I(\beta + \delta_2)\big)x$ for $i = 1, 2$.

Lemma 3.14. *Consider a positive real number ε_1 such than $\varepsilon_1 < \frac{1}{2(1+|\beta|)}$. Suppose that $\rho(\mathcal{A}, \mathcal{A}_{\delta_1, \delta_2}) < \varepsilon_1$ then*

$$|\delta_1| < \frac{2|\alpha - \beta|\beta^2}{|\alpha - \beta| - 2\varepsilon_1 |\beta|(1+|\beta|)}\varepsilon_1 \qquad and \qquad |\delta_2| < \frac{2(1+|\beta| + |\alpha - \beta|)\beta^2}{|\alpha - \beta| - 2\varepsilon_1 |\beta|(1+|\beta|)}\varepsilon_1.$$

Proof. Consider the absolute values of the coefficients at y^2 and at xy for the difference of Markoff-Davenport forms for the MCRS-groups $\mathcal{A}$ and $\mathcal{A}_{\delta_1, \delta_2}$. By the conditions of the lemma these coefficients are less then ε_1:

$$\left|\frac{\delta_2 - \delta_1}{2\beta(\beta + \delta_2)}\right| < \varepsilon_1 \quad \text{and} \quad \left|\frac{\alpha\delta_2 - \beta\delta_1}{2\beta(\beta + \delta_2)}\right| < \varepsilon_1.$$

Hence we have

$$\left|\frac{(\alpha - \beta)\delta_2}{2\beta(\beta + \delta_2)}\right| \leq + \left|\frac{\alpha\delta_2 - \beta\delta_1}{2\beta(\beta + \delta_2)}\right| + |\beta|\left|\frac{\delta_2 - \delta_1}{2\beta(\beta + \delta_2)}\right| < (1 + |\beta|)\varepsilon_1.$$

This gives us the estimate for δ_2.

For δ_1 we have

$$|\delta_1| < 2|\beta| \left| |\beta| + \frac{2(1+|\beta|)\beta^2}{|\alpha-\beta|-2\varepsilon_1|\beta|(1+|\beta|)}\varepsilon_1 \right| \varepsilon_1 + \frac{2(1+|\beta|)\beta^2}{|\alpha-\beta|-2\varepsilon_1|\beta|(1+|\beta|)}\varepsilon_1 = \frac{2(1+|\beta|+|\alpha-\beta|)\beta^2}{|\alpha-\beta|-2\varepsilon_1|\beta|(1+|\beta|)}\varepsilon_1.$$

The proof is completed. $\qquad\square$

Lemma 3.15. *Let ε_2 be a positive real number. Suppose $|\delta_1| < \varepsilon_2$ and $|\delta_2| < \varepsilon_2$, then*

$$\rho(\mathcal{A}, \mathcal{A}_{\delta_1,\delta_2}) < \frac{\max\left(2, 2(|\alpha|+|\beta|), |\alpha^2-\beta^2|+2|\alpha\beta|+2|\beta|\varepsilon_2\right)}{|\beta|(|\beta|+\varepsilon_2)}\varepsilon_2.$$

Proof. The statement of lemma follows directly form the estimate of the coefficients for the difference of Markoff-Davenport forms for the MCRS-groups $\mathcal{A}$ and $\mathcal{A}_{\delta_1,\delta_2}$. $\qquad\square$

Proof of Theorem 3.13. The remaining part of the proof almost completely repeats the end of the proof of Theorem 3.1, so we omit it here. $\qquad\square$

3.2.2. *Technique of calculation of best approximations in the hyperbolic case.* Here we show a general technique of calculation of best approximations for an arbitrary MCRS-group $\mathcal{A}$ with eigenspaces $y = (\alpha \pm I\beta)x$ for real number α and positive real β.

Proposition 3.16. *Let a satisfy $|\alpha + I\beta| < \varepsilon_3$, then the following holds:*

$$|(\alpha + I\beta) - a| > \frac{2\beta|\Phi_{\mathcal{A}}(1, a)|}{2\beta + \varepsilon_3}.$$

Proof. We have $|(\alpha + I\beta) - a| = \frac{|(\alpha+I\beta)-a|((\alpha-I\beta)-a)}{(\alpha-I\beta)-a} = \frac{2\beta|\Phi_{\mathcal{A}}(1,a)|}{|((\alpha+I\beta)-a)-2I\beta|} > \frac{2\beta|\Phi_{\mathcal{A}}(1,a)|}{2\beta+\varepsilon_3}.$ $\qquad\square$

Procedure of best approximation calculation.

1). Find best Diophantine approximations of α and β using continued fractions in the square $N \times N$. Suppose for α and β it are m_1/n_1, and m_2/n_2, and the next best approximation are m_1'/n_1', and m_2'/n_2'.

2). Consider the MCRS-group $\overline{\mathcal{A}}$ with invariant lines $y = \left(\frac{m_1}{n_1} \pm I\frac{m_2}{n_2}\right)x$. By Lemma 3.15 we get an upper bound for $\rho(\mathcal{A}, \overline{\mathcal{A}})$ (where $\varepsilon_2 = \max(1/(n_1 n_1'), 1/(n_2 n_2'))$).

3). Now having the estimate on discrepancy we use Lemma 3.14 to get estimates C_1 and C_2 for the best approximation of $\mathcal{A}$: $\left|\alpha - \frac{p_1}{q_1}\right|$ and $\left|\beta - \frac{p_2}{q_2}\right|$ respectively.

4). By Proposition 3.16 we write an estimate for $\left|\Phi_{\mathcal{A}}\left(1, \frac{p_1}{q_1} + I\frac{p_2}{q_2}\right)\right|$.

5). Finally we compare the discrepancies for all MCRS-groups that satisfies the estimates obtained in 4).

4. SIMULTANEOUS APPROXIMATIONS IN $\mathbb{R}^3$ AND MCRS-GROUP APPROXIMATIONS

Theory of simultaneous approximation of a real vector by vectors with rational coefficients can be considered as a special case of MCRS-group approximations similarly to the Diophantine case. In this section we study several examples of simultaneous approximations in frames of MCRS-group approximations. The first example is an eigen-direction of a hyperbolic operator (see in Subsection 3.2) and the second is an eigen-direction of a nonhyperbolic operator (see in Subsection 3.3).

4.1. **General construction.** Let $[a, b, c]$ be a vector in $\mathbb{R}^3$. Consider the maximal commutative subgroup $\mathcal{A}[a, b, c]$ defined by three vectors

$$(a, b, c), \quad (0, 1, I), \quad (0, 1, -I).$$

The problem of approximation here is in approximation of the subgroup $\mathcal{A}[a, b, c]$ by $\mathcal{A}[a', b', c']$ for integer vectors (a', b', c'). For this case we have:

$$\Phi_{\mathcal{A}[a,b,c]}(x, y, z) = I\left(-\frac{b^2 + c^2}{2a^2}x^3 + \frac{b}{a}x^2 y + \frac{c}{a}x^2 z - \frac{1}{2}xy^2 - \frac{1}{2}xz^2\right).$$

Therefore,

$$\rho\big(\mathcal{A}[a, b, c], \mathcal{A}[a', b', c']\big) = \min\left(\max\left(\left|\tfrac{b}{a} - \tfrac{b'}{a'}\right|, \left|\tfrac{c}{a} - \tfrac{c'}{a'}\right|, \left|\tfrac{b^2 + c^2}{2a^2} - \tfrac{b'^2 + c'^2}{2a'^2}\right|\right),\right.$$
$$\left.\max\left(\left|\tfrac{b}{a} + \tfrac{b'}{a'}\right|, \left|\tfrac{c}{a} + \tfrac{c'}{a'}\right|, \left|\tfrac{b^2 + c^2}{2a^2} + \tfrac{b'^2 + c'^2}{2a'^2}\right|\right)\right).$$

4.2. **A ray of non-hyperbolic operator.** Consider the non-hyperbolic algebraic operator

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This operator is in some sense the simplest non-hyperbolic operator we can have (see for more information [21]).

Denote the eigenvalues of E_1 by ξ_1, ξ_2, and ξ_3 such that ξ_1 is real, ξ_2 and ξ_3 are complex conjugate. Notice also that

$$|\xi_1| > |\xi_2| = |\xi_3|.$$

We approximate the eigenspace corresponding to ξ_1. Let v_{ξ_1} be the vector in this eigenspace having the first coordinate equal to 1. Note that

$$\xi_1 \approx 1.3247179573 \quad \text{and} \quad v_{\xi_1} \approx (1, .5698402911, .7548776662).$$

The set of best approximations $\mathcal{A}_N$ with $N \leq 10^6$ contains of 48 elements. These elements are of type $B^{n_i}(1, 0, 0)$ where $n_1 = 4$, and for $2 \leq i \leq 48$ we have $n_i = i + 4$. We conjecture that all the set of best approximations coincide with the set of points $B^k(1, 0, 0)$ where $k = 4$, or $k \geq 6$, the approximation rate in this case is $CN^{-3/2}$.

4.3. **Two-dimensional golden ratio.** Let us consider an algebraic operator

$$G = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

This operator is usually called *two-dimensional golden ratio*. It is the simplest hyperbolic operator from many points of view, his two-dimensional continued fraction in the sense of Klein was studied in details by E. I. Korkina in [26] and [27].

The group of all integer operators of $GL(3, \mathbb{Z})$ commuting with G is generated by the following two operators:

$$E_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Note that $G = E_1^2$ and $E_2 = (E_1 - Id)^{-1}$, where Id is an identity operator. Operator E_1 is a *three-dimensional Fibonacci operator*.

Denote the eigenvalues of E_1 by ξ_1, ξ_2, and ξ_3 in such a way that the following holds:

$$|\xi_1| > |\xi_2| > |\xi_3|.$$

Let us approximate the eigenspace corresponding to ξ_1. Denote by v_{ξ_1} the vector of this eigenspace having the last coordinate equal to 1. Note that

$$\xi_1 \approx 2.2469796037 \quad \text{and} \quad v_{\xi_1} \approx (2.2469796037, 1.8019377358, 1).$$

The set of best approximations $\mathcal{A}_N$ with $N \leq 10^6$ contains 40 elements. These elements are in the set

$$\left\{ E_1^m E_2^n (1, 0, 0) \,\Big|\, m, n \in \mathbb{Z} \right\}.$$

All the points of the sequence can be found from the next table. In the column c we get $m = m_c$, $n = n_c$ for the approximation $E_1^{m_c} E_2^{n_c} (1, 0, 0)$.

i	1	2	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
m	1	2	3	3	4	4	5	5	6	6	6	7	7	8	8	9	9	10	10	11	11
n	1	1	2	1	2	1	3	2	3	2	1	3	2	3	2	4	3	4	3	5	4

i	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
m	11	12	12	13	13	14	14	15	15	15	16	16	17	17	18	18	19	19	19
n	3	4	3	5	4	5	4	6	5	4	5	4	6	5	6	5	7	6	5

In addition to this table we have $\mathcal{A}_3 = (3, 2, 1)$ as best approximation.

We conjecture that all the set of best approximations except $\mathcal{A}_3$ is contained in the set of all points of type $E_1^m E_2^n (1, 0, 0)$, the approximation rate in this case is $CN^{-3/2}$.

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E-mail address, Oleg N. Karpenkov: `karpenk@mccme.ru`

E-mail address, Anatoliy M. Vershik: `vershik@pdmi.ras.ru`