GEOMETRY OF LATTICE ANGLES, POLYGONS, AND CONES SUMMARY

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Contents

1. Introduction

When some interested person asks my PhD adviser Prof. Vladimir Arnold, who is one of the famous mathematicians of our times, to name his preferable branch of mathematics, he always says that mathematics is a single entity which is beautiful in total. It is very often that papers are in the intersection of several branches of mathematics, in some cases they give birth to new branches or modify the existing ones. In addition he usually says, smiling thoughtfully, that the difference between physics and mathematics is also very small, namely in the cost of experiments. Mathematicians do the experiments on paper or computers and therefore the experiments are usually relatively cheap, while in physics the experiments are very expensive like in the case of the Large Hadron Collider.

As a former student of Prof. Vladimir Arnold I totally agree with his point of view on mathematics. Certainly I do not have works in many topics of mathematics, but still I have written papers on subjects in several distinct fields. Most of my papers are on geometric continued fractions [4], [5], [6], and [9]. This subject is in the intersection of number theory and geometry. In addition [H5] is touching the theory of dynamic systems. In [10], we use topological methods to prove consistency of some continued fraction algorithm. Further, in papers [2], [3], and [7] we study variational principles of functional of energies of knots, we apply principles of variational calculus to topology. The subjects of papers [12] and [13]

Date: 3 December 2009.

The author is partially supported by RFBR SS-709.2008.1 grant and by FWF grant No. S09209.

(tensegrities and meshes) are in the fields of discrete geometry with several applications to differential geometry. In a small note [1] and a further paper [11] we solve a problem of singularity theory which in this particular case can be considered as a combinatorial problem in algebraic geometry. In the note [8] we describe the difference operators over finite fields.

As a theme for the habilitation thesis I have chosen *geometry of lattices* which is considered to be in the intersection of number theory (in particular, Diophantine approximations) and discrete geometry.

Lattice geometry. Geometry in general can be interpreted in many different ways. Ancient Greeks defined geometry as science of land measurement and their geometry was mostly plane geometry. Still not all measurements can be made within the framework of plane geometry. Of course in ancient Greece the scientists did not know that the Earth has the shape of a sphere and hence it is impossible to apply the laws of plane geometry to measure regions of land that have large area. This is one of the reasons to introduce spherical (and further, hyperbolic) geometry. Now the notion of geometry itself takes its own place in mathematics.

We take one of the classical definitions of geometry: geometry is a *set of objects* and a *congruence relation* for these objects (sometimes it is just called equivalence between objects). Usually a congruence relation is defined by some group of transformations. For instance in *Euclidean geometry* in the plane the objects are points, lines, segments, polygons, circles, etc, and the congruence relation is defined by the orthogonal group $O(2,\mathbb{R})$ of distance preserving transformations.

In *lattice geometry* we consider a full rank lattice in $\mathbb{R}^n$ which one can think of as of the set of all points in the space $\mathbb{R}^n$ with integer coordinates, i.e., it is $\mathbb{Z}^n$. It is interesting to note that we can introduce a group structure for the integer lattice: we take the coordinate-wise addition as the group operation. Then it is clear that the sum of two lattice points is again a lattice point, and the inverse of a lattice point is found by reflection in the origin.

The objects of interest would be lattice points, *lattice segments* with lattice endpoints, *lattice lines* passing through a couple of lattice points, *ordinary lattice angles* and *cones* with vertex in a lattice point, *lattice polygons* and *lattice polytopes* having all vertices in the lattice. The congruence relation in lattice geometry is the group of affine transformations of $\mathbb{R}^n$ preserving the lattice. This group is generated by all linear transformations that preserve the integer lattice (it is actually the group $SL(n,\mathbb{Z})$ of all integer matrices with determinant equal to 1 that acts on the space by a matrix-vector multiplication) and by translations which preserve the lattice.

It is interesting to compare lattice geometry and Euclidean geometry. Let us fix an integer lattice in the plane. On one hand these two geometries have a lot in common, but on the other hand their properties sometimes behave in unexpectedly different ways. For instance, twice Euclidean area of a polygon is always equivalent to its lattice area, but there is no such relation between Euclidean length and lattice length. The sine formula works in both geometries, but the set of values of sines for Euclidean angles is the interval

$[0,1]$, while the lattice sines can take all non-negative integers as a value. The arctangents and lattice arctangents (geometrically) coincide, nevertheless the arithmetics of angles are different: one can add two angles in lattice geometry in infinitely many ways and all the resulting angles are non-congruent, which is not the case in Euclidean geometry. We admit also that the angles ABC and CBA are congruent in Euclidean plane, but they are not always congruent in lattice case. In addition in lattice case there are only 6 regular basic lattice polygons in the plane (2 triangles, 2 quadrangles, and 2 hexagons), while in Euclidean plane we get infinitely many (a regular n-gon for any $n \geq 3$). Still in some dimensions the amount of lattice-regular polytopes can be arbitrarily large (in Euclidean case we have 3 distinct polytopes for dimensions greater than 4: a symplex, a cube, and a generalized octahedron).

Relations between lattice geometry and other branches of mathematics. Lattice geometry is interesting by itself, still it has many relations to other branches of mathematics. Within algebraic geometry, it translates to the geometry of toric varieties which is an important part of algebraic geometry. For instance we have the following list of correspondences. The polygons and polytopes of lattice geometry are toric varieties, their angles and cones are singularities. The singularities of lattice-congruent angles are algebraically the same. The inverse is almost true (actually the angles ABC and CBA define the same singularity, but still they can be lattice non-congruent). So any description of lattice polygons and polytopes leads to an analogous algebro-geometric description of toric varieties. For instance, varieties that correspond to regular polytopes therefore have the maximal possible group of symmetries.

In the context of toric geometry one of the first aims is to study singularities, i.e. lattice angles and cones. There is a deep relation between lattice angles and geometric continued fractions corresponding to these angles. This relation was generalized by F. Klein to the multidimensional case.

Multidimensional continued fractions in the sense of Klein have many connections with other branches of mathematics. For example, J.-O. Moussafir and O. N. German studied the connection between the sails of multidimensional continued fractions and Hilbert bases. H. Tsuchihashi found the relationship between periodic multidimensional continued fractions and multidimensional cusp singularities, which generalizes the relationship between ordinary continued fractions and two-dimensional cusp singularities. M. L. Kontsevich and Yu. M. Suhov discussed the statistical properties of the boundary of a random multidimensional continued fraction. The geometric generalization of Lagrange's theorem was obtained by E. I. Korkina, an algebraic generalization was given by G. Lachaud. For the algorithms of constructing multidimensional continued fractions, see the papers of R. Okazaki, J.-O. Moussafir and the author. E. Korkina, G. Lachaud, A. D. Bruno and V. I. Parusnikov, and the author produced a large number of fundamental domains for periodic algebraic two-dimensional continued fractions.

V. I. Arnold presented a survey of geometrical problems and theorems associated with one-dimensional and multidimensional continued fractions. In particular, he posed several problems on an analog of Gauss-Kuzmin formula for statistics of the elements of ordinary continued fractions and a question on algebraic multidimensional continued fractions for the simplest operators. We study the first question in [H5], the answer for the second question in the case of three dimensions was given by E. Korkina, in [H6] we give the answer in the case of four dimensions.

In addition I would like to say a few words about a relation between cones and their geometry and Gauss reduction theory. It turns out that ordinary continued fractions give a good description of conjugacy classes in the group $SL(2,\mathbb{Z})$. There are almost no answers for the case of $SL(n,\mathbb{Z})$ where $n \geq 3$. The structure of the set of conjugacy classes is very complicated (note that for closed fields such classes are described by Jordan Normal Forms). Multidimensional continued fractions in the sense of Klein is a strong invariant of such classes, which distinguish the classes up to some simple relation. Their lattice characteristics are good invariants for the conjugacy classes.

Organization of this habilitation thesis. We investigate lattice properties of lattice objects. In the first three papers we mostly study the two-dimensional case, and in the last three papers we work with multidimensional objects. In papers [H1] and [H2] we introduce lattice trigonometric functions and show that the lattice tangent is a complete invariant for ordinary lattice angles with respect to the group of lattice preserving affine transformations. Lattice tangents are certain continued fractions whose elements are constructed by lattice invariants of angles. We find the formula for lattice tangents of the sums of the angles and show the necessary condition for angles to be the angles of some polygon. For the case of triangles we get a necessary and sufficient condition, which was announced in [H3] and studied in a more detailed way in [H1]. Further, in [H4] we give a classification of all lattice-regular lattice polytopes with lattice lengths of the edges equal 1 (all the remaining polyhedra are multiple of these). Finally in the last two papers we deal with lattice cones in the multidimensional case. A multidimensional continued fraction in the sense of Klein is a natural geometric generalization of an ordinary continued fraction that is a complete invariant of lattice cones. In paper [H5] we study the statistical questions related to the generalization of the Gauss-Kuzmin distribution. Finally in paper [H6] we construct the first algebraic examples of four-dimensional periodic continued fractions. These fractions are supposed to be the simplest continued fractions in four-dimensional case.

Acknowledgements. The author is grateful to Prof. J. Wallner, Prof. V. I. Arnold, and Prof. A. B. Sossinski for constant attention to this work. This work was done at Technische Universität Graz.

2. Description of main results

2.1. **Lattice angles and their trigonometric functions.** One of the main themes of ancient geometry was the study of properties of right triangles (like the triangle with edges equal 3, 4, and 5 units of measure). The relation between the measurements of the edges of the right triangles is basically the subject of *trigonometry*. In trigonometry we traditionally use the trigonometric functions *sine*, *cosine*, and *tangent* to describe such relations. In papers [H1], [H2], and [H3] we study the lattice analog of the trigonometric functions.

The study of lattice angles is an essential part of modern lattice geometry. Invariants of lattice angles are used in the study of convex lattice polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the work of F. Klein, V. I. Arnold, E. Korkina, M. Kontsevich, Yu. Suhov, and G. Lachaud).

As we have already mentioned before, lattice angles are in the limelight of complex projective toric varieties: they corresponds to singularities. It turns out that these singularities in many cases define toric varieties that are actually lattice polygons in lattice geometry. So the study of toric varieties is reduced to the study of convex lattice polygons. The first classical open problem in this area is on the description of convex lattice polygons. It is only known that the number of such polygons with lattice area $\leq n$ growths exponentially in $n^{1/3}$ as n tends to infinity (this was studied by V. Arnold, I. Bárány, and A. M. Vershik).

We develop new methods to study lattice angles, which are based on the introduction of trigonometric functions. A general study of lattice trigonometric functions is given in papers [H1] and [H2].

Continued fractions. We start with the necessary definitions from the theory of continued fractions. For any finite sequence of integers $(a_0, a_1, \ldots, a_n)$ we associate the expression

$$q \quad = \quad a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \quad \vdots \atop a_{n-1} + \frac{1}{a_n}}}$$

and denote it by $]a_0, a_1, \ldots, a_n[$. If $a_1, \ldots, a_n$ are positive and all the elements are integers, then the expression for q is called the *ordinary continued fraction*.

Proposition. *For any rational number there exists a unique ordinary continued fraction with an odd number of elements, and another one with an even number of elements.*

So there are only two ordinary continued fractions for, say, 7/5:

$$7/5 =]1, 2, 2[=]1, 2, 1, 1[.$$

Lattice trigonometric functions. Let us give definitions for lattice trigonometric functions. In many cases we choose the lattice to be the integer lattice (whose lattice points has all integer coordinates).

We start with several preliminary definitions. A *lattice length* of a segment AB (denoted by $\mathrm{l}\ell(AB)$) is the ratio of its Euclidean length and the minimal Euclidean length of lattice vectors with vertices in AB. A *lattice area* of a triangle ABC (denoted by $\mathrm{lS}(AB)$) is the index of the sublattice generated by vectors AB and AC in the whole lattice.

Consider an arbitrary lattice angle α with lattice vertex V (having some lattice points distinct to V on both his edges). The boundary of the convex hull of the set of all lattice points except V in the angle α is called the *sail* of the angle. The sail of the angle is a

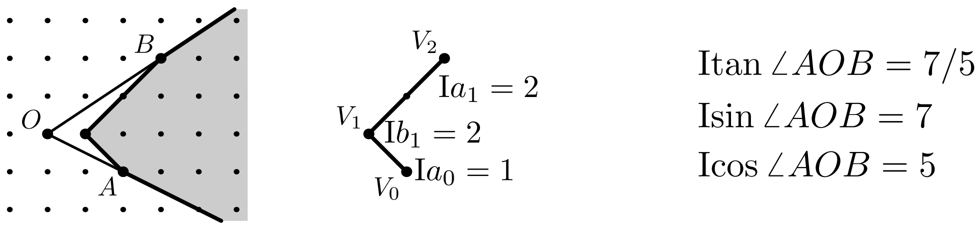


FIGURE 1. Left: a lattice angle OAB. Center: the sail of the angle OAB. Right: the lattice trigonometric functions for OAB. Note that we have $\frac{7}{5} = 1 + \frac{1}{2+1/2}$.

finite broken line with the first and the last vertices on different edges of the angle. Let us orient the broken line in the direction from the first ray to the second ray of the angle and denote the vertices of this broken line by: $A_0, \ldots, A_{m+1}$. Let

$$a_i = \mathrm{l}\ell(A_iA_{i+1}) \qquad \text{for } i = 0, \ldots, m;$$
$$b_i = \frac{\mathrm{lS}(A_{i-1}A_iA_{i+1})}{\mathrm{l}\ell(A_{i-1}A_i)\,\mathrm{l}\ell(A_iA_{i+1})} \qquad \text{for } i = 1, \ldots, m.$$

Definition 2.1. The *lattice tangent* of the angle α is the following rational number:

$$\mathrm{ltan}\,\alpha :=]a_0, b_1, a_1, b_2, a_2, \ldots, b_m, a_m[.$$

The *lattice sine* is the numerator of the irreducible fraction for the rational number $\mathrm{ltan}\,\alpha$, denote it by $\mathrm{lsin}\,\alpha$.
The *lattice cosine* is the denominator of the irreducible fraction for $\mathrm{ltan}\,\alpha$, denote it by $\mathrm{lcos}\,\alpha$ (see the example in Figure 1).

Note that there are several equivalent definitions of the lattice trigonometric functions. Here we choose the shortest one.

Results on lattice trigonometry. In papers [H1], [H2], and [H3] we study basic properties of lattice trigonometric functions.

In paper [H1] we define ordinary lattice angles, and the lattice sine, tangent, cosine, and arctangent (defined for rational numbers ≥ 1). Further we introduce the sum formula for the lattice tangents of ordinary lattice angles of lattice triangles. The sum formula is a lattice generalization of the Euclidean statement on the sum of three angles of a triangle being equal to π. Then we introduce the notion of extended lattice angles and find their normal forms. This lead to the definition of sums of extended and ordinary lattice angles. In particular this gives a new extension of the notion of sails in the sense of Klein: we define and study oriented broken lines at unit distance from lattice points. We give a necessary and sufficient condition for an ordered n-tuple of angles to be the angles of some convex lattice polygon. We conclude this paper with criterions of lattice congruence for lattice triangles.

In paper [H2] we introduce trigonometric functions for angles whose vertices are lattice, but whose edges may not contain lattice points other than the vertex (we call such angles *irrational*). Further we study equivalence classes (with respect to the group of affine lattice preserving transformations) of irrational angles and find normal forms for such classes. Finally in several cases we give definitions of sums of irrational angles.

Paper [H3] is a small note in which we announce the result on sum of lattice angles in the triangle, which is proved in [H1].

One of the results of paper [H1]**.** Let us focus on one particular application of lattice trigonometry, the description of all lattice triangles (see [H1] and [H3]). In toric geometry this result is a description of all possible sets of singularities of complex projective toric surfaces whose Euler characteristic equals 3.

Let us consider, for rational q_i, $i = 1, \ldots, k$, the ordinary continued fractions with an odd number of elements:

$$q_i =]a_{i,0}, a_{i,1}, \ldots, a_{i,2n_i}[.$$

Denote by $]q_1, q_2, \ldots, q_k[$ the element

$$]a_{1,0}, a_{1;1}, \ldots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \ldots, a_{2,2n_2}, \ldots a_{k,0}, a_{k,1}, \ldots, a_{k,2n_k}[.$$

The Euclidean condition

$$\alpha + \beta + \gamma = \pi$$

can be written in terms of tangents of angles as follows:

$$\begin{cases} \tan(\alpha+\beta+\gamma) = 0 \\ \tan(\alpha+\beta) \notin [0; \tan\alpha] \end{cases}$$

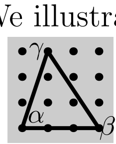
(without loss of generality, here we suppose that α is acute). The lattice version of this Euclidean sum formula for the triangles is as follows:

Theorem 2.2. [H1] **a).** *Let α_0, α_1, and α_2 be an ordered triple of lattice angles. There exists an oriented lattice triangle with the consecutive angles lattice-equivalent to the angles α_0, α_1, and α_2 if and only if there exists $j \in \{0,1,2\}$ such that the angles $\alpha = \alpha_j$, $\beta = \alpha_{j+1(\mathrm{mod}\,3)}$, and $\gamma = \alpha_{j+2(\mathrm{mod}\,3)}$ satisfy the following conditions:*

i) $]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta, -1, \mathrm{ltan}\,\gamma[= 0$; *ii)* $]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta[\notin [0; \mathrm{ltan}\,\alpha]$.

b). *Two lattice triangles with the same sequences of integer tangents are integer-homothetic.*

We illustrate the theorem by an example:



$\mathrm{ltan}\,\alpha = 3 =]3[$;
$\mathrm{ltan}\,\beta = 9/7 =]1,3,2[$;
$\mathrm{ltan}\,\gamma = 3/2 =]1,1,1[$.

i) $]3,-1,1,3,2,-1,1,1,1[= 0$;
ii) $]3,-1,1,3,2[= -3/2 \notin [0;3]$.

2.2. **Lattice-regular polygons and polytopes.** Let us say a few words about paper [H4].

We define a polyhedron to be *lattice-regular* if for any two complete flags of this polyhedron there exists a lattice preserving transformation taking the first flag to the second one. In [H4] we develop a complete classification of lattice-regular polytopes in all dimensions. It turns out that in the plane there are only 2 nonequivalent regular integer triangles, 2 quadrangles, and 2 hexagons. In three-dimensional space we have 3 regular tetrahedra, 3 regular octahedra, and 3 regular cubes. In dimension 4 we have 2 simplices, 3 generalized octahedra, 3 cubes and 2 hyperdiamonds (24-cells). Finally, in dimension n we always get k simplices, 3 cubes, and 3 octahedra, where k is the number of positive integer divisors of $n+1$. Lattice-regular three-dimensional polygons of different types are shown on Figure 2.

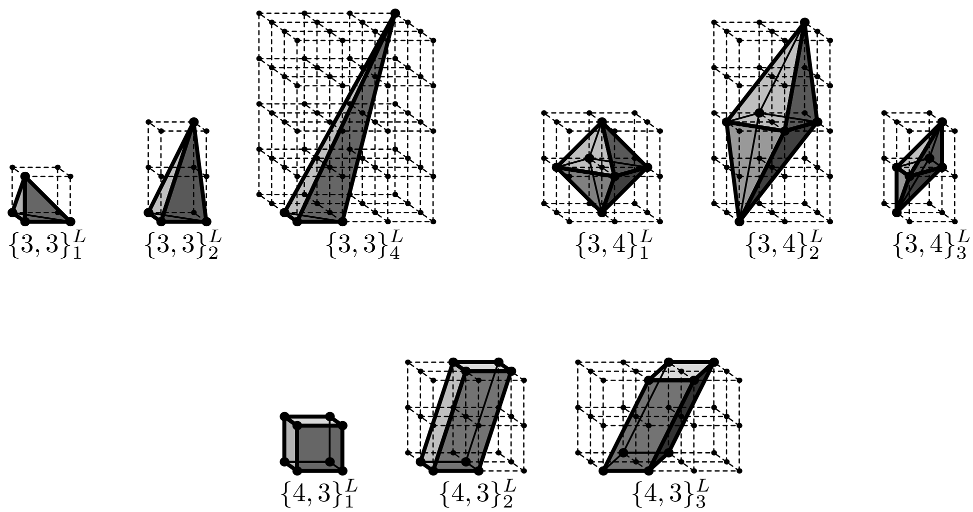


FIGURE 2. Three-dimensional lattice-regular polytopes.

2.3. **Geometry of simplicial lattice cones and multidimensional continued fractions.** Here we discuss the papers [H5] and [H6].

Simplicial lattice cones are in one to one correspondence to Klein's multidimensional continued fractions (up to the action of the group of affine lattice transformations). The problem of generalization of ordinary continued fractions to the higher-dimensional case was posed by C. Hermite in 1839. A large number of attempts to solve this problem lead to the birth of several different remarkable theories of multidimensional continued fractions. We consider the geometric generalization of ordinary continued fractions to the multidimensional case presented by F. Klein in 1895.

The complement of the union of these hyperplanes consists of 2^{n+1} open cones. Let us choose an arbitrary cone.

Definition 2.3. Consider a set of $n+1$ hyperplanes of $\mathbb{R}^{n+1}$ passing through the origin in general position and take one of the connected components of the complement to the unions of these hyperplanes (which is a cone). The boundary of the convex hull of all integer points except the origin in the closure of the cone is called the *sail*. The collection of 2^{n+1} sails for all the connected components in the complement is called the *n-dimensional continued fraction* associated to the given $n+1$ hyperplanes.

Two n-dimensional continued fractions are said to be *equivalent* if there exists a linear transformation that preserves the integer lattice of $\mathbb{R}^{n+1}$ and maps the sails of the first continued fraction to the sails of the other.

A few words about my PhD-thesis. In my PhD-thesis (which was supervised by Prof. V. I. Arnold) I studied infinite series of two-dimensional continued fractions [4]. Further I made a description of polygonal faces lying in planes on integer distance greater then 1 to the origin [9]. This leads to an effective algorithm to construct periodic multidimensional continued fractions [10].

Gauss-Kuzmin formula and Möbius measure, generalization to multidimensional case ([H5]**).** For the first time the statement on statistics of numbers as elements of ordinary continued fractions was formulated by C. F. Gauss in his letters to P. S. de Laplace. This statement was proven further by R. O. Kuzmin, and further was proven one more time by P. Lévy. Later investigations in this direction were made by E. Wirsing. In 1989 V. I. Arnold generalized statistical problems to the case of one-dimensional and multidimensional continued fractions in the sense of Klein.

The one-dimensional case was studied in details by M. O. Avdeeva and B. A. Bykovskii. In the two-dimensional and multidimensional cases, V. I. Arnold formulated many problems on statistics of sail characteristics of multidimensional continued fractions such as an amount of triangular, quadrangular faces and so on, such as their integer areas, and length of edges, etc. A major part of these problems is open nowadays, while some are almost completely solved.

M. L. Kontsevich and Yu. M. Suhov in their work proved the existence of the above mentioned statistics. In [H5] we explicitly construct a natural Möbius measure of the manifold of all n-dimensional continued fractions in the sense of Klein and introduce new integral formulae for the statistics.

Simplest four-dimensional examples ([H6]**).** The problem of investigation of the simplest n-dimensional continued fraction for $n \geq 2$ was posed by V. I. Arnold. The answer for the case of $n = 2$ can be found in the work of E. Korkina and G. Lachaud. We have studied the case of $n = 3$ in [H6]. We constructed three examples of three-dimensional continued fractions that for many reasons (such as additional symmetries, simplicity of the fundamental domains, characteristic polynomials of special types) seems to be the simplest examples tree-dimensional continued fractions.

2.4. **A few words about my plans for further research.** The results obtained in papers [H1], [H2], and [H3] give the complete global description of singularities of complex toric varieties whose Euler characteristic is 3. I am planning to continue the research aimed on the global description of the singularities for general case. Actually there are many other open problems concerning lattice trigonometry, even elementary ones. For instance, it is interesting to find the lattice analog of the cosine formula, etc.

The study of lattice-regular polytopes in [H4] is the first step toward the solution of a famous classical problem of classification of empty tetrahedra in the four-dimensional case. Recall that White's theorem gives the complete description of empty lattice tetrahedra in

$\mathbb{R}^3$. We plan to continue to investigate the problem in four dimensions. The next step is to classify lattice-regular cones over lattice-regular polytopes.

There are several interesting questions related to geometry of cones and corresponding multidimensional continued fractions. In particular, I am planning to study the problem of description of conjugacy classes in the group $SL(n, \mathbb{Z})$ using the corresponding cones. Gauss Reduction Theory gives the answer for the case $n = 2$, but for $n \geq 3$ the problem is still open. I propose a new approach to this problem based on reduction to reduced Hessenberg matrices. An important tool used in here is to determine minima of Markoff-Davenport characteristics at the vertices of Klein-Voronoi continued fractions. Markoff-Davenport forms play an important role in approximation theory of maximal commutative subgroups. To develop this idea further is one of the possible directions for further study.

In addition my current interests are in stability of meshes and tensegrities. In the frames of this subject I am planning to study necessary conditions of flexibility for semidiscrete surfaces.

3. Teaching experience

I started teaching in 1998, when I was a second year student of the Moscow State University (MSU) and, in parallel, of the Independent University of Moscow (IUM). As an absolute winner of the Moscow Mathematical Olympiad and one of the winners of the XXII Russian Federation Mathematical Olympiad, I was invited to give classes on solution of olympiad problems. These traditional classes for pupils are organized in MSU. I was one of the teachers at the classes over 3 academic years. We taught the pupils *to express their solution in an understandable way, acceptable for their peers*. An additional task for us was *to explain a difference between correct and wrong arguments*.

My further teaching encounter was in School 57, which I graduated myself. School N57 is one of the best special mathematical schools for senior pupils in Moscow. One should successively pass a 7-step interview to enter the school. This explains why every year its pupils make up at least 75 percent of a Moscow team taking part in the Russian mathematical olympiad. The main strategy of School 57 is to introduce talented pupils to the beauty of higher mathematics. Therefore, standard school courses in Geometry and Algebra are usually supplemented by more advanced courses in Calculus and Linear Algebra which are actually at university level. For 4 years, I was one of the Calculus and Linear Algebra teachers in this school. The main challenges that I faced as a teacher there were as follows. *The material of the courses should be presented to the students in a vivid and colorful way*, otherwise they will not get involved into the subject. This will not only slow down their progress in the studies but will also cause problems with the discipline. Moreover, a teacher must *give attractive problems that challenge pupils' creativity and open up their imagination. Proposed problems should interest pupils and stimulate them to find a solution even if the problems are very hard.* In 2000 and 2001 I was a deputy head of the Moscow teams at the Russian Federation Mathematical Olympiads. I was responsible for the mathematical and psychological support of the team.

In 2002, I became a PhD-student, again simultaneously, of the MSU and IMU. From that moment, I started assisting teaching at the IMU. The duties included conducting problem sessions and preparing and checking exams. I assisted classes in Calculus, Topology I, and Complex Analysis. A new experience for me was *working with a larger audience.* Here I was not able to control every single student, so I tried to estimate the average rate of understanding of the audience. In such situation, it is useful to have a collection of supplementary problems and examples in case certain moments are not sufficiently well good understood by some students.

During four semesters I was conducting “Topology I” problem sessions in English within the framework of the International Program “Math in Moscow” for under- and postgraduate students. The majority of the students in the program are coming to Moscow after selection by American and Canadian Universities. There are also occasional students from Europe and sometimes even from India. The diversity of the mathematical backgrounds, habits, and mentalities of the students is one of the most complicated difficulties for the lecturers. For example, once we had a bright student from India who did not know Calculus and some other basic mathematical courses but was able to solve complicated combinatorial and analytic problems. We devoted some extra time to intense the study of the basics of Calculus, Algebra, etc. After a semester his level became comparable with the level of the others. So one of the main tasks of a lecturer in the program was *to single out students' difficulties and give them appropriate advice*. We also *paid attention to the educational priorities of our students and made efforts to stimulate their interest in the studies*. This is essentially important since the study motivation varies a lot from a country to a country.

During two academic years from winter 2006 till summer 2008 I was working as a postdoc in mathematics in Leiden University; In summer 2007 I developed and taught an advanced course “Topology II” for bachelor and master students in Leiden. The Mathematical Department in Leiden is not very big and has main interests in Algebraic Geometry, Number Theory, and Statistics. Due to that, most of my students were those who wanted to widen their mathematical knowledge. I identified my main tasks in the course as follows: *to give an exposition of the main ideas of the subject and introduce — if necessary — appropriate technique*, and *to explain the interrelations between modern topology and various branches of mathematics.*

In winter 2007 I assisted a Calculus course for students of the Biology department, where I *explained basic mathematical methods to non-mathematical students in an understandable way, and showed how powerful they are in applications.* I found that it is good to mention examples from everyday life. Generally, I think that mathematical notions and theorems normally should be accompanied by emphasizing their physical meanings.

Starting from winter 2008, I have the pleasure to teach at TU Graz. I started with the exercises part of Algebraic topology, which I taught together with Prof. J. Wallner. In summer 2009, I read a course on knot theory. It was interesting to read this course especially since knot theory develops at high speed nowadays. Hence it is not enough to use only the classical textbooks in order to read a good course. The main task for me was *to give a colorful introduction to the material which is very fresh and sometimes technical to interest the students*. This semester I am reading a course on geometry of continued fractions. The topic of this course is particularly based on the first three papers of the current habilitation thesis, as well as on additional supplementary material.

General consideration of teaching. Let us conclude with a small summary. It does not seem to make sense to develop common strategies for courses in general, I prefer to think of any of them separately. Of course, the background of the audience partially identifies the material of the course. If the level of students is not very high, I reduce the density of the material and spend time on examples and additional explanations. Otherwise, if the level is high enough, I would propose to students to listen or to read some additional material, pose more complicated problems, and show advanced techniques. As the same audience can be prepared well enough for one subject, and can be completely surprised by another, it is necessary to control the audience's level of comprehension. Usually, it is good to pay attention to the "strongest" and to the "weakest" student. The style of lectures varies a lot with respect to the quantity of students. For non-mathematical students I prefer to explain ideas of mathematics, to give a clear understanding of methods, and to teach how to test the obtained result (ideally after the courses, the students should be capable of communicating with mathematicians).

Usually while teaching a course, I use notes or follow a good book, so the students do not need to spend much time for rewriting the material. Of course this depends on the established rules but in my opinion it is useful to make mid-term exams, since sometimes students suppose that they understand the material much better than they really do and start working only a week before the final examination.

Following the ideas of my teacher Prof. Vladimir Arnold, I think that the progress in education is usually achieved rather with solving problems and studying examples than with memorizing general theory. In this case the students enjoy the beauty of the theory and at the same time they feel the challenge of discovery itself. In my opinion this leads to the establishment of a taste to Mathematics that helps the students in future life, work, and research.

Papers submitted as habilitation thesis

[H1] O. Karpenkov, *Elementary notions of lattice trigonometry*, Math. Scand., vol. 102(2008), no. 2, pp. 161–205.
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