VLADIMIR IGOREVICH ARNOLD

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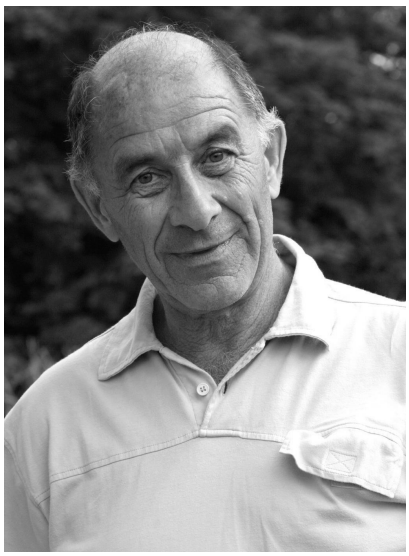

On June 3, 2010, the outstanding mathematician Vladimir Igorevich Arnold died in Paris at the age of 72. He was an international member of the French Academy of Science since 1983, an international member of the U.S. National Academy of Sciences since 1984, a corresponding member/member of the Russian Academy of Sciences since 1986/1990, a member of the Academy of Arts and Sciences of the United States since 1987, a member of the London Royal Society since 1988, an international member of Accademia dei Lincei in Rome since 1988, and a member of the Russian Academy of Natural Sciences since 1991. From 1996, V. I. Arnold was President of the Moscow Mathematical Society.

Vladimir Arnold started his mathematical life with finding a solution of the 13-th Hilbert problem at the age of twenty. For this result he received the Prize of the Moscow Mathematical Society in 1958. He had a great influence on many branches of mathematics and its applications. He is one of the co-authors of the theory of small denominators (KAM-theory), developed a theory of Lagrangian singularities, and he laid down the basics of symplectic topology. Vladimir Arnold has influenced differential equations and partial differential equations, singularity theory, topology, theory of braids, real algebraic geometry, magneto-hydrodynamics, the theory of multidimensional continued fractions, finite projective geometry, and combinatorics.

Together with his scientific advisor A. N. Kolmogorov, V. I. Arnold received the Lenin prize of the Soviet Government in 1965, for work on celestial mechanics. In 1982 he received the Crafoord Prize of the Swedish Academy of Sciences (with L. Nirenberg) for his theory of nonlinear differential equations. In 1994 he was awarded the Technion's Harvey prize for a *basic contribution to the stability theory of Dynamical Systems, his pioneering work on singularity theory and seminal contributions to analysis and geometry.* Further, in 2001, he received the Dannie Heineman Prize for Mathematical Physics for *his fundamental contributions to our understanding of dynamics and of singularities of maps with profound consequences for mechanics, astrophysics, statistical mechanics, hydrodynamics and optics* and the Wolf Prize in Mathematics. In 2007 he was awarded the State Prize of the Russian Federation in science and technology for *outstanding success in mathematics.*

Together with L. D. Faddeev, V. I. Arnold in 2008 received the Shaw Prize *for their contributions to mathematical physics*.

Supervising Ph.D. theses. I thank my lucky stars that I had an opportunity to be supervised by Vladimir Arnold and to work with him for almost ten years. Despite all the academic honors and rewards, Vladimir Arnold always kept in touch with students. Anyone was able to become a student of Vladimir Arnold, I did not remember that he refused to work with someone. However, only a few students continue. It was not complicated to become his student, but it took an effort to stay his student and later to write a Ph.D. thesis under his supervision.

The first task a student receives from Vladimir Arnold is to solve the exercises of the so-called *mathematical trivium*. This is a list of 100 selected exercises, that any master student should be able to solve. In addition, Vladimir Arnold said that any of these problems should take at most 5 minutes. They cover many branches of mathematics, and each of them is dedicated to some bright idea in the corresponding area. Let us formulate one of them.

$$\textit{Problem 2: Compute } \lim_{x\to 0} \frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}.$$

This famous exercise has two essentially different solutions. One is straightforward: one should apply the rule of de L'Hôpital. However, the calculation of derivatives by hand can take several hours or even days depending on the technical ability of the student. The second solution is very elegant. I will skip it in order to give the reader the opportunity to discover the idea and to enjoy the process of solving the exercise. Usually each exercise takes much more than just 5 minutes to solve. In my case it was only 10 exercises per week. Vladimir Arnold checked the exercises only in written form. If there is even a very small non-clear or non-accurate statement in the solution, the exercise is returned unaccepted with partially humorous remarks. As I heard, there is no one who successfully solved all (or even 90 of 100) exercises – it is almost impossible. While probably many of these exercises were collected by Vladimir Arnold from folklore, several definitely belong to himself.

When the majority of exercises is done, Vladimir Arnold has collected evidence that he works with a strong and diligent student. At that time he already has some information on the abilities and mathematical priorities of his new follower. Nevertheless he never proposed certain problems to his students, he told them that they can choose any problem he (Arnold) is currently interested in. When I asked him for a problem for my Ph.D., he told me that *to choose a problem is as personal as to choose a wife*, so to ask this question to someone else is just useless. Each year he announced many new problems and new directions that appeared in mathematics, many of which are collected in the book "Arnold's Problems" (*Springer Verlag, Berlin; PHASIS, Moscow, 2004*), where they are supplemented by comments on their state at that time by mathematicians who personally contributed to them. The half-life of such a problem turned out to be approximately 8 years. Many are still open or only partially solved.

As a rule, Vladimir Arnold worked in a certain area for several years. He did there as much as he could, formulating problems and conjectures for his students and colleagues for further study. Then he usually switched to another area, giving to his followers the opportunity to enjoy the beauty of mathematics. Such a system

works perfectly, since students do not compete with other former students, and the knowledge of the members of Arnold's school is complementary but not competitive.

It was usually hard to write a Ph.D. thesis under the supervision of Vladimir Arnold. When you have enough material for the thesis and finally bring a text to Vladimir Arnold to read (waiting for his praise and admiration), then within one week he gives the thesis back to you, together with a review for the PhD defense, looking at you in a very amicable and gentle-ironical way. So you are almost ready to go and defend the thesis, but then you suddenly have a closer look at the returned text and you find that all the empty spaces on each page and sometimes between the lines are filled by Arnold's remarks. The amount of remarks and corrections was really comparable to the amount of the text itself. The last hope for easy money is lost when you read his review. First he describes the area of the thesis and its importance in science and applications, but then he continues with the following: *unfortunately the student did not understand all this, he did not solve the original problems, and the text is completely unreadable.* Finally he concludes with an exposition of currently unsolved problems in this area for further investigation and wishes you good luck in the PhD defense. Of course it is useless even to try to defend the thesis with such a review. So after this trip back to the earth you work hard for several weeks and prepare a completely revised text (that is really an improvement on the first one). Then you give the second version to Vladimir Arnold, and he returns within one week with new comments and a review. In this review he softens some of the original statements, but usually it is still not sufficient for a PhD defense. So you should continue to improve your text. When I had rewritten my thesis 7 times, I finally received a review with which I could try to defend the thesis. Later I found out that I was actually lucky: in some cases the number of iterations was 20! Here I should admit that after each correction cycle the quality of the text in general improves, and its mathematical worth increases as well. In addition, the student learns much on how to write. All of Vladimir Arnold's articles and books are written in perfect language, understandable to specialists from different areas: he was aiming to educate them in the same way as his students.

Outlook on Mathematics. Vladimir Arnold liked to joke that mathematicians often experience life in the way of a person much younger than they actually are. They somehow finish their 'growth' at certain age. The younger the age of maturity, the more brilliant the professional characteristics of a mathematician usually are. Then he states that he himself stopped growing at the age of a 12–13 year old boy. This is approximately the age when a child is wondering about everything he sees. He is trying to play with everything and to make simple experiments which do not make sense for an adult, with the purpose to answer thousands of 'why' questions. Usually when children grow older, they are less and less interested in such things.

Experiments formed an essential part of Vladimir Arnold's research. He stated that mathematics is an experimental science. If one is aiming to solve some problem first he should collect information. Many laws just cannot be seen without an exhaustive study of examples.

In addition Vladimir Arnold did not like to distinguish the branches of mathematics, since it is based on bright ideas that appear in theorems of different areas of mathematics. This can be easily seen by means of the idea of Euclid's algorithm and its realization in terms of ordinary continued fractions. The algorithm

was invented in number theory to find the greater common divisor $\gcd(p,q)$ of two integers p and q, and works as follows. The first step is to find integers a_0 and r_1 with $q > r_1 \geq 0$ such that

$$p = a_0 q + r_1.$$

Suppose we have completed $k-1$ steps, resulting in integers a_{k-2} and r_{k-1}. Then we find a_{k-1} and r_k where $r_{k-1} > r_k \geq 0$ such that

$$r_{k-2} = a_{k-1} r_{k-1} + r_k.$$

The algorithm stops after $n+1$ steps, and we have $\gcd(p,q) = r_n$. The key point of the algorithm is computation of the numbers $a_0, \ldots, a_n$. It is interesting to note that these numbers form an ordinary continued fraction for p/q. Indeed,

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; + \cfrac{1}{a_n}}}}.$$

To calculate $a_0, \ldots, a_n$ for a positive rational number p/q, we only employ the two basic operations $s \to s-1$ and $s \to s^{-1}$:

$$\frac{p}{q} = 1 + \left(\frac{p}{q} - 1\right) = \ldots = a_0 + \left(\frac{p}{q} - \left\lfloor \frac{p}{q} \right\rfloor\right) = a_0 + \frac{1}{\frac{1}{\frac{p}{q} - \left\lfloor \frac{p}{q} \right\rfloor}} = a_0 + \frac{1}{1 + \left(\frac{1}{\frac{p}{q} - \left\lfloor \frac{p}{q} \right\rfloor} - 1\right)} = \ldots$$

Here $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Similar operations appear in different problems, and interpretations of Euclid's algorithm are used in Anosov systems in dynamic theory, in classifications of rational knots and 3-dimensional manifolds in topology, Hirzebruch-Jung theory of singularity resolutions. Simplest continued fractions appear in the spiral patterns of sunflowers and cones of trees. This is also due to the fact that the corresponding biological laws somehow refer to the mentioned operations.

In the opinion of Vladimir Arnold, the difference between mathematics and physics is only in the total cost of experiments. In physics, they may cost millions like in the case of the Large Hadron Collider, while in mathematics they cost almost nothing: one can perform them with a pen and paper. In the past centuries certain areas of mathematics, mechanics, and astronomy were close and essentially dependent on each other.

Vladimir Arnold payed extra attention to the history of science and knew many details from the original classical texts of the great mathematicians of the past centuries: L. Euler, H. Poincairé, F. Klein, etc. He liked to know by whom a theorem was discovered first. The Arnold principle states that *theorems, theories, principles, etc. in science are usually not named after their authors*. While saying this he always added that *Arnold's principle applies to itself*. He explained to us that there are two ways to avoid wrong attributions: One should either tell all discoveries fairly to everyone, or to keep them secret up to the end. In both cases there is no opportunity for another party to steal the results. He mentioned that he prefers and follows only the first strategy and taught us not to be lazy to give talks and to make them in such a way that everyone can understand them, even if he works in another field of mathematics.

Arnold's seminars. Vladimir Igorevich was running two permanent seminars: one in Moscow and one in Paris. I attended the Paris seminar only a small number of times, but I was regularly participating in the Moscow seminar which took place in room 14-14 of the Main Building of Moscow State University. This building is situated on top of a spectacular green hill (Vorobyevi Gori) to the South-West of the center of Moscow near the Moscow river. It is one of seven Stalinist neoclassic-style skyscrapers built in 1953. It has 36 floors, the Mathematical Department being located at floors 12–16, with a magnificent view of the city. A seminar usually started on Tuesdays at around 4 p.m. and officially lasted for one and a half hours. In former times it could continue for two or three more hours, depending on the subject. When I started to attend the seminar in 1998, it lasted usually no more than two hours, since many of its participants, including Arnold himself, were involved in the meetings of the Moscow Mathematical Society scheduled two hours after the seminar.

The first talk of each winter semester was always given by Vladimir Arnold himself. This was the most important event in the life of the seminar. In this talk he discussed new problems for the seminar that he had collected recently. The list usually contained 10–20 problems on 4 or 5 different subjects. Some of them were rather elementary and perfectly suitable for students starting their research, while other problems were dealing with more complicated subjects that sometimes even gave a new sight on classical problems and conjectures. These problems form the core of the volume "Arnold's Problems" mentioned above.

Apart from the problems, Vladimir Arnold usually distributed 'tasks' to his former students. He brought recent papers that attracted his attention and asked the members of the seminar if someone is interested to read some of them and give their opinion. In the majority of cases the papers were interesting, and we spent one seminar or half of a seminar to listen to the participants who had read the papers.

The remaining seminar talks were given by his students and colleagues who presented their new results, and in part by distinguished colleagues who gave survey talks. This seminar did not have a certain fixed subject. Vladimir Arnold himself contributed to many branches of mathematics, including applied mathematics and mechanics. So one could expect talks on current problems all over mathematics and its applications. It was felt to be an honor to speak at the seminar and to receive comments of Vladimir Arnold and his former students, many of whom are now holding professor positions at Moscow State University or some other institutes or universities in Moscow. Almost always the feedback received from the audience led to new insights, additional references, and ideas for future work.

It was really a peculiar experience for a speaker to have Vladimir Igorevich in the audience. He would interrupt any speaker, especially in his seminar. When the talk started without taking care of all important definitions and notions, he would ask for all the necessary details and himself explain the importance of these details to the speaker, mentioning examples when different understandings or definitions of certain notions lead to completely different results. Such a speaker would be lucky to give the main definitions and the main result in one and a half hours. In some sense, of course, this is the minimal amount of information necessary for the audience. So at the end of the talk it was completely clear what was the result and what are the open questions (which are often not mentioned by the

speakers, but are extremely interesting for the audience). While listening to such a presentation, Vladimir Arnold did not distinguish between a young unknown student and a famous academician. Of course not all people would appreciate such an attitude and from time to time they got quite angry (Vladimir Igorevich did not pay attention to their complaints). All this made the talks extremely useful for the audience and in many cases for the speaker himself. Sometimes, when the speaker ignored his remarks, Arnold would start to correct his students' papers or think about his own research. Thus the speaker was in some sense 'punished': his talk had become an ordinary talk, with the majority of the audience not paying attention, either switching to their own businesses, or diligently continuing looking at the blackboard with eyes wide open.

On abstraction and teaching. Vladimir Arnold was against what he called 'algebraization' of mathematics which is very popular in our days among many people whom he would refer to as the *devils of algebra*, while he attributed himself as the *angel of geometry*. One cannot conclude from this that he was 'against' certain areas of mathematics, but he was against eliminating the thinking process by formal calculations, which apparently he thought certain followers of N. Bourbaki are guilty of.

Once I was reading a preprint of a school textbook, and there was a standard question on percentages: *with a monthly interest rate of 1%, by how many percent has the capital increased after one year?* The solution given by the author was around 20000%. This is clearly not true, and you would never write such nonsense if you were thinking while writing the book. Unfortunately similar situations sometimes present themselves in mathematical articles where catching mistakes is not as easy.

Vladimir Igorevich did always care about the future of mathematics. On the one hand he was a person with great knowledge which he was trying to transfer to his colleagues, students, or even persons met by chance. On the other hand, he clearly understood that the roots of mathematics lie in education. If we neglect the quality of teaching in schools, we will get a generation unable to expand, usefully employ, or even to keep the achievements of our and previous generations. A recent issue here is the use of computers which in many cases makes it possible to press the correct buttons instead of thinking. As a result we get pupils unable to do simple operations without a computer. Here are my favorite examples of 'wrong' answers given by pupils which I know from Vladimir Arnold. The first one is coming from some American test where it turned out that the majority of schoolchildren did not know to handle fractions: They did computations like

$$\frac{2}{3}+\frac{1}{2}=\frac{2+1}{3+2}=\frac{3}{5}.$$

The second example is related to the traditional emphasis on axiomatics in France. Vladimir Arnold asked a pupil with an excellent record in mathematics: *what is* $2+3$? After a minute of thinking the pupil replied that

$$2+3=3+2,$$

since addition is a commutative operation. Vladimir Arnold was constantly trying to improve mathematical education, writing many articles, critical reviews, and giving public talks with the purpose of educating a new generation of mathematicians.

Private life and work. It is hard to believe how many things Vladimir Arnold apparently managed to do. The reviewing of theses and articles of his students and colleagues was just a small part of his work. In the last years he wrote what must have been up to 100 handwritten pages per day. This includes new articles, books on mathematics, articles on popular mathematics and recent problems of education. It is interesting to know that Vladimir Arnold did not like to work on a computer in his office, but he preferred to do research with pencil and paper, occupying one of his favorite outdoor places: sitting under the oldest tree in Paris or somewhere in the Bois de Vincennes; somewhere in the rocky region of Trieste, where he took part in archeological research in the local caves; at his dacha near Moscow in an area with beautiful lakes, rivers and forests, berries and mushrooms.

I have experienced Vladimir Arnold as a perfect person to discuss any subject. Communication with him was a powerful stimulus for research, and yielded inspiration for work even months after it actually took place. He preferred to meet people in an informal environment, for instance in his dacha, where the meeting could last all evening. His wife Eleonora Aleksandrovna is a hospitable, very friendly and kind person, who used to prepare delicious dinners for their guests. After such a dinner Vladimir Arnold invited his guests for a round trip in the forest, on foot, on skis, or by bike. In Paris he usually invited his guests and colleagues for 4–5 hour excursions and gave bright expositions of history that were full of stories about the adventures of G. Casanova, or the guillotine, or or the poisons of Catherine de Medici.

In Russia, Vladimir Arnold spent most of the time at his famous dacha situated in the academician suburban settlement not far from Moscow. To go there, one takes the bus from the local train station *Perkhushkovo* and alights at bus stop "Dacha No. 1" (Dacha No. 1 is one of the dachas belonging to Stalin). In such settlements academicians can meet and talk to each other privately and informally, and you could meet leading people of all branches of science. Doubtless this invitation to multidisciplinary collaboration was appreciated by many Russian scientists living there.

When Vladimir Arnold stayed in Moscow, his students, friends, and colleagues had an opportunity to meet another challenge: It sounds quite simple to run 40 kilometers on cross-country skies together with Vladimir Arnold and his older students in in the forests on both sides of Moscow river. But the main obstacle to completing this particular course is that it starts on one side of the river and finishes on the other. The river in this place is quite close to its origin and not very wide, but fast enough not to freeze in most winters. Approximately halfway, Vladimir Arnold crossed the river by swimming, with the majority of his group following him (I should add at this point that the only thing Vladimir Arnold was wearing were his shorts). In his best days, Vladimir Arnold was in very good shape physically: he used to go by bike but stated that mere bike roads were too simple, he would swim 10km from Luminy to Marseille, or sail in a canoe for several days. Of course like every man of his age, he had some health problems. Still all who knew him were sure that he would celebrate his 80-th, 90-th, and 100-th anniversaries. He died unexpectedly. Several days before his death he was full of plans, new ideas and enthusiasm. Now he has left us. But we still have his rich intellectual legacy contained in his many lecture notes, books, articles and students, who were inspired

by the enthusiastic, optimistic, and always young genius which Vladimir Arnold definitely was.

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