ON EXISTENCE AND UNIQUENESS CONDITIONS OF LATTICE TRIANGLE WITH GIVEN ANGLES. [1]

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The problem of description in integer-invariant terms of integer convex polygons is still open. At present it is only known that the number of convex integer polygons with lattice area bounded from above by n growths exponentially in $n^{1/3}$ (see the works [1] and [2]). In this note we give a complete description for the case of integer triangles. The author is grateful to V. I. Arnold, I. Bárány, and A. G. Khovanskii for attention to this work and useful remarks.

General definitions. Consider a two-dimensional oriented real affine plane. Fix some system of coordinates OXY in this plane. A point of the plane is said to be *integer* if all its coordinates are integers. The convex hull of a finite number of integer points that do not contained in the same line is said to be an *integer convex polygon.* Consider a minimal set of points defining a given polygon. The points of this set are called *vertices* of the polygon. Since all vertices are at the boundary of the convex hull, the vertices can be ordered in a cyclic counterclockwise or clockwise way: $A_1, \ldots, A_n$. Let us call such polygon *positively-oriented* or *negative-oriented* respectively and denote it by $A_1 \ldots A_n$.

By an *angle* we mean the ordered set of two closed rays with common vertex that do not contained in the same line. The rays are called the *edges* of the angle, and their common vertex is the vertex of the angle. An angle is called *integer* if its vertex is integer and both its edges contain integer points distinct from the vertex. An angle $\angle ABC$ of an oriented integer polygon with consecutive vertices A, B, and C is the integer angle with integer vertex B and edges BA and BC.

The affine transformation of the plane is called *integer-affine* if it preserves the set of all integer points. Polygons $A_1 \ldots A_n$ and $B_1 \ldots B_n$ (angles $\angle A_1A_2A_3$ and $\angle B_1B_2B_3$) are said to be *integer-equivalent* if there exist an integer-affine transformation of the plane taking the points A_i to B_i, for $i = 1, \ldots, n$ (respectively, rays A_2A_1 and A_2A_3 to the rays B_2B_1 and B_2B_3).

For any positive integer n and a point $A(x, y)$ denote by nA the point with the coordinates (nx, ny). A polygon $nA_0 \ldots nA_k$ is called *n-homothetic* to the polygon $P = A_0 \ldots A_k$ and denoted by nP. Polygons P_1 and P_2 are said to be integer-homothetic if there exist positive integers m_1 and m_2 such that m_1P_1 is integer-equivalent to m_2P_2.

Finite continued fractions. Let us expand the set of rationals with operations $+$ and $1/*$ by the element ∞ end denote this expansion by $\overline{\mathbb{Q}}$. We say that $q \pm \infty = \infty$, $1/0 = \infty$, $1/\infty = 0$ (the expressions $\infty \pm \infty$ are not defined).

For any finite sequence of integers $(a_0, a_1, \ldots, a_n)$ we associate an element $a_0 + 1/(a_1 + 1/(a_2 + \ldots) \ldots))$ of $\overline{\mathbb{Q}}$ and denote it by $]a_0, a_1, \ldots, a_n[$. If the elements of the sequence $a_1, \ldots, a_n$ are positive, then the expression for q is called the *ordinary continued fraction.*

Proposition. *For any rational there exists a unique ordinary continued fraction with odd number of elements.*

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Let us consider for $q_i \in \mathbb{Q}$, $i = 1, \ldots, k$ the ordinary continued fractions with odd number of elements: $q_i =]a_{i,0}, a_{i,1}, \ldots, a_{i,2n_i}[$. Denote by $]q_1, q_2, \ldots, q_k[$ the element

$$]a_{1,0}, a_{1;1}, \ldots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \ldots, a_{2,2n_2}, \ldots a_{k,0}, a_{k,1}, \ldots, a_{k,2n_k}[\in \overline{\mathbb{Q}}.$$

Integer tangents. An *integer length* of the segment AB (denoted by $\mathrm{l}\ell(AB)$) is the ratio of its Euclidean length and the minimal Euclidean length of integer vectors with vertices in AB. An *integer* (*non-oriented*) *area* of the polygon P is the doubled Euclidean area of the polygon, it is denoted by $\mathrm{lS}(P)$.

Consider an arbitrary integer angle $\angle ABC$. The boundary of the convex hull of the set of all integer points except B in the convex hull of the angle $\angle ABC$ is called the *sail* of the orthant. The sail of the angle is a finite broken line with the first and the last vertices on different edges of the angle. Let us orient the broken line in the direction from the ray BA to the ray BC and denote its vertices: $A_0, \ldots, A_{m+1}$. Denote $a_i = \mathrm{l}\ell(A_iA_{i+1})$ for $i = 0, \ldots, m$, and also $b_i = \mathrm{lS}(A_{i-1}A_iA_{i+1})$ for $i = 1, \ldots, m$. The following rational is called the *integer tangent of the angle* $\angle ABC$:

$$]a_0, b_1, a_1, b_2, a_2, \ldots, b_m, a_m[, \quad \text{we denote:} \quad \mathrm{ltg}\,\angle ABC.$$

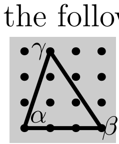
Formulation of the theorem. In Euclidean geometry on the plane the existence condition for the triangle with given angles can be written with tangents of angles in the following way. There exists a triangle with angles α, β, and γ iff $\mathrm{tg}(\alpha+\beta+\gamma) = 0$ and $\mathrm{tg}(\alpha+\beta) \notin [0; \mathrm{tg}\,\alpha]$ (without lose of generality, here we suppose that α is acute). Let us show the integer analog of the last statement.

Theorem **a).** *Let α_0, α_1, and α_2 be an ordered triple of integer angles. There exists an oriented integer triangle with the consecutive angles integer-equivalent to the angles α_0, α_1, and α_2 iff there exists $j \in \{1, 2, 3\}$ such that the angles $\alpha = \alpha_j$, $\beta = \alpha_{j+1 (\mathrm{mod}\, 3)}$, and $\gamma = \alpha_{j+2 (\mathrm{mod}\, 3)}$ satisfy the following conditions:*

i) $]\,\mathrm{ltg}\,\alpha, -1, \mathrm{ltg}\,\beta, -1, \mathrm{ltg}\,\gamma[= 0$; *ii)* $]\,\mathrm{ltg}\,\alpha, -1, \mathrm{ltg}\,\beta[\notin [0; \mathrm{ltg}\,\alpha]$.

b). *Two integer triangles with the same sequences of integer tangents are integer-homothetic.*

Note that for the conditions for the theorem we always take ordinary continued fractions with odd number of elements for tangents of angles. Let us illustrate the theorem with the following particular example:



$\mathrm{ltg}\,\alpha = 3 =]3[$;
$\mathrm{ltg}\,\beta = 9/7 =]1, 3, 2[$;
$\mathrm{ltg}\,\gamma = 3/2 =]1, 1, 1[$.

i) $]3, -1, 1, 3, 2, -1, 1, 1, 1[= 0$;
ii) $]3, -1, 1, 3, 2[= -3/2 \notin [0; 3]$.

REFERENCES

[1] V. I. Arnold, *Statistics of integer convex polygons*, Func. an. appl., vol.14(1980), n. 2, pp. 1-3. [2] I. Bárány, A. M. Vershik, *On the number of convex lattice polytopes*, Geom. Funct. Anal. v. 2(4), 1992, pp. 381-393.