ON INVARIANT MÖBIUS MEASURE AND GAUSS-KUZMIN FACE DISTRIBUTION.

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Dedicated to my teacher
Vladimir Igorevich Arnold.

Introduction

Consider an n-dimensional real vector space with lattice of integer points in it. The boundary of the convex hull of all integer points contained inside one of the n-dimensional invariant cones for a hyperbolic n-dimensional linear operator without multiple eigenvalues is called a *sail* in the sense of Klein. The set of all sails of such n-dimensional operator is called $(n-1)$-*dimensional continued fraction* in the sense of Klein (see in more details in Section 2). Any sail is a polyhedral surface. In this work we study frequencies of faces of multidimensional continued fractions.

There exists and is unique up to multiplication by a constant function a form of the highest dimension on the manifold of n-dimensional continued fractions in the sense of Klein, such that the form is invariant under the natural action of the group of projective transformations $PGL(n+1)$. A measure corresponding to the integral of such form is called a *Möbius measure*. In the present paper we deduce an explicit formulae to calculate invariant forms in special coordinates. These formulae allow to give answers to some statistical questions of theory of multidimensional continued fractions. As an example, we show in this work the results of approximate calculations of frequencies for certain two-dimensional faces of two-dimensional continued fractions.

A problem of generalization of ordinary continued fractions was posed by C. Hermite [13] in 1839. One of the most interesting geometrical generalizations was introduced by F. Klein in 1895 in his works [22] and [23]. Unfortunately, the computational complexity of multidimensional continued fractions did not allow to make significant advances in studies of their properties one hundred years ago. V. I. Arnold originally studying A-graded algebras [1] faced with theory of multidimensional continued fractions in the sense of Klein. Since 1989 he has formulated many problems on geometry and statistics of multidimensional continued fractions, reviving an interest to the study of multidimensional continued fractions (see the works [2] and [5]).

Multidimensional continued fractions in the sense of Klein are in use in different branches of mathematics. J.-O. Moussafir [33] and O. N. German [12] studied the connection between the sails of multidimensional continued fractions and Hilbert bases. In [38] H. Tsuchihashi established the relationship between periodic multidimensional continued fractions and multidimensional cusp singularities. This relationship generalizes the classical relationship between

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ordinary continued fractions and two-dimensional cusp singularities known before. The combinatorial topological multidimensional generalization of Lagrange theorem for ordinary continued fractions was obtained by E. I. Korkina in [26] and the corresponding algebraic generalization by G. Lachaud, see [30].

A large number of examples of multidimensional periodic continued fraction were constructed by E. Korkina in [25], [27], and [28], G. Lachaud in [30], and [31], A. D. Bruno and V. I. Parusnikov in [9], and [37], and also by the author in [15] and [16]. A portion of these two-dimensional continued fractions is introduced at the web-site [8] by K. Briggs. A few examples of three-dimensional continued fractions in four-dimensional space were constructed by the author in [21]. The algorithms for constructing multidimensional continued fractions are described in the works of R. Okazaki [36], J.-O. Moussafir [34] and the author [17].

For the first time the statement on statistics of numbers as elements of ordinary continued fractions was formulated by K. F. Gauss in his letters to P. S. Laplace (see in [11]). This statement (see in the first section) was proven further by R. O. Kuzmin [29], and further was proven one more time by P. Lévy [32]. Further investigations in this direction were made by E. Wirsing in [39]. (A basic notions of theory of ordinary continued fractions is described in the books [14] by A. Ya. Hinchin and [5] by V. I. Arnold.) In 1989 V. I. Arnold generalized statistical problems to the case of one-dimensional and multidimensional continued fractions in the sense of Klein, see in [4], [2], and [3].

One-dimensional case was studied in details by M. O. Avdeeva and B. A. Bykovskii in the works [6] and [7]. In two-dimensional and multidimensional cases V. I. Arnold formulated many problems on statistics of sail characteristics of multidimensional continued fractions such as an amount of triangular, quadrangular faces and so on, such as their integer areas, and length of edges, etc. A major part of these problems is open nowadays, while some are almost completely solved.

M. L. Kontsevich and Yu. M. Suhov in their work [24] proved the existence of the mentioned above statistics. Recently V. A. Bykovskii and M. A. Romanov used Monte Carlo method to calculate frequencies for some types of faces of sails. At present paper we write down in special coordinates a natural Möbius measure of the manifold of all n-dimensional continued fractions in the sense of Klein. In particular, this allows to make approximate calculations of relative frequencies of multidimensional faces of multidimensional continued fractions.

Note that the Möbius measure is used also in theory of energies of knots and graphs, see in the works of Freedman M. H., He Z. -H., and Wang Z. [10], J. O'Hara [35] and the author [18]. For the case of one-dimensional continued fractions the Möbius measure is induced by the relativistic measure of three-dimensional de Sitter world.

This work is organized as follows. In the first section we give necessary notions of theory of ordinary continued fractions. In particular, we give the definition of Gauss-Kuzmin statistics. Further in the second section we describe the smooth manifold structure for the set of all n-dimensional continued fractions and define Möbius measure on it. In the third section we study relative frequencies of faces of one-dimensional continued fractions. These frequencies are proportional to the frequencies of Gauss-Kuzmin statistics. In the fourth section we study relative frequencies of faces of multidimensional continued fractions. Finally, in the fifth section we show approximate calculation results of relative frequencies for some faces of two-dimensional continued fractions.

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1. One-dimensional continued fractions and Gauss-Kuzmin statistics

Let α be an arbitrary rational. Suppose that

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + \ldots + 1/(a_{n-1} + 1/a_n)\ldots)),$$

where a_0 is integer, and the remaining a_i, $i = 1, \ldots, n$ are positive integers. An expression on the right side of this equality is called a *decomposition of α into a finite ordinary continued fraction* and denoted by $[a_0, a_1, \ldots, a_n]$. If $n+1$ — the total number of the elements of the decomposition is even, then the continued fraction is said to be *even*, and if this number is odd, then the continued fraction is said to be *odd*.

Let a_0 be integer, and $a_1, \ldots, a_n, \ldots$ be infinite sequence of positive integers. Denote by r_n the rational $[a_0, \ldots, a_{n-1}]$. For such integers a_i, the sequence (r_n) always converges to some real α. The limit

$$\lim_{n\to\infty} [a_0, a_1, \ldots, a_{n-1}]$$

is called the *decomposition of α into a infinite ordinary continued fraction* and denoted by $[a_0, a_1, a_2, \ldots]$.

Ordinary continued fractions possess the following basic properties.

Proposition 1.1. a). *Any rational has exactly two distinct decompositions into a finite ordinary continued fraction, one of them is even, and the other is odd.*
b). *Any irrational has a unique decomposition into an infinite ordinary continued fraction.*
c). *A decomposition into finite ordinary continued fraction is rational.*
d). *A decomposition into infinite ordinary continued fraction is irrational.*

Notice, that for any finite continued fraction $[a_0, a_1, \ldots, a_n]$, where $a_n \neq 1$, the following holds:

$$[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1].$$

This equality determines a one-to-one correspondence between the sets of even and odd finite continued fractions.

Let α be some irrational between zero and unity, and let $[0, a_1, a_2, a_3, \ldots]$ be its ordinary continued fraction. Denote by $z_n(\alpha)$ the real $[0, a_n, a_{n+1}, a_{n+2}, \ldots]$.

Let $m_n(\alpha)$ denote the measure of the set of reals α contained in the segment $[0; 1]$, such that $z_n(\alpha) < x$. In his letters to P. S. Laplace K. F. Gauss formulated without proofs the following theorem. It was further proved by R. O. Kuzmin [29], and then proved one more time by P. Lévy [32].

Theorem 1.2. Gauss-Kuzmin. *For $0 \le x \le 1$ the following holds:*

$$\lim_{n\to\infty} m_n(x) = \frac{\lg(1+x)}{\lg 2}.$$

Denote by $P_n(k)$ for an arbitrary integer $k > 0$ the measure of the set of all reals α of the segment $[0; 1]$, such that each of them has the number k at n-th position. A limit $\lim\limits_{n\to\infty} P_n(k)$ is called a *frequency of k* for ordinary continued fractions and denoted by $P(k)$.

Corollary 1.3. *For any positive integer k the following holds*

$$P(k) = \frac{1}{\ln 2} \ln \left(1 + \frac{1}{k(k+2)} \right).$$

Proof. Notice, that $P_n(k) = m_n(\frac{1}{k}) - m_n(\frac{1}{k+1})$. Now the statement of the corollary follows from Gauss-Kuzmin theorem. □

The problem of V. I. Arnold on the asymptotic behaviours of frequencies of integers as elements of ordinary continued fractions for rationals with bounded numerators and denominators was completely studied by V. A. Bykovskii and M. O. Avdeevain the works [6] and [7]. It turns out that such frequencies coincide with frequencies $P(k)$ defined above.

2. Multidimensional continued fractions in the sense of Klein

2.1. **Geometry of ordinary continued fractions.** Consider a two-dimensional plane with standard Euclidean coordinates. A point is said to be *integer*, if both its coordinates are integers. An *integer length* of the segment AB with integer vertices A and B is the ration of its Euclidean length and the minimal Euclidean length for integer vectors contained in the segment AB, we denote it by $\mathrm{l}\ell(AB)$. An *integer (non-oriented) area* of the polygon P is the ratio of its Euclidean area and the minimal Euclidean area for the triangles with integer vertices, we denote it by $\mathrm{lS}(P)$. The quantity $\mathrm{lS}(P)$ coincides with doubled Euclidean area of the polygon P.

For an arbitrary real $\alpha \geq 1$ we consider an angle in the first orthant defined by the rays $\{(x,y)|y=0, x\geq 0\}$ and $\{(x,y)|y=\alpha x, x \geq 0\}$. The boundary of the convex hull of the set of all integer points in the closure of this angle except the origin O is a broken line, consisting of segments and possible of a ray or two rays contained in the sides of the angle. The union of all segments of that broken line is called the *sail* of the angle. The sail of the angle is a finite broken line for rational α and it is an infinite broken line for irrationals. Denote the point with coordinates $(1,0)$ by A_0, and denote all the others vertices of the broken line consequently by A_1, $A_2, \ldots$ Let $a_i = \mathrm{l}\ell(A_iA_{i+1})$ for $i=0,1,2,\ldots$, let also $b_i = \mathrm{lS}(A_{i-1}A_iA_{i+1})$ for $i=1,2,3,\ldots$, then the following equality holds

$$\alpha = [a_0, b_1, a_1, b_2, a_2, b_3, a_3, \ldots].$$

On Figure 1 we examine an example of $\alpha = 7/5 = [1,2,2]$.

2.2. **Definition of multidimensional continued fractions.** Based on geometrical construction that we describe in the previous subsection F. Klein introduced the following geometrical generalization of ordinary continued fractions to the multidimensional case (see [22] and [23]).

Consider arbitrary $n+1$ hyperplanes in $\mathbb{R}^{n+1}$, such that their intersection consists of a unique point — of the origin. The complement to the union of these hyperplanes consist of 2^{n+1} open orthants. Consider one of them. The boundary of the convex hull for the set of all integer points of the closure of the orthant except the origin is called the *sail* of the orthant. The set of all 2^{n+1} sails is called the *n-dimensional continued fraction*, related to the given $n+1$ hyperplanes. An intersection of a hyperplane with the sail is said to be a *k-dimensional face of the sail* if it is contained in some k-dimensional plane and is homeomorphic to k-dimensional disc. (See also [16].)

Two multidimensional faces of multidimensional continued fractions are said to be *integer-linear (-affine) equivalent*, if there exist a linear (affine) integer lattice preserving transformation

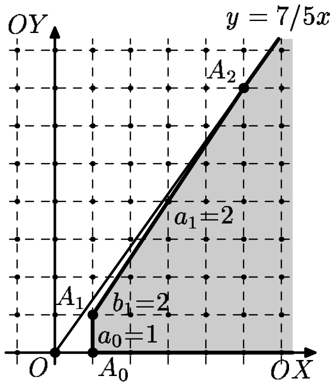


FIGURE 1. The sail for the continued fraction of $7/5 = [1, 2, 2]$.

taking one face to the other. A class of all integer-linear (-affine) equivalent faces is called an *integer-linear (-affine) type* of any face of this class.

Let us define one useful integer-linear invariant of a plane. Consider an arbitrary k-dimensional plane π not containing the origin, whose integer vectors generates a sublattice of rank k in the lattice of all integer vectors. Let the Euclidean distance from the origin to the plane π equal ℓ. Denote by ℓ_0 the minimal nonzero Euclidean distance to π from integer points of the plane (of dimension $k+1$) spanning the given plane π and the origin. The ratio ℓ/ℓ_0 is called the *integer distance* from the origin to the plane π.

Let us now describe one of the original problems of V. I. Arnold on statistics of faces of multidimensional continued fractions. Note that for any real hyperbolic operator with distinct eigenvalues there exists a unique corresponding multidimensional continued fraction. One should take invariant hyperplanes for the action of the operator as hyperplanes defining the corresponding multidimensional continued fraction. Let us consider only three-dimensional hyperbolic operators, that are defined by integer matrices with rational eigenvalues. Denote the set of all such operators by A_3. A continued fraction for any operator of A_3 consists of finitely many faces. Denote by $A_3(m)$ the set of all the operators of A_3 with bounded above by m sums of absolute values of all its coefficients. The number of such operators is finite. Let us calculate the number of triangles, quadrangles and so on among continued fractions, constructed for the operators of $A_3(m)$. While m tends to infinity we have a general distribution of the frequencies for triangles, quadrangles and so on. The problem of V. I. Arnold includes the study of the properties of such distribution (for instant, *what is more frequent: triangles or quadrangles, what is the frequency of integer points inside the faces*, etc.). Note that this problem still has not been completely studied. Surely, the questions formulated above can be easily generalized to the multidimensional case.

V. I. Arnold has also formulated statistical problems for special algebraic periodic multidimensional continued fractions. For more information see [2] and [3].

2.3. **Smooth manifold of n-dimensional continued fractions.** Denote the set of all continued fractions of dimension n by CF_n. Let us describe a natural structure of a smooth nonsingular non-closed manifold on the set CF_n.

Consider an arbitrary continued fraction, that is defined by unordered collection of hyperplanes $(\pi_1, \ldots, \pi_{n+1})$. The enumeration of planes here is relative, without any ordering. Denote by l_i for $i = 1, \ldots, n{+}1$ the intersection of all the above hyperplanes except the hyperplane π_i. Obviously, $l_1, \ldots, l_{n+1}$ are *independent* straight lines (i.e. they are not contained in a hyperplane) passing through the origin. These straight lines form an unordered collection of independent straight lines. From the other side, any unordered collection of $n{+}1$ independent straight lines uniquely determines some continued fraction.

Denote the sets of all ordered collections of $n{+}1$ independent and dependent straight lines by FCF_n and Δ_n respectively. We say that FCF_n is a space of n-dimensional *framed continued fractions.* Also denote by S_{n+1} the permutation group acting on ordered collections of $n{+}1$ straight lines. In this notation we have:

$$FCF_n = \big(\underbrace{\mathbb{R}P^n \times \mathbb{R}P^n \times \ldots \times \mathbb{R}P^n}_{n+1 \text{ times}}\big) \setminus \Delta_n \qquad \text{and} \qquad CF_n = FCF_n / S_{n+1}.$$

Therefore, the sets FCF_n and CF_n admit natural structures of smooth manifolds that are identified by the structure of the Cartesian product of $n{+}1$ projective spaces $\mathbb{R}P^n$. Note also, that FCF_n is an $((n{+}1)!)$-fold covering of CF_n. We call the map of "forgetting" of the order in the ordered collections the *natural projection* of the manifold FCF_n to the manifold CF_n and denote it p, $p: FCF_n \to CF_n$.

2.4. **Möbius measure on the manifolds of continued fractions.** A group $PGL(n{+}1, \mathbb{R})$ of transformations of $\mathbb{R}P^n$ takes the set of all straight lines passing through the origin of $(n{+}1)$-dimensional space into itself. Hence, $PGL(n{+}1, \mathbb{R})$ naturally acts on the manifolds CF_n and FCF_n. Furthermore, the action of $PGL(n{+}1, \mathbb{R})$ is transitive, i. e. it takes any (framed) continued fraction to any other. Note that for any n-dimensional (framed) continued fraction the subgroup of $PGL(n{+}1, \mathbb{R})$ taking this continued fraction to itself is of dimension n.

Definition 2.1. A form of the manifold CF_n (respectively FCF_n) is said to be a *Möbius form* if it is invariant under the action of $PGL(n{+}1, \mathbb{R})$.

Transitivity of the action of $PGL(n{+}1, \mathbb{R})$ implies that all n-dimensional Möbius forms of the manifolds CF_n and FCF_n are proportional if exist.

Let ω be some volume form of the manifold M. Denote by μ_ω a measure of the manifold M that at any open measurable set S contained at the same piece-wise connected component of M is defined by an equality:

$$\mu_\omega(S) = \left| \int_S \omega \right|.$$

Definition 2.2. A measure μ of the manifold CF_n (FCF_n) is said to be a *Möbius measure* if there exist a Möbius form ω of CF_n (FCF_n) such that $\mu = \mu_\omega$.

Note that any two Möbius measures of CF_n (FCF_n) are proportional.

Remark 2.3. The projection p projects the Möbius measures of the manifold FCF_n to the Möbius measures of the manifold CF_n. That establishes an isomorphism between the spaces of Möbius measures for CF_n and FCF_n. Since the manifold of framed continued fractions possesses simpler chart system, all formulae of the work are given for the case of framed continued fractions manifold. To calculate a measure of some set F of the unframed continued fractions manifold one should: take $p^{-1}(F)$; calculate Möbius measure of the obtained set of the manifold of framed continued fractions; divide the result by $(n{+}1)!$.

3. One-dimensional case

3.1. **Explicit formulae for the Möbius form.** Let us write down Möbius forms of the framed one-dimensional continued fractions manifold FCF_1 explicitly in special charts.

Consider a vector space $\mathbb{R}^2$ equipped with standard metrics on it. Let l be an arbitrary straight line in $\mathbb{R}^2$ that does not pass through the origin, let us choose some Euclidean coordinates O_lX_l on it. Denote by $FCF_{1,l}$ a chart of the manifold FCF_1 that consists of all ordered pairs of straight lines both intersecting l. Let us associate to any point of $FCF_{1,l}$ (i. e. to a collection of two straight lines) coordinates (x_l, y_l), where x_l and y_l are the coordinates on l for the intersections of l with the first and the second straight lines of the collection respectively. Denote by $|\overline{v}|_l$ the Euclidean length of a vector $\overline{v}$ in the coordinates $O_lX_lY_l$ of the chart $FCF_{1,l}$. Note that the chart $FCF_{1,l}$ is a space $\mathbb{R}\times\mathbb{R}$ minus its diagonal.

Consider the following form in the chart $FCF_{1,l}$:

$$\omega_l(x_l, y_l) = \frac{dx_l \wedge dy_l}{|x_l - y_l|_l^2}.$$

Proposition 3.1. *The measure μ_{ω_l} coincides with the restriction of ssome Möbius measures to $FCF_{1,l}$.*

Proof. Any transformation of the group $PGL(2,\mathbb{R})$ is in the one-to-one correspondence with the set of all projective transformations of the straight line l projectivization. Note that the expression

$$\frac{\Delta x_l \Delta y_l}{|x_l - y_l|_l^2}$$

is an infinitesimal cross-ratio of four point with coordinates x_l, y_l, $x_l{+}\Delta x_l$ and $y_l{+}\Delta y_l$. Hence the form $\omega_l(x_l, y_l)$ is invariant for the action of transformations (of the everywhere dense set) of the chart $FCF_{1,l}$, that are induced by projective transformations of l. Therefore, the measure μ_{ω_l} coincides with the restriction of some Möbius measures to $FCF_{1,l}$. □

Corollary 3.2. *A restriction of an arbitrary Möbius measure to the chart $FCF_{1,l}$ is proportional to μ_{ω_l}.*

Proof. The statement follows from the proportionality of any two Möbius measures. □

Consider now the manifold FCF_1 as a set of ordered pairs of distinct points on a circle $\mathbb{R}/\pi\mathbb{Z}$ (this circle is a one-dimensional projective space obtained from unit circle by identifying antipodal points). The doubled angular coordinate φ of the circle $\mathbb{R}/\pi\mathbb{Z}$ inducing by the coordinate x of straight line $\mathbb{R}$ naturally defines the coordinates (φ_1, φ_2) of the manifold FCF_1.

Proposition 3.3. *The form $\omega_l(x_l, y_l)$ is extendable to some form ω_1 of FCF_1. In coordinates (φ_1, φ_2) the form ω_1 can be written as follows:*

$$\omega_1 = \frac{1}{4}\operatorname{ctg}^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) d\varphi_1 \wedge d\varphi_2.$$

We leave a proof of Proposition 3.3 as an exercise for the reader.

3.2. **Relative frequencies of faces of one-dimensional continued fractions.** Without loose of generality in this subsection we consider only Möbius form ω_1 of Proposition 3.3. Denote the natural projection of the form μ_{ω_1} to the manifold of one-dimensional continued fractions CF_1 by μ_1.

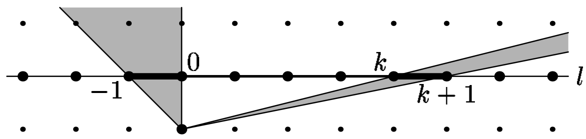


FIGURE 2. Rays defining a continued fraction should lie in the domain colored in gray.

Consider an arbitrary segment F with vertices at integer points. Denote by $CF_1(F)$ the set of continued fractions that contain the segment F as a face.

Definition 3.4. The quantity $\mu_1(CF_1(F))$ is called *relative frequency* of the face F.

Note that the relative frequencies of faces of the same integer-linear type are equivalent. Any face of one-dimensional continued fraction is at unit integer distance from the origin. Thus, integer-linear type of a face is defined by its integer length (the number of inner integer points plus unity). Denote the relative frequency of the edge of integer length k by $\mu_1(''k'')$.

Proposition 3.5. *For any positive integer k the following holds:*

$$\mu_1(''k'') = \ln\left(1+\frac{1}{k(k+2)}\right).$$

Proof. Consider a particular representative of an integer-linear type of the length k segment: the segment with vertices $(0,1)$ and $(k,1)$. One-dimensional continued fraction contains the segment as a face iff one of the straight lines defining the fraction intersects the interval with vertices $(-1,1)$ and $(0,1)$ while the other straight line intersects the interval with vertices $(k,1)$ and $(k{+}1,1)$, see on Figure 2.

For the straight line l defined by the equation $y=1$ we calculate the Möbius measure of Cartesian product of the described couple of intervals. By the last subsection it follows that this quantity coincides with relative frequency $\mu_1(''k'')$. So,

$$\mu_1(''k'') = \int\limits_{-1}^{0}\int\limits_{k}^{k+1}\frac{dx_l dy_l}{(x_l-y_l)^2} = \int\limits_{k}^{k+1}\left(\frac{1}{y_l}-\frac{1}{y_l+1}\right)dy_l =$$
$$\ln\left(\frac{(k+1)(k+1)}{k(k+2)}\right) = \ln\left(1+\frac{1}{k(k+2)}\right).$$

This proves the proposition. □

Remark 3.6. Note that the argument of the logarithm $\frac{(k+1)(k+1)}{k(k+2)}$ is a cross-ratio of points $(-1,1)$, $(0,1)$, $(k,1)$, and $(k{+}1,1)$.

Corollary 3.7. *Relative frequency $\mu_1(''k'')$ up to the factor*

$$\ln 2 = \int\limits_{-1}^{0}\int\limits_{1}^{+\infty}\frac{dx_l dy_l}{(x_l-y_l)^2}$$

coincides with Gauss-Kuzmin frequency $P(k)$ for k to be an element of continued fraction. □

4. Multidimensional case

4.1. **Explicit formulae for the Möbius form.** Let us now write down explicitly Möbius forms for the manifold of framed n-dimensional continued fractions FCF_n for arbitrary n.

Consider $\mathbb{R}^{n+1}$ with standard metrics on it. Let π be an arbitrary hyperplane of the space $\mathbb{R}^{n+1}$ with chosen Euclidean coordinates $OX_1 \ldots X_n$, let also π does not pass through the origin. By the chart $FCF_{n,\pi}$ of the manifold FCF_n we denote the set of all collections of $n{+}1$ ordered straight lines such that any of them intersects π. Let the intersection of π with i-th plane is a point with coordinates $(x_{1,i}, \ldots, x_{n,i})$ at the plane π. For an arbitrary tetrahedron $A_1 \ldots A_{n+1}$ in the plane π we denote by $V_\pi(A_1 \ldots A_{n+1})$ its oriented Euclidean volume in the coordinates $OX_{1,1} \ldots X_{n,1}X_{1,2} \ldots X_{n,n+1}$ of the chart $FCF_{n,\pi}$. Denote by $|\overline{v}|_\pi$ the Euclidean length of the vector $\overline{v}$ in the coordinates $OX_{1,1} \ldots X_{n,1}X_{1,2} \ldots X_{n,n+1}$ of the chart $FCF_{n,\pi}$. Note that the map $FCF_{n,\pi}$ is everywhere dense in $(\mathbb{R}^n)^{n+1}$.

Consider the following form in the chart $FCF_{n,\pi}$:

$$\omega_\pi(x_{1,1}, \ldots, x_{n,n+1}) = \frac{\bigwedge\limits_{i=1}^{n+1}\left(\bigwedge\limits_{j=1}^{n} dx_{j,i}\right)}{V_\pi(A_1 \ldots A_{n+1})^{n+1}}.$$

Proposition 4.1. *The measure μ_{ω_π} coincides with the restriction of some of Möbius measure to $FCF_{n,\pi}$.*

Proof. Any transformation of the group $PGL(n{+}1, \mathbb{R})$ is in the one-to-one correspondence with the set of all projective transformations of the plane π. Let us show that the form ω_π is invariant for the action of transformations (of the everywhere dense set) of the chart $FCF_{n,\pi}$, that are induced by projective transformations of hyperplane π.

Let us at each point of the tangent space to $FCF_{n,\pi}$ define a new basis corresponding to the directions of edges of the corresponding tetrahedron in π. Namely, consider an arbitrary point $(x_{1,1}, \ldots, x_{n,n+1})$ of the chart $FCF_{n,\pi}$ and the tetrahedron $A_1 \ldots A_{n+1}$ in hyperplane π corresponding to the point. Let

$$\overline{f}_{ij} = \frac{\overline{A_jA_i}}{|\overline{A_jA_i}|_\pi}, \qquad i,j = 1, \ldots, n{+}1; \quad i \neq j.$$

The basis constructed above continuously depends on the point of the chart $FCF_{n,\pi}$. By dv_{ij} we denote the 1-form corresponding to the coordinate along the vector $\overline{f}_{ij}$ of $FCF_{n,\pi}$.

Denote by $A_i = A_i(x_{1,i}, \ldots, x_{n,i})$ the point depending on the coordinates of the plane π with coordinates $(x_{1,i}, \ldots, x_{n,i})$, $i = 1, \ldots, n{+}1$. Let us rewrite the form ω_π in new coordinates.

$$\omega_\pi(x_{1,1}, \ldots, x_{n,n+1}) =$$

$$\prod_{i=1}^{n+1}\left(\frac{V_\pi(A_i, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots A_{n+1})}{\prod\limits_{k=1,k\neq i}^{n+1} |\overline{A_kA_i}|_\pi}\right) \cdot \frac{dv_{21} \wedge dv_{31} \wedge \ldots \wedge dv_{n,n+1}}{V_\pi(A_1 \ldots A_{n+1})^{n+1}} =$$

$$(-1)^{[\frac{n+3}{4}]} \cdot \frac{dv_{21} \wedge dv_{12}}{|\overline{A_1A_2}|_\pi^2} \wedge \frac{dv_{32} \wedge dv_{23}}{|\overline{A_2A_3}|_\pi^2} \wedge \ldots \frac{dv_{n+1,n} \wedge dv_{n,n+1}}{|\overline{A_{n+1}A_n}|_\pi^2},$$

here by $[a]$ we denote the maximal integer not exceeding a.

Like in one-dimensional case the expression

$$\frac{\Delta v_{ij}\Delta v_{ji}}{|\overline{A_iA_j}|^2}$$

for the infinitesimal small Δv_{ij} and Δv_{ji} is the infinitesimal cross-ratio of four points: A_i, A_j, $A_i{+}\Delta v_{ji}\overline{f}_{ji}$, and $A_j{+}\Delta v_{ij}\overline{f}_{ij}$ of the straight line A_iA_j. Therefore, the form ω_π is invariant for the action of transformations (of the everywhere dense set) of the chart $FCF_{n,\pi}$, that are induced by projective transformations of hyperplane π. Hence the measure μ_{ω_π} coincides with the restriction of some Möbius measure to $FCF_{n,\pi}$. □

Corollary 4.2. *A restriction of an arbitrary Möbius measure to the chart $FCF_{n,\pi}$ is proportional to μ_{ω_π}.*

Proof. The statement follows from the proportionality of any two Möbius measures. □

Let us fix an origin O_{ij} for the straight line A_iA_j. The integral of the form dv_{ij} (respectively dv_{ij}) for the segment $O_{ij}P$ defines the *coordinate* v_{ij} (v_{ij}) of the point P contained in the straight line A_iA_j. As in one-dimensional case consider a projectivization of the straight line A_iA_j. Denote the angular coordinates by φ_{ij} and φ_{ji} respectively. In this coordinates it holds:

$$\frac{dv_{ij}\wedge dv_{ji}}{|\overline{A_iA_j}|^2_\pi}=\frac{1}{4}\operatorname{ctg}^2\left(\frac{\varphi_{ji}-\varphi_{ij}}{2}\right)d\varphi_{ij}\wedge d\varphi_{ji}.$$

Then, the following is true.

Corollary 4.3. *The form ω_π extends to some form ω_n of FCF_n. In coordinates v_{ij} the form ω_n is as follows:*

$$\omega_n=\frac{(-1)^{[\frac{n+3}{4}]}}{2^{n(n+1)}}\left(\prod_{i=1}^{n+1}\prod_{j=i+1}^{n+1}\operatorname{ctg}^2\left(\frac{\varphi_{ij}-\varphi_{ji}}{2}\right)\right)\cdot\left(\bigwedge_{i=1}^{n+1}\left(\bigwedge_{j=i+1}^{n+1}d\varphi_{ij}\wedge d\varphi_{ji}\right)\right).$$

4.2. **Relative frequencies of faces of multidimensional continued fractions.** As in one-dimensional case without loose of generality we consider the form ω_n of Corollary 4.3. Denote by μ_n the projection of the measure μ_{ω_n} to the manifold of multidimensional continued fractions CF_n.

Consider an arbitrary polytope F with vertices at integer points. Denote by $CF_n(F)$ the set of n-dimensional continued fractions that contain the polytope F as a face.

Definition 4.4. The value $\mu(CF_n(F))$ is called the *relative frequency* of a face F.

Relative frequencies of faces of the same integer-linear type are equivalent.

Problem 1. Find integer-linear types of n-dimensional continued fractions with the highest relative frequencies. Is it true that the number of integer-linear types of faces with relative frequencies bounded above by some constant is finite? Find its asymptotics for the constant tending to infinity.

Problem 1 is open for $n\geq 2$.

Conjecture 2. Relative frequencies of faces are proportional to the frequencies of faces in the sense of Arnold (see in Subsection 2.2).

This conjecture is checked in the present work for the case of one-dimensional continued fractions. It is still open for the n-dimensional case for $n\geq 2$.

5. Examples of calculation of relative frequencies for faces in two-dimensional case

5.1. **A method of relative frequencies computation.** Let us describe a method of relative frequencies computation in two-dimensional case more detailed.

Consider a space $\mathbb{R}^3$ with standard metrics on it. Let π be an arbitrary plane in $\mathbb{R}^3$ not passing through the origin and with fixed system of Euclidean coordinates $O_\pi X_\pi Y_\pi$. Let $FCF_{2,\pi}$ be the corresponding chart of the manifold FCF_2 (see the previous section). For an arbitrary triangle ABC of the plane π we denote by $S_\pi(ABC)$ its oriented Euclidean area in the coordinates $O_\pi X_1Y_1X_2Y_2X_3Y_3$ of the chart $FCF_{2,\pi}$. Denote by $|\overline{v}|_\pi$ the Euclidean length of the vector $\overline{v}$ in the coordinates $O_\pi X_1Y_1X_2Y_2X_3Y_3$ of the chart $FCF_{2,\pi}$. Consider the following form in the chart $FCF_{2,\pi}$:

$$\omega_\pi(x_1,y_1,x_2,y_2,x_3,y_3) = \frac{dx_1\wedge dy_1\wedge dx_2\wedge dy_2\wedge dx_3\wedge dy_3}{S_\pi((x_1,y_1)(x_2,y_2)(x_3,y_3))^3}.$$

Note that the oriented area S_π of the triangle $(x_1,y_1)(x_2,y_2)(x_3,y_3)$ can be expressed in the coordinates x_i, y_i as follows:

$$S_\pi((x_1,y_1)(x_2,y_2)(x_3,y_3)) = \frac{1}{2}\big(x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1 + x_2y_1 - x_1y_2\big).$$

For the approximate computations of relative frequencies of faces it is useful to rewrite the form ω_π in the dual coordinates (see Remark 5.2). Define a triangle ABC in the plane π by three straight lines l_1, l_2, and l_3, where l_1 passes through B and C, l_2 passes through A, and C, and l_3 passes through A, and B. Define the straight line l_i $(i = 1,2,3)$ in π by the equation (preliminary we make a translation of π in such a way that the origin is taken to some inner point of the triangle)

$$a_ix + b_iy = 1$$

in x and y variables. Then if we know the 6-tuple of numbers $(a_1,b_1,a_2,b_2,a_3,b_3)$ we can restore the triangle in the unique way.

Proposition 5.1. *In coordinates a_1,b_1,a_2,b_2,a_3,b_3 the form ω_π can be written as follows:*

$$-\frac{8da_1\wedge db_1\wedge da_2\wedge db_2\wedge da_3\wedge db_3}{(a_3b_2 - a_2b_3 + a_1b_3 - a_3b_1 + a_2b_1 - a_1b_2)^3}.$$

□

So, we reduce the computation of relative frequency for the face F, i. e. the value of $\mu_2(CF_2(F))$ to the computation of measure $\mu_{\omega_2}(p^{-1}(CF_2(F)))$. Consider some plane π in $\mathbb{R}^3$ not passing through the origin. By Corollary 4.3

$$\mu_{\omega_2}(p^{-1}(CF_2(F))) = \mu_{\omega_\pi}(p^{-1}(CF_2(F))\cap(FCF_{2,\pi}).$$

Finally the computation should be made for the set $\mu_{\omega_\pi}(p^{-1}(CF_2(F))\cap(FCF_{2,\pi})$ in dual coordinates a_i,b_i (see Proposition 5.1).

Remark 5.2. In a_i,b_i coordinates the computation of value of the relative frequency often reduces to the estimation of the integral on the disjoint union of the finite number of six-dimensional Cartesian products of three triangles in a_i,b_i coordinates (see Proposition 5.1). The integration over such a simple domain greatly fastens the speed of approximate computations. In particular, the integration can be reduced to the integration over some 4-dimensional domain.

5.2. **Some results.** In conclusion of the work we give some results of relative frequencies calculation for some two-dimensional faces of two-dimensional continued fractions.

Explicit calculations of relative frequencies for the faces seems not to be realizable. Nevertheless it is possible to make approximations of the corresponding integrals. Normally, the greater area of the integer-linear type of the polygon is, the lesser its relative frequency. The most complicated approximation calculations correspond to the most simple faces, such as an empty triangle.

On Figure 3 we show examples of the following faces: triangular $(0,0,1)$, $(0,1,1)$, $(1,0,1)$ and $(0,0,1)$, $(0,2,1)$, $(2,0,1)$ and quadrangular $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,0,1)$. For each face it is shown the plane containing the face. The points painted in light-gray correspond to the points at which the rays defining the two-dimensional continued fraction can intersect the plane of the chosen face.

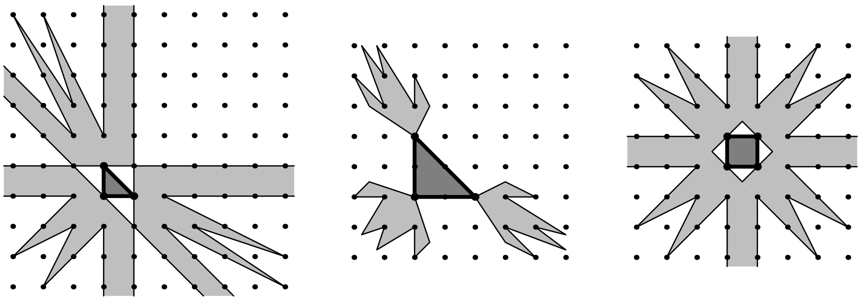


FIGURE 3. The points painted in light-gray correspond to the points at which the rays defining the two-dimensional continued fraction can intersect the plane of the chosen face.

Faces of two-dimensional continued fractions for the majority of integer-linear types lie at unit integer distance from the origin. Only three infinite series and three partial examples of faces lie at integer distances greater or equal to two from the origin, see a detailed description in [19] and [20]. If the distance to the face is increasing, then the frequency of faces is reducing on average. The average rate of reducing the frequency is unknown to the author.

In Table 1 we show the results of relative frequencies calculations for 12 integer-linear types of faces. In a column "N°" we write a special sign for integer-affine type of a face. The index denotes the integer distance from the corresponding face to the origin. In a column "face" we draw a picture of integer-affine type of the face. Further in a column "lS" we write down integer areas of faces, and in a column "ld" we write down integer distances from the planes of faces to the origin. Finally in a column "μ_2" we show the results of the approximate relative frequency calculations for the corresponding integer-linear types of faces.

Note that in the given examples the integer-affine type and integer distance to the origin determines the integer-linear type of the face.

In conclusion of this section we give two simple statements on relative frequencies of faces.

Statement 5.3. *Faces of the same affine-linear type at integer distance to the origin equivalent to* 1 *and at integer distance to the origin equivalent to* 2 *always have the same relative frequencies* (*see for example* $\mathbf{VI}_1$ *and* $\mathbf{VI}_2$ *of Table 1*).

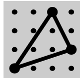
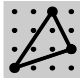
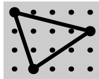
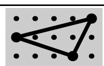
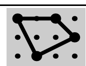
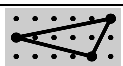
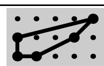
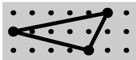
N°	face	lS	ld	μ_2	N°	face	lS	ld	μ_2
$\mathbf{I}_1$		3	1	$1.3990 \cdot 10^{-2}$	$\mathbf{VI}_1$		7	1	$3.1558 \cdot 10^{-4}$
$\mathbf{I}_3$		3	3	$1.0923 \cdot 10^{-3}$	$\mathbf{VI}_2$		7	2	$3.1558 \cdot 10^{-4}$
$\mathbf{II}_1$		5	1	$1.5001 \cdot 10^{-3}$	$\mathbf{VII}_1$		11	1	$3.4440 \cdot 10^{-5}$
$\mathbf{III}_1$		7	1	$3.0782 \cdot 10^{-4}$	$\mathbf{VIII}_1$		7	1	$5.6828 \cdot 10^{-4}$
$\mathbf{IV}_1$		9	1	$9.4173 \cdot 10^{-5}$	$\mathbf{IX}_1$		7	1	$1.1865 \cdot 10^{-3}$
$\mathbf{V}_1$		11	1	$3.6391 \cdot 10^{-5}$	$\mathbf{X}_1$		6	1	$9.9275 \cdot 10^{-4}$

TABLE 1. Some results of calculations of relative frequencies.

Denote by A_n the triangle with vertices $(0,0,1)$, $(n,0,1)$, and $(0,n,1)$. Denote by B_n the square with vertices $(0,0,1)$, $(n,0,1)$, $(n,n,1)$, and $(0,n,1)$.

Statement 5.4. *The following holds*

$$\lim_{n\to\infty} \frac{\mu(CF_n(A_n))}{\mu(CF_n(B_n))} = 8.$$

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