Habilitation Thesis

Geometry of Lattice Angles, Polygons, and Cones

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GEOMETRY OF LATTICE ANGLES, POLYGONS, AND CONES INTRODUCTION

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1. General description of the topic

Lattice geometry. Geometry in general can be interpreted in many different ways. We take one of the classical definitions of geometry: geometry is a *set of objects* and a *congruence relation* for these objects. Usually a congruence relation is defined by some group of transformations. For instance in *Euclidean geometry* in the plain the objects are points, lines, segments, polygons, circles, etc, and the congruence relation is defined by the orthogonal group $O(2,\mathbb{R})$.

In *lattice geometry* we have a full rank lattice in $\mathbb{R}^n$. The set of objects would be lattice points, *lattice segments* with lattice endpoints, *lattice lines* passing through a couple of lattice points, *ordinary lattice angles* and *cones* with vertex in a lattice point, *lattice polygons* and *lattice polytopes* having all vertices in the lattice. The congruence relation in lattice geometry is the group of affine transformations of $\mathbb{R}^n$ preserving the lattice. This group is isomorphic to a semidirect product of $GL(n,\mathbb{Z})$ and the group of integer lattice preserving translations.

It is interesting to compare lattice plane geometry and Euclidean plane geometry. Let us fix an integer lattice in the plane. From one hand these two geometries has a lot in common, from the other hand their properties sometimes behave in unexpectedly different ways. For instance twice Euclidean area of a polygon is always equivalent to its lattice area, but there is no such relations between Euclidean and lattice lengths. Sine formula works in both geometries, but the set of values of sines for Euclidean angles is the segment

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$[0, 1]$, while the values of integer sines are non-negative integers. Further if we consider angles defined by the rays $\{y = 0, x \geq 0\}$, and $\{y = \alpha x, x \geq 0\}$ for some $\alpha \geq 1$ then their tangents and lattice tangents coincide, nevertheless the arithmetics of angles is different: one can add two angles in lattice geometry in infinitely many ways (and all the resulting angles are non-congruent). We admit also that the angles ABC and CBA are congruent in Euclidean plane, but they are not always congruent in lattice case. In addition in lattice case there is less lattice regular polygons in the plane, but in some dimensions the amount of lattice regular polytopes can be arbitrarily large (in Euclidean case we have 3 distinct polytopes for dimensions greater than 4).

Relations between lattice geometry and other branches of mathematics. Lattice geometry is interesting by itself, still it has many applications to other branches of mathematics. In translation to algebraic geometry lattice geometry coincides with toric geometry. The polygons and polytopes in lattice geometry are toric varieties, their angles and cones are singularities. The singularities of lattice congruent angles are algebraically the same. The inverse is almost true, actually the angles ABC and CBA define the same singularity, but still they can be lattice non-congruent. So any description of lattice polygons and polytopes leads to the analogous algebro-geometric description for toric varieties. For instance, varieties that correspond to regular polytopes has, therefore, the maximal possible groups of symmetries.

In the context of toric geometry one of the first aims is to study singularities, i.e. lattice angles and cones. There is a deep relation between lattice angles and geometric continued fractions corresponding to these angles. This relation was generalized by F. Klein to the multidimensional case. Later V. I. Arnold formulated many questions on lattice cones (or toric singularities) and their multidimensional continued fractions. In particular, he posed several problems on an analog of Gauss-Kuzmin formula for statistics of the elements of ordinary continued fractions and a question on algebraic multidimensional continued fractions for the simplest operators. We study the first question in [H5], the answer for the second question in the case of three dimensions was given by E Korkina, in [H6] we give the answer in the case of four dimensions.

In addition I would like to say a few words about a relation between cones and their geometry and Gauss reduction theory. It turns out that ordinary continued fractions give a good description of conjugacy classes in the group $SL(2, \mathbb{Z})$. There are almost no answers for the case of $SL(n, \mathbb{Z})$ where $n \geq 3$. The structure of the set of conjugacy classes is very complicated (we remind that for closed fields such classes are described by Jordan Normal Forms). Multidimensional continued fractions in the sense of Klein is a strong invariant of such classes, which distinguish the classes up to some simple relation. Their lattice characteristic are good invariants for the conjugacy classes.

Organization of this habilitation thesis. We investigate lattice properties of lattice objects. In the first three papers we mostly study the two-dimensional case, and in the last three papers we work with multidimensional objects. In papers [H1] and [H2] we introduce lattice trigonometric functions and show that the lattice tangent is a complete invariant for ordinary lattice angles with respect to the group of lattice preserving affine transformations. Lattice tangents are certain continued fractions whose elements are constructed by lattice invariants of angles. We find the formula for lattice tangents of the sums of the angles and show the necessary condition for angles to be the angles of some polygon. For the case of triangles we get the necessary and sufficient condition, it was announced in [H3] and studied in a more detailed way in [H1]. Further in [H4] we give a classification of all lattice regular lattice polytopes with lattice lengths of the edges equal 1 (all the rest polyhedra are multiple to these). Finally in the last two papers we deal with lattice cones in multidimensional case. A multidimensional continued fraction in the sense of Klein is a natural geometric generalization of an ordinary continued fraction that is a complete invariant of lattice cones. In paper [H5] we study the statistical questions related to the generalization of the Gauss-Kuzmin distribution. Finally in paper [H6] we construct the first algebraic examples of four-dimensional periodic continued fractions. These fractions are supposed to be the simplest continued fractions in four-dimensional case.

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2. Short description of new results

2.1. **Lattice angles and their trigonometric functions.** In this subsection we start with the observation of papers [H1], [H2], and [H3].

The study of lattice angles is an essential part of modern lattice geometry. Invariants of lattice angles are used in the study of lattice convex polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the works of F. Klein [22], V. I. Arnold [3], E. Korkina [27], M. Kontsevich and Yu. Suhov [24], G. Lachaud [30]).

Lattice angles are in the limelight of complex projective toric varieties (see for more information the works of V. I. Danilov [11], G. Ewald [12], T. Oda [35], and W. Fulton [13]), they corresponds to singularities there. The singularities in many cases define toric varieties, that are actually lattice polygons in lattice geometry. So the study of toric varieties is reduced to the study of convex lattice polygons. The first classical open problem here is on description of convex lattice polygons. It is only known that the number of such polygons with lattice area bounded from above by n growths exponentially in $n^{1/3}$, while n tends to infinity (see the works of V. Arnold [7], and of I. Bárány and A. M. Vershik [8]).

We develop new methods to study lattice angles, these methods are based on introduction of trigonometric functions. General study of lattice trigonometric functions is given in papers [H1] and [H2].

Continued fractions. We start with necessary definitions from theory of continued fractions. For any finite sequence of integers $(a_0, a_1, \ldots, a_n)$ we associate an element

$$q \quad = \quad a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \quad \vdots \atop \qquad a_{n-1} + \frac{1}{a_n}}}$$

and denote it by $]a_0, a_1, \ldots, a_n[$. If the elements of the sequence $a_1, \ldots, a_n$ are positive, then the expression for q is called the *ordinary continued fraction*.

PROPOSITION. *For any rational number there exists a unique ordinary continued fraction with odd number of elements.*

Lattice trigonometric functions. Let us give definitions of lattice trigonometric functions. In many cases we choose lattice to be the integer lattice (whose lattice points has all integer coordinates).

We start with several preliminary definitions. A *lattice length* of a segment AB (denoted by $\mathrm{l}\ell(AB)$) is the ratio of its Euclidean length and the minimal Euclidean length of lattice vectors with vertices in AB. A *lattice area* of a triangle ABC is the index of a lattice generated by the vertices AB and AC in the whole lattice.

Consider an arbitrary lattice angle α with lattice vertex V (and having some lattice points distinct to V on both his edges). The boundary of the convex hull of the set of all lattice points except V in the angle α is called the *sail* of the angle. The sail of the angle is a finite broken line with the first and the last vertices on different edges of the angle. Let us orient the broken line in the direction from the first ray to the second ray of the angle and denote the vertices of this broken line as follows: $A_0, \ldots, A_{m+1}$. Denote

$$a_i = \mathrm{l}\ell(A_iA_{i+1}) \qquad \text{for } i = 0, \ldots, m;$$
$$b_i = \frac{\mathrm{lS}(A_{i-1}A_iA_{i+1})}{\mathrm{l}\ell(A_{i-1}A_i)\,\mathrm{l}\ell(A_iA_{i+1})} \qquad \text{for } i = 1, \ldots, m.$$

Definition 2.1. The *lattice tangent* of the angle α is the following rational number:

$$]a_0, b_1, a_1, b_2, a_2, \ldots, b_m, a_m[, \quad \text{we denote:} \quad \mathrm{ltan}\,\alpha.$$

The *lattice sine* is the numerator of rational number $\mathrm{ltan}\,\alpha$, denote it by $\mathrm{lsin}\,\alpha$.
The *lattice cosine* is the denominator of $\mathrm{ltan}\,\alpha$, denote it by $\mathrm{lcos}\,\alpha$.

Note that there are several equivalent definitions of the lattice trigonometric functions. We chose here the shortest one.

Results on lattice trigonometry. In papers [H1], [H2], and [H3] we study basic properties of lattice trigonometric functions.

In paper [H1] we define ordinary lattice angles, and the functions of lattice sine, tangent, and cosine, and lattice arctangent (defined for rational numbers greater than or equal 1). Further we introduce the sum formula for the lattice tangents of ordinary lattice angles of lattice triangles. The sum formula is a lattice generalization of the following Euclidean statement on three angles of the triangle being equal to π . Then we introduce the notion of extended lattice angles and find their normal forms. This lead to the definition of sums of extended and ordinary lattice angles. In particular this gives a new extension of the notion of sails in the sense of Klein: we define and study oriented broken lines at unit distance from lattice points. We give a necessary and sufficient condition for an ordered n-tuple of angles to be the angles of some convex lattice polygon. We conclude this paper with criterions of lattice congruence for lattice triangles.

In paper [H2] we introduce trigonometric functions for angles whose vertices are lattice but edges may not contain lattice points other than the vertex, we call such angles *irrational*. Further we study equivalence classes (with respect to the group of affine lattice preserving transformations) of irrational angles and find normal forms for such classes. Finally in several cases we give definitions of sums of irrational angles.

Paper [H3] is a small note in which we announce the result on sum of lattice angles in the triangle, that is further proved in [H1].

One of the results of paper [H1]**.** Let us focus on one particular application of lattice trigonometry, the description of all lattice triangles (see in [H1] and [H3]). In toric geometry this result is a description of all possible sets of singularities of complex projective toric surfaces whose Euler characteristic equals 3.

Let us consider for a rational q_i, $i = 1, \ldots, k$ the ordinary continued fractions with odd number of elements:

$$q_i =]a_{i,0}, a_{i,1}, \ldots, a_{i,2n_i}[.$$

Denote by $]q_1, q_2, \ldots, q_k[$ the element

$$]a_{1,0}, a_{1;1}, \ldots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \ldots, a_{2,2n_2}, \ldots a_{k,0}, a_{k,1}, \ldots, a_{k,2n_k}[.$$

The Euclidean condition

$$\alpha + \beta + \gamma = \pi$$

can be written with tangents of angles as follows:

$$\begin{cases} \tan(\alpha+\beta+\gamma) = 0 \\ \tan(\alpha+\beta) \notin [0; \tan\alpha] \end{cases}$$

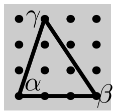
(without lose of generality, here we suppose that α is acute). The lattice version of Euclidean sum formula for the triangles is as follows.

Theorem 2.2. [H1] **a).** *Let α_0, α_1, and α_2 be an ordered triple of lattice angles. There exists an oriented lattice triangle with the consecutive angles lattice-equivalent to the angles α_0, α_1, and α_2 if and only if there exists $j \in \{0, 1, 2\}$ such that the angles $\alpha = \alpha_j$, $\beta = \alpha_{j+1(\mathrm{mod}\,3)}$, and $\gamma = \alpha_{j+2(\mathrm{mod}\,3)}$ satisfy the following conditions:*

i) $]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta, -1, \mathrm{ltan}\,\gamma[= 0$; *ii)* $]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta[\notin [0; \mathrm{ltan}\,\alpha]$.

b). *Two lattice triangles with the same sequences of integer tangents are integer-homothetic.*

We illustrate the theorem with the example of [H3]:



$\operatorname{ltan}\alpha = 3 =]3[;$
$\operatorname{ltan}\beta = 9/7 =]1,3,2[;$
$\operatorname{ltan}\gamma = 3/2 =]1,1,1[.$

i) $]3,-1,1,3,2,-1,1,1,1[= 0;$
ii) $]3,-1,1,3,2[= -3/2 \notin [0;3].$

2.2. **Lattice regular polygons and polytopes.** Let us say a few words about paper [H4].

We say that a polyhedron is *lattice regular* if for any two complete flags of this polyhedron there exists a lattice preserving transformation taking the first flag to the second one. In [H4] we develop a complete classification of lattice regular polytopes in all dimensions. It turns out that in the plain there are only 2 nonequivalent regular integer triangles, 2 quadrangles, and 2 hexagons. In the three-dimensional space we have 3 regular polyhedra, 3 regular octahedra, and 3 regular cubes. In dimension 4 we have 2 simplices, 3 generalized octahedra, 3 cubes and 2 hyperdiamonds (or 24-cells). Finally in dimension n we always get k simplices, 3 cubes, 3 octahedra, where k is the number of positive integer divisors of $n+1$.

2.3. **Geometry of lattice simplicial cones, multidimensional continued fractions.** In this subsection we observe the papers [H5] and [H6].

Lattice simplicial cones are in one to one corresponding to Klein's multidimensional continued fractions (up to the action of the group of lattice affine transformations). The problem of generalization of ordinary continued fractions to the higher-dimensional case was posed by C. Hermite [16] in 1839. A large number of attempts to solve this problem lead to the birth of several different remarkable theories of multidimensional continued fractions (see in [40], [38], etc.). We consider the geometric generalization of ordinary continued fractions to the multidimensional case presented by F. Klein in 1895 and published by him in [22] and [23].

Consider a set of $n{+}1$ hyperplanes of $\mathbb{R}^{n+1}$ passing through the origin in general position. The complement to the union of these hyperplanes consists of 2^{n+1} open cones. Let us choose an arbitrary cone.

Definition 2.3. The boundary of the convex hull of all integer points except the origin in the closure of the cone is called the *sail*. The set of all 2^{n+1} sails of the space $\mathbb{R}^{n+1}$ is called the *n-dimensional continued fraction* associated to the given $n{+}1$ hyperplanes in general position in $(n{+}1)$-dimensional space.

Two n-dimensional continued fractions are said to be *equivalent* if there exists a linear transformation that preserves the integer lattice of the $(n{+}1)$-dimensional space and maps the sails of the first continued fraction to the sails of the other.

Multidimensional continued fractions in the sense of Klein have many connections with other branches of mathematics. For example, J.-O. Moussafir [32] and O. N. German [15] studied the connection between the sails of multidimensional continued fractions and Hilbert bases. In [39] H. Tsuchihashi found the relationship between periodic multidimensional continued fractions and multidimensional cusp singularities, which generalizes the relationship between ordinary continued fractions and two-dimensional cusp singularities. M. L. Kontsevich and Yu. M. Suhov discussed the statistical properties of the boundary of a random multidimensional continued fraction in [24]. The combinatorial topological generalization of Lagrange theorem was obtained by E. I. Korkina in [26] and its algebraic generalization by G. Lachaud [29].

V. I. Arnold presented a survey of geometrical problems and theorems associated with one-dimensional and multidimensional continued fractions in his article [6] and his book [3]). For the algorithms of constructing multidimensional continued fractions, see the papers of R. Okazaki [34], J.-O. Moussafir [33] and the author [20].

E. Korkina in [25] and [27] and G. Lachaud in [29], [30], A. D. Bruno and V. I. Parusnikov in [10], [36], and [37], the author in [18] and [19] produced a large number of fundamental domains for periodic algebraic two-dimensional continued fractions. A nice collection of two-dimensional continued fractions is given in the work [9] by K. Briggs.

A few words about my PhD-thesis. In my PhD-thesis (that was supervised by V. I. Arnold) I studied infinite series of two-dimensional continued fractions [18]. Further I made a description of polygonal faces lying in planes on integer distance greater than 1 to the origin [21]. This leads to an effective algorithm to construct periodic multidimensional continued fractions [20].

Gauss-Kuzmin formula and Möbius measure, generalization to multidimensional case ([H5]). For the first time the statement on statistics of numbers as elements of ordinary continued fractions was formulated by K. F. Gauss in his letters to P. S. Laplace (see in [14]). This statement was proven further by R. O. Kuzmin [28], and further was proven one more time by P. Lévy [31]. Further investigations in this direction were made by E. Wirsing in [42]. (A basic notions of theory of ordinary continued fractions is described in the books [17] by A. Ya. Hinchin and [3] by V. I. Arnold.) In 1989 V. I. Arnold generalized statistical problems to the case of one-dimensional and multidimensional continued fractions in the sense of Klein, see in [5] and [4].

One-dimensional case was studied in details by M. O. Avdeeva and B. A. Bykovskii in the works [1] and [2]. In two-dimensional and multidimensional cases V. I. Arnold formulated many problems on statistics of sail characteristics of multidimensional continued fractions such as an amount of triangular, quadrangular faces and so on, such as their integer areas, and length of edges, etc. A major part of these problems is open nowadays, while some are almost completely solved.

M. L. Kontsevich and Yu. M. Suhov in their work [24] proved the existence of the mentioned above statistics. In [H5] we write explicitly a natural Möbius measure of the manifold of all n-dimensional continued fractions in the sense of Klein and introduced new integral formulae for the statistics.

Simplest four-dimensional examples ([H6]). The problem of investigation of the simplest algebraic n-dimensional cones and their continued fraction for $n \geq 2$ was posed by V. Arnold. The answers for the case of $n = 2$ were given by E. Korkina and G. Lachaud. We have studied the case of $n = 3$ in [H6]. The two three-dimensional continued fractions for the cones proposed in the paper seems to be the simplest examples for many reasons (such as existence of additional symmetries, simplicities of fundamental domains and characteristic polynomials).

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ELEMENTARY NOTIONS OF LATTICE TRIGONOMETRY

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Introduction

0.1. The goals of this paper and some background

Consider a two-dimensional oriented real vector space and fix some full-rank lattice in it. A triangle or a polygon is said to be *lattice* if all its vertices belong to the lattice. The angles of any lattice triangle are said to be *lattice*.

In this paper we introduce and study *lattice trigonometric* functions of lattice angles. The lattice trigonometric functions are invariant under the action of the group of lattice-affine transformations (i.e. affine transformations preserving the lattice), like the ordinary trigonometric functions are invariant under the action of the group of Euclidean length preserving transformations of Euclidean space.

One of the initial goals of the present article is to make a complete description of lattice triangles up to the lattice-affine equivalence relation (see Theorem 2.2). The classification problem of convex lattice polygons becomes now classical. There is still no a good description of convex polygons. It is only known that the number of such polygons with lattice area bounded from above by n growths exponentially in $n^{1/3}$, while n tends to infinity (see the works of V. Arnold [2], and of I. Bárány and A. M. Vershik [3]).

We extend the geometric interpretation of ordinary continued fractions to define lattice sums of lattice angles and to establish relations on lattice tangents of lattice angles. Further, we describe lattice triangles in terms of *lattice sums* of lattice angles.

In present paper we also show a lattice version of the sine formula and introduce a relation between the lattice tangents for angles of lattice triangles and the numbers of lattice points on the edges of triangles (see Theorem 1.15).

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We conclude the paper with applications to toric varieties and some unsolved problems.

The study of lattice angles is an essential part of modern lattice geometry. Invariants of lattice angles are used in the study of lattice convex polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the works of F. Klein [14], V. I. Arnold [1], E. Korkina [16], M. Kontsevich and Yu. Suhov [15], G. Lachaud [17], and the author [10]).

Lattice polygons and polytopes of the lattice geometry are in the limelight of complex projective toric varieties (see for more information the works of V. I. Danilov [4], G. Ewald [5], T. Oda [18], and W. Fulton [6]). To illustrate, we deduce (in Appendix A) from Theorem 2.2 the corresponding global relations on the toric singularities for projective toric varieties associated to integer-lattice triangles. We also show the following simple fact: for any collection with multiplicities of complex-two-dimensional toric algebraic singularities there exists a complex-two-dimensional toric projective variety with the given collection of toric singularities (this result seems to be classical, but it is missing in the literature).

The studies of lattice angles and measures related to them were started by A. G. Khovanskii, A. Pukhlikov in [12] and [13] in 1992. They introduced and investigated special additive polynomial measure for the extended notion of polytopes. The relations between sum-formulas of lattice trigonometric functions and lattice angles in Khovanskii-Pukhlikov sense are unknown to the author.

0.2. Some distinctions between lattice and Euclidean cases

Lattice trigonometric functions and Euclidean trigonometric functions have much in common. For example, the values of lattice tangents and Euclidean tangents coincide in a special natural system of coordinates. Nevertheless, lattice geometry differs a lot from Euclidean geometry. We show this with the following four examples.

1. The angles $\angle ABC$ and $\angle CBA$ are always congruent in Euclidean geometry, but not necessary lattice-congruent in lattice geometry.

2. In Euclidean geometry for any $n \geq 3$ there exist a regular polygon with n vertices, and any two regular polygons with the same number of vertices are homothetic to each other. In lattice geometry there are only six non-homothetic regular lattice polygons: two triangles (distinguished by lattice tangents of angles), two quadrangles, and two octagons. (See a more detailed description in [11].)

3. In Appendix B we will consider three natural criteria for triangle congruence in Euclidean geometry. Only the first criterion can be taken to the case of

lattice geometry. The others two are false in lattice trigonometry. (We refer to Appendix B.)

4. There exist two non-congruent right angles in lattice geometry. (See Corollary 1.12.)

0.3. Description of the paper

This paper is organized as follows.

We start in Section 1 with some general notation of lattice geometry. We define ordinary lattice angles, and the functions lattice sine, tangent, and cosine on the set of ordinary lattice angles, and lattice arctangent for rationals greater than or equal 1. Further we indicate their basic properties. We proceed with the geometrical interpretation of lattice tangents in terms of ordinary continued fractions. In conclusion of Section 1 we study the basic properties of angles in lattice triangles.

In Section 2 we introduce the sum formula for the lattice tangents of ordinary lattice angles of lattice triangles. The sum formula is a lattice generalization of the following Euclidean statement: three angles are the angles of some triangle iff their sum equals π.

Further in Section 3 we introduce the notion of extended lattice angles and their normal forms and give the definition of sums of extended and ordinary lattice angles. Here we extend the notion of sails in the sense of Klein: we define and study oriented broken lines at unit distance from lattice points.

In Section 4 we finally prove the first statement of the theorem on sums of lattice tangents for angles in lattice triangles. In this section we also describe some relations between continued fractions for lattice oriented broken lines and the lattice tangents for the corresponding extended lattice angles. Further we give a necessary and sufficient condition for an ordered n-tuple of angles to be the angles of some convex lattice polygon.

We conclude this paper with three appendices. In Appendix A we describe applications to theory of complex projective toric varieties mentioned above. Further in Appendix B we formulate criterions of lattice congruence for lattice triangles. Finally in Appendix C we give a list of unsolved problems and questions.

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1. Definitions and elementary properties of lattice trigonometric functions

1.1. Preliminary notions and definitions

By $\gcd(n_1, \ldots, n_k)$ and by $\operatorname{lcm}(n_1, \ldots, n_k)$ we denote the greater common divisor and the less common multiple of the nonzero integers $n_1, \ldots, n_k$ respectively. Suppose that a, b be arbitrary integers, and c be an arbitrary positive integer. We write that $a \equiv b \pmod c$ if the reminders of a and b modulo c coincide.

1.1.1. Lattice notation. Here we define the main objects of lattice geometry, their lattice characteristics, and the relation of $\mathscr{L}$-congruence (lattice-congruence).

Consider R^2 and fix some orientation and some lattice in it. A straight line is said to be *lattice* if it contains at least two distinct lattice points. A ray is said to be *lattice* if its vertex is a lattice point, and it contains lattice points distinct from its vertex. An angle (i.e. the union of two rays with the common vertex) is said to be *ordinary lattice* (or just *ordinary* for short) if the rays defining it are lattice. A segment is called *lattice* if its endpoints are lattice points.

By a *convex polygon* we mean a convex hulls of a finite number of points that do not lie in a straight line. A straight line π is said to be *supporting* a convex polygon P, if the intersections of P and π is not empty, and the whole polygon P is contained in one of the closed half-planes bounded by π. An intersection of a polygon P with its supporting straight line is called a *vertex* or an *edge* of the polygon if the dimension of intersection is zero, or one respectively.

A triangle (or convex polygon) is said to be *lattice* if all its vertices are lattice points. A lattice triangle is said to be *simple* if the vectors corresponding to its edges generate the lattice.

The affine transformation is called *$\mathscr{L}$-affine* if it preserves the set of all lattice points. Consider two arbitrary (not necessary lattice in the above sense) sets. We say that these two sets are *$\mathscr{L}$-congruent* to each other if there exist a $\mathscr{L}$-affine transformation of R^2 taking the first set to the second.

DEFINITION 1.1. The *lattice length* of a lattice segment AB is the ratio between the Euclidean length of AB and the length of the basic lattice vector for the straight line containing this segment. We denote the lattice length by $\mathrm{l}\ell(AB)$.

By the (non-oriented) *lattice area* of the convex polygon P we will call the ratio of the Euclidean area of the polygon and the area of any lattice simple triangle, and denote it by $\mathrm{lS}(P)$.

Two lattice segments are $\mathscr{L}$-congruent iff they have equal lattice lengths. The lattice area of the convex polygon is well-defined and is proportional to the Euclidean area of the polygon.

1.1.2. Finite ordinary continued fractions. For any finite sequence $(a_0, a_1, \ldots, a_n)$ where the elements $a_1, \ldots, a_n$ are positive integers and a_0 is an arbitrary integer we associate the following rational number q:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \quad \vdots \atop a_{n-1} + \cfrac{1}{a_n}}}.$$

This representation of the rational q is called an *ordinary continued fraction* for q and denoted by $[a_0, a_1, \ldots, a_n]$.

An ordinary continued fraction $[a_0, a_1, \ldots, a_n]$ is said to be *odd* if $n+1$ is odd, and *even* if $n+1$ is even. Note that if $a_n \neq 1$ then $[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1]$. Let us formulate the following classical theorem.

THEOREM 1.2. *For any rational there exist exactly one odd ordinary continued fraction and exactly one even ordinary continued fraction.*

1.2. Definition of lattice trigonometric functions

In this subsection we define the functions lattice sine, tangent, and cosine on the set of ordinary lattice angles and formulate their basic properties. We describe a geometric interpretation of lattice trigonometric functions in terms of ordinary continued fractions. Then we give the definitions of ordinary angles that are adjacent, transpose, and opposite interior to the given angles. We use the notions of adjacent and transpose ordinary angles to define ordinary lattice right angles.

Let A, O, and B be three lattice points that do not lie in the same straight line. We denote the ordinary angle with the vertex at O and the rays OA and OB by $\angle AOB$.

One can chose any other lattice point C in the open lattice ray OA and any lattice point D in the open lattice ray OB. For us the angle $\angle AOB$ *coincides* with $\angle COD$. We denote this by $\angle AOB = \angle COD$.

DEFINITION 1.3. Two ordinary angles $\angle AOB$ and $\angle A'O'B'$ are said to be *$\mathscr{L}$-congruent* if there exist a $\mathscr{L}$-affine transformation that takes the point O to O' and the rays OA and OB to the rays $O'A'$ and $O'B'$ respectively. We denote this as follows: $\angle AOB \cong \angle A'O'B'$.

Here we note that the relation $\angle AOB \cong \angle BOA$ holds only for special ordinary angles. (See also below in Subsubsection 1.2.4.)

1.2.1. Definition of lattice sine, tangent, and cosine for an ordinary lattice angle. Consider an arbitrary ordinary angle $\angle AOB$. Let us associate a special basis to this angle. Denote by $\overline{v}_1$ and by $\overline{v}_2$ the lattice vectors generating the rays of the angle:

$$\overline{v}_1 = \frac{\overline{OA}}{\mathrm{l}\ell(OA)}, \qquad \text{and} \qquad \overline{v}_2 = \frac{\overline{OB}}{\mathrm{l}\ell(OB)}.$$

The set of lattice points at unit lattice distance from the lattice straight line OA coincides with the set of all lattice points of two lattice straight lines parallel to OA. Since the vectors $\overline{v}_1$ and $\overline{v}_2$ are linearly independent, the ray OB intersects exactly one of the above two lattice straight lines. Denote this straight line by l. The intersection point of the ray OB with the straight line l divides l into two parts. Choose one of the parts which lies in the complement to the convex hull of the union of the rays OA and OB, and denote by D the lattice point closest to the intersection of the ray OB with the straight line l (see Figure 1).

Now we choose the vectors $\overline{e}_1 = \overline{v}_1$ and $\overline{e}_2 = \overline{OD}$. These two vectors are linearly independent and generate the lattice. The basis $(\overline{e}_1, \overline{e}_2)$ is said to be *associated* to the angle $\angle AOB$.

Since $(\overline{e}_1, \overline{e}_2)$ is a basis, the vector $\overline{v}_2$ has a unique representation of the form:

$$\overline{v}_2 = x_1\overline{e}_1 + x_2\overline{e}_2,$$

where x_1 and x_2 are some integers.

DEFINITION 1.4. In the above notation, the coordinates x_2 and x_1 are said to be the *lattice sine* and the *lattice cosine* of the ordinary angle $\angle AOB$ respectively. The ratio of the lattice sine and the lattice cosine (x_2/x_1) is said to be the lattice tangent of $\angle AOB$.

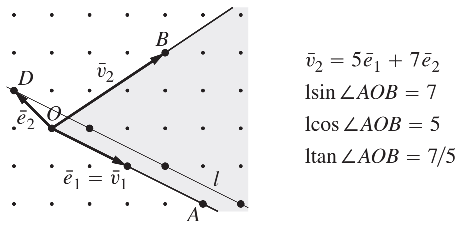


FIGURE 1. An ordinary angle $\angle AOB$ and its lattice trigonometric functions.

Figure 1 shows an example of lattice angle with the lattice sine equals 7 and the lattice cosine equals 5.

Let us briefly enumerate some elementary properties of lattice trigonometric functions.

PROPOSITION 1.5. a) *The lattice sine and cosine of any ordinary angle are relatively-prime positive integers.*

b) *The values of lattice trigonometric functions for $\mathcal{L}$-congruent ordinary angles coincide.*

c) *The lattice sine of an ordinary angle coincide with the index of the sublattice generated by all lattice vectors of two angle rays in the lattice.*

d) *For any ordinary angle α the following inequalities hold:*

$$\operatorname{lsin}\alpha \geq \operatorname{lcos}\alpha, \qquad \textit{and} \qquad \operatorname{ltan}\alpha \geq 1.$$

The equalities hold iff the lattice vectors of the angle rays generate the whole lattice.

e) (Description of lattice angles) *Two ordinary angles α and β are $\mathcal{L}$-congruent iff* $\operatorname{ltan}\alpha = \operatorname{ltan}\beta$.

1.2.2. Lattice arctangent. Let us fix the origin O and a lattice basis $\bar{e}_1$ and $\bar{e}_2$.

DEFINITION 1.6. Consider an arbitrary rational $p \geq 1$. Let $p = m/n$, where m and n are positive integers. Suppose $A = O + \bar{e}_1$, and $B = O + n\bar{e}_1 + m\bar{e}_2$. The ordinary angle $\angle AOB$ is said to be the *arctangent of p in the fixed basis* and denoted by $\operatorname{larctan}(p)$.

The invariance of lattice tangents immediately implies the following properties.

PROPOSITION 1.7. a) *For any rational $s \geq 1$, we have:* $\operatorname{ltan}(\operatorname{larctan} s) = s$.

b) *For any ordinary angle α the following holds:* $\operatorname{larctan}(\operatorname{ltan}\alpha) \cong \alpha$.

1.2.3. Lattice tangents, length-sine sequences, sails, and continued fractions. Let us start with the notion of sails for the ordinary angles. This notion is taken from theory of multidimensional continued fractions in the sense of Klein (see, for example, the works of F. Klein [14], and V. Arnold [1]).

Consider an ordinary angle $\angle AOB$. Let also the vectors $\overline{OA}$ and $\overline{OB}$ be linearly independent, and of unit lattice length. Denote the closed convex solid cone for the ordinary angle $\angle AOB$ by $C(AOB)$. The boundary of the convex hull of all lattice points of the cone $C(AOB)$ except the origin is homeomorphic to the straight line. This boundary contains the points A and B. The closed part of this boundary contained between the points A and B is called the *sail* for the cone $C(AOB)$.

A lattice point of the sail is said to be a *vertex* of the sail if there is no lattice segment of the sail containing this point in the interior. The sail of the cone $C(AOB)$ is a broken line with a finite number of vertices and without self intersections. Let us orient the sail in the direction from A to B, and denote the vertices of the sail by V_i (for $0 \le i \le n$) according to the orientation of the sail (such that $V_0 = A$, and $V_n = B$).

DEFINITION 1.8. Let the vectors $\overline{OA}$ and $\overline{OB}$ of the ordinary angle $\angle AOB$ be linearly independent, and of unit lattice length. Let V_i, where $0 \le i \le n$, be the vertices of the corresponding sail. The sequence of lattice lengths and sines

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \mathrm{l}\ell(V_1V_2), \mathrm{lsin}\,\angle V_1V_2V_3, \\ \dots, \mathrm{l}\ell(V_{n-2}V_{n-1}), \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n))$$

is called the *lattice length-sine sequence* for the ordinary angle $\angle AOB$. Further we say *LLS-sequence* for short.

REMARK 1.9. The elements of the lattice LLS-sequence for any ordinary angle are positive integers. The LLS-sequences of $\mathscr{L}$-congruent ordinary angles coincide.

THEOREM 1.10. *Let $(a_0, a_1, \dots, a_{2n-3}, a_{2n-2})$ be the LLS-sequence for the ordinary angle $\angle AOB$. Then the lattice tangent of the ordinary angle $\angle AOB$ equals to the value of the following ordinary continued fraction*

$$[a_0, a_1, \dots, a_{2n-3}, a_{2n-2}].$$

On Figure 2 we show an example of an ordinary angle with tangent equivalent to $7/5$.

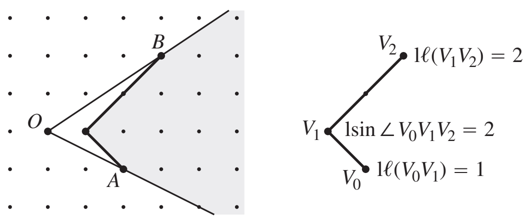


FIGURE 2. $\mathrm{ltan}\,\angle AOB = \frac{7}{5} = 1 + \frac{1}{2+1/2}$.

Further in Theorem 3.5 we formulate and prove a general statement for generalized sails and signed lattice length-sine sequences. In the proof of Theorem 3.5 we refer only on the preceding statements and definitions of Subsection 3.1, that are independent of the statements and theorems of all previous sections. For these reasons we skip now the proof of Theorem 1.10 (see also Remark 3.6).

1.2.4. Adjacent, transpose, and opposite interior ordinary angles. An ordinary angle $\angle BOA$ is said to be *transpose* to the ordinary angle $\angle AOB$. We denote it by $(\angle AOB)^t$. An ordinary angle $\angle BOA'$ is said to be *adjacent* to an ordinary angle $\angle AOB$ if the points A, O, and A' are contained in the same straight line, and the point O lies between A and A'. We denote the ordinary angle $\angle BOA'$ by $\pi - \angle AOB$. The ordinary angle is said to be *right* if it is $\mathscr{L}$-congruent to the adjacent and to the transpose ordinary angles.

It immediately follows from the definition, that for any ordinary angle α the angles $(\alpha^t)^t$ and $\pi - (\pi - \alpha)$ are $\mathscr{L}$-congruent to α.

In the next theorem we use the following notion. Suppose that some integers a, b and c, where $c \geq 1$, satisfy the following: $ab \equiv 1 \pmod c$. Then we denote $a \equiv \big(b \pmod c\big)^{-1}$.

THEOREM 1.11. *Consider an ordinary angle α. If $\alpha \cong \operatorname{larctan}(1)$, then*

$$\alpha^t \cong \pi - \alpha \cong \operatorname{larctan}(1).$$

Suppose now, that $\alpha \ncong \operatorname{larctan}(1)$, then

$$\operatorname{lsin}(\alpha^t) = \operatorname{lsin}\alpha, \qquad \operatorname{lcos}(\alpha^t) \equiv \big(\operatorname{lcos}\alpha \pmod{\operatorname{lsin}\alpha}\big)^{-1};$$
$$\operatorname{lsin}(\pi - \alpha) = \operatorname{lsin}\alpha, \qquad \operatorname{lcos}(\pi - \alpha) \equiv \big(-\operatorname{lcos}\alpha \pmod{\operatorname{lsin}\alpha}\big)^{-1}.$$

Note also, that $\pi - \alpha \cong \operatorname{larctan}^t\big(\frac{\operatorname{ltan}\alpha}{\operatorname{ltan}(\alpha)-1}\big)$.

Theorem 1.11 (after applying Theorem 1.10) immediately reduces to the theorem of P. Popescu-Pampu. We refer the readers to his work [19] for the proofs.

1.2.5. Right ordinary lattice angles. It turns out that in lattice geometry there exist exactly two lattice non-equivalent right ordinary angles.

COROLLARY 1.12. *Any ordinary right angle is $\mathscr{L}$-congruent to exactly one of the following two angles:* $\operatorname{larctan}(1)$, *or* $\operatorname{larctan}(2)$.

Consider two lattice parallel distinct straight lines AB and CD, where A, B, C, and D are lattice points. Let the points A and D be in different open half-planes with respect to the straight line BC. Then the ordinary angle $\angle ABC$ is called *opposite interior* to the ordinary angle $\angle DCB$. Further we use the following proposition on opposite interior ordinary angles.

PROPOSITION 1.13. *Two opposite interior to each other ordinary angles are $\mathscr{L}$-congruent.*

The proof is left for the reader as an exercise.

1.3. Basic lattice trigonometry of lattice angles in lattice triangles

In this subsection we introduce the sine formula for angles and edges of lattice triangles. Further we show how to find the lattice tangents of all angles and the lattice lengths of all edges of any lattice triangle, if the lattice lengths of two edges and the lattice tangent of the angle between them are given.

Let A, B, C be three distinct and not collinear lattice points. We denote the lattice triangle with the vertices A, B, and C by $\triangle ABC$. The lattice triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *$\mathscr{L}$-congruent* if there exist a $\mathscr{L}$-affine transformation which takes the point A to A', B to B', and C to C' respectively. We denote: $\triangle ABC \cong \triangle A'B'C'$.

PROPOSITION 1.14 (The sine formula for lattice triangles). *The following holds for any lattice triangle $\triangle ABC$.*

$$\frac{\mathrm{l}\ell(AB)}{\mathrm{lsin}\,\angle BCA} = \frac{\mathrm{l}\ell(BC)}{\mathrm{lsin}\,\angle CAB} = \frac{\mathrm{l}\ell(CA)}{\mathrm{lsin}\,\angle ABC} = \frac{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)\,\mathrm{l}\ell(CA)}{\mathrm{lS}(\triangle ABC)}.$$

PROOF. The statement of Proposition 1.14 follows directly from the definition of lattice sine.

Suppose that we know the lattice lengths of the edges AB, AC and the lattice tangent of $\angle BAC$ in the triangle $\triangle ABC$. Now we show how to restore the lattice length and the lattice tangents for the the remaining edge and ordinary angles of the triangle.

For the simplicity we fix some lattice basis and use the system of coordinates OXY corresponding to this basis (denoted $(*, *)$).

THEOREM 1.15. *Consider some triangle $\triangle ABC$. Let*

$$\mathrm{l}\ell(AB) = c, \qquad \mathrm{l}\ell(AC) = b, \qquad \textit{and} \qquad \angle CAB \cong \alpha.$$

Then the ordinary angles $\angle BCA$ and $\angle ABC$ are defined in the following way.

$$\angle BCA \cong \begin{cases} \pi - \mathrm{larctan}\left(\frac{c\,\mathrm{lsin}\,\alpha}{c\,\mathrm{lcos}\,\alpha - b}\right) & \textit{if } c\,\mathrm{lcos}\,\alpha > b \\ \mathrm{larctan}(1) & \textit{if } c\,\mathrm{lcos}\,\alpha = b \\ \mathrm{larctan}^t\left(\frac{c\,\mathrm{lsin}\,\alpha}{b - c\,\mathrm{lcos}\,\alpha}\right) & \textit{if } c\,\mathrm{lcos}\,\alpha < b, \end{cases}$$

$$\angle ABC \cong \begin{cases} \left(\pi - \mathrm{larctan}\left(\frac{b\,\mathrm{lsin}(\alpha^t)}{b\,\mathrm{lcos}(\alpha^t) - c}\right)\right)^t & \textit{if } b\,\mathrm{lcos}(\alpha^t) > c \\ \mathrm{larctan}(1) & \textit{if } b\,\mathrm{lcos}(\alpha^t) = c \\ \mathrm{larctan}\left(\frac{b\,\mathrm{lsin}(\alpha^t)}{c - b\,\mathrm{lcos}(\alpha^t)}\right) & \textit{if } b\,\mathrm{lcos}(\alpha^t) < c. \end{cases}$$

For the lattice length of the edge CB we have

$$\frac{\mathrm{l}\ell(CB)}{\mathrm{lsin}\,\alpha} = \frac{b}{\mathrm{lsin}\,\angle ABC} = \frac{c}{\mathrm{lsin}\,\angle BCA}.$$

PROOF. Let $\alpha \cong \mathrm{larctan}(p/q)$, where $\gcd(p,q) = 1$. Then $\triangle CAB \cong \triangle DOE$ where $D = (b,0)$, $O = (0,0)$, and $E = (qc, pc)$. Let us now find the ordinary angle $\angle EDO$. Denote by Q the point $(qc, 0)$. If $qc - b = 0$, then $\angle BCA = \angle EDO = \mathrm{larctan}\,1$. If $qc - b \neq 0$, then we have

$$\angle QDE \cong \mathrm{larctan}\left(\frac{cp}{|cq-b|}\right) \cong \mathrm{larctan}\left(\frac{c\,\mathrm{lsin}\,\alpha}{|c\,\mathrm{lcos}\,\alpha - b|}\right).$$

The expression for $\angle BCA$ follows directly from the above expression for $\angle QDE$, since $\angle BCA \cong \angle QDE$. (See Figure 3: here $\mathrm{l}\ell(OD) = b$, $\mathrm{l}\ell(OQ) = c\,\mathrm{lcos}\,\alpha$, and therefore $\mathrm{l}\ell(DQ) = |c\,\mathrm{lcos}\,\alpha - b|$.)

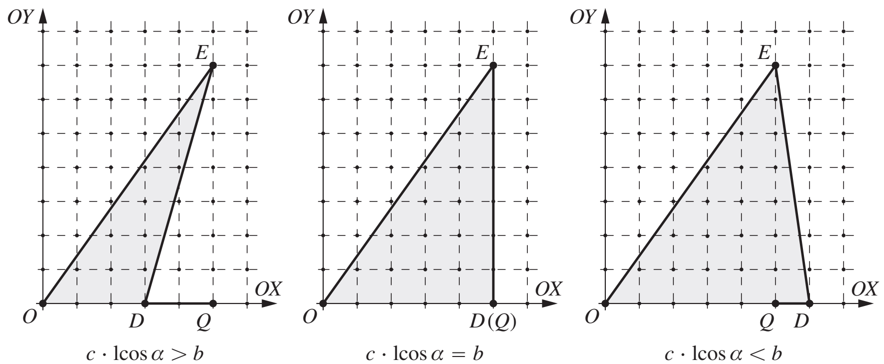


FIGURE 3. Three possible configuration of points O, D, and Q.

To obtain the expression for $\angle ABC$ we consider the triangle $\triangle BAC$. Calculate $\angle CBA$ and then transpose all ordinary angles in the expression. Since

$$\begin{aligned}\mathrm{lS}(ABC) &= \mathrm{l}\ell(AB)\,\mathrm{l}\ell(AC)\,\mathrm{lsin}\,\angle CAB\\ &= \mathrm{l}\ell(BA)\,\mathrm{l}\ell(BC)\,\mathrm{lsin}\,\angle BCA\\ &= \mathrm{l}\ell(CB)\,\mathrm{l}\ell(CA)\,\mathrm{lsin}\,\angle ABC,\end{aligned}$$

we have the last statement of the theorem.

2. Theorem on sum of lattice tangents for the ordinary lattice angles of lattice triangles. Proof of its second statement

Throughout this section we fix some lattice basis and use the system of coordinates OXY corresponding to this basis.

2.1. Finite continued fractions with not necessary positive elements

We start this section with the notation for finite continued fractions with not necessary positive elements. Let us extend the set of rationals $\mathbb{Q}$ with the operations $+$ and $1/*$ on it with the element ∞. We pose $q \pm \infty = \infty$, $1/0 = \infty$, $1/\infty = 0$ (we do not define $\infty \pm \infty$ here). Denote this extension by $\overline{\mathbb{Q}}$.

For any finite sequence of integers $(a_0, a_1, \ldots, a_n)$ we associate an element q of $\overline{\mathbb{Q}}$:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \quad \vdots \atop a_{n-1} + \cfrac{1}{a_n}}}.$$

and denote it by $]a_0, a_1, \ldots, a_n[$.

Let q_i be some rationals, $i = 1, \ldots, k$. Suppose that the odd continued fraction for q_i is $[a_{i,0}, a_{i,1}, \ldots, a_{i,2n_i}]$ for $i = 1, \ldots, k$. We denote by $]q_1, q_2, \ldots, q_n[$ the following number

$$]a_{1,0}, a_{1;1}, \ldots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \ldots, a_{2,2n_2}, \ldots a_{k,0}, a_{k,1}, \ldots, a_{k,2n_k}[.$$

2.2. Formulation of the theorem and proof of its second statement

In Euclidean geometry the sum of Euclidean angles of the triangle equals π. For any 3-tuple of angles with the sum equals π there exist a triangle with these angles. Two Euclidean triangles with the same angles are homothetic. Let us show a generalization of these statements to the case of lattice geometry.

Let n be an arbitrary positive integer, and $A = (x, y)$ be an arbitrary lattice point. Denote by nA the point (nx, ny).

DEFINITION 2.1. Consider any convex polygon or broken line with vertices $A_0, \ldots, A_k$. The polygon or broken line $nA_0 \ldots nA_k$ is called *n-multiple* (or *multiple*) to the given polygon or broken line.

THEOREM 2.2 (On sum of lattice tangents of angles in lattice triangles). a) *Let* $(\alpha_1, \alpha_2, \alpha_3)$ *be an ordered* 3*-tuple of ordinary angles. There exists a triangle with three consecutive ordinary angles* $\mathscr{L}$*-congruent to* α_1*,* α_2*, and*

α_3 *iff there exists* $i \in \{1, 2, 3\}$ *such that the angles* $\alpha = \alpha_i$, $\beta = \alpha_{i+1 \pmod 3}$, *and* $\gamma = \alpha_{i+2 \pmod 3}$ *satisfy the following conditions:*

i) *for* $A =]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta[$ *the following holds* $A < 0$, *or* $A > \mathrm{ltan}\,\alpha$, *or* $A = \infty$;

ii) $]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta, -1, \mathrm{ltan}\,\gamma[= 0$.

b) *Let the consecutive ordinary angles of some triangle be* α, β, *and* γ. *Then this triangle is multiple to the triangle with vertices* $A_0 = (0, 0)$, $B_0 = (\lambda_2 \,\mathrm{lcos}\,\alpha, \lambda_2 \,\mathrm{lsin}\,\alpha)$, *and* $C_0 = (\lambda_1, 0)$, *where*

$$\lambda_1 = \frac{\mathrm{lcm}(\mathrm{lsin}\,\alpha, \mathrm{lsin}\,\beta, \mathrm{lsin}\,\gamma)}{\gcd(\mathrm{lsin}\,\alpha, \mathrm{lsin}\,\gamma)}, \quad \textit{and} \quad \lambda_2 = \frac{\mathrm{lcm}(\mathrm{lsin}\,\alpha, \mathrm{lsin}\,\beta, \mathrm{lsin}\,\gamma)}{\gcd(\mathrm{lsin}\,\alpha, \mathrm{lsin}\,\beta)}.$$

Let us say a few words about the essence of the theorem. In Euclidean geometry on the plane the condition on the angles of triangles can be rewritten with tangent functions in the following way. A triangle with angles exists α, β, and γ iff $\tan(\alpha+\beta+\gamma) = 0$ and $\tan(\alpha+\beta) \notin [0; \tan\alpha]$ (here without lose of generality we suppose that α is acute). Theorem 2.2 is a translation of this condition into lattice case.

In addition we say that there is no a good description of lattice polygons terms of lattice invariants at present. Theorem 2.2 gives such description for the case of triangles.

At this moment we do not have the necessary notation to prove the first statement of Theorem 2.2. For a proof we need first to define extended angles and their sums, and study their properties. We give a proof further in Subsections 4.2 and 4.3. We prove the second statement of the theorem below in this subsection.

REMARK 2.3. Note that the statement of Theorem 2.2a holds only for odd continued fractions for the tangents of the correspondent angles. We illustrate this with the following example. Consider a lattice triangle with the lattice area equals 7 and all angles $\mathscr{L}$-congruent to larctan $7/3$. If we take the odd continued fractions $7/3 = [2, 2, 1]$ for all lattice angles of the triangle, then we have

$$]2, 2, 1, -1, 2, 2, 1, -1, 2, 2, 1[= 0.$$

If we take the even continued fractions $7/3 = [2, 3]$ for all angles of the triangle, then we have

$$]2, 3, -1, 2, 3, -1, 2, 3[= \frac{35}{13} \neq 0.$$

PROOF OF THE SECOND STATEMENT OF THEOREM 2.2. Consider a triangle $\triangle ABC$ with ordinary angles α, β, and γ (at vertices at A, B, and C respectively). Suppose that for any $k > 1$ and any lattice triangle $\triangle KLM$ the triangle $\triangle ABC$ is not $\mathscr{L}$-congruent to the k-multiple of $\triangle KLM$. In other world, we have

$$\gcd\big(\mathrm{l}\ell(AB), \mathrm{l}\ell(BC), \mathrm{l}\ell(CA)\big) = 1.$$

Suppose that S is the lattice area of $\triangle ABC$. Then by the sine formula the following holds

$$\begin{cases} \mathrm{l}\ell(AB)\,\mathrm{l}\ell(AC) = S/\operatorname{lsin}\alpha \\ \mathrm{l}\ell(BC)\,\mathrm{l}\ell(BA) = S/\operatorname{lsin}\beta \\ \mathrm{l}\ell(CA)\,\mathrm{l}\ell(CB) = S/\operatorname{lsin}\gamma \end{cases}.$$

Since $\gcd(\mathrm{l}\ell(AB), \mathrm{l}\ell(BC), \mathrm{l}\ell(CA)) = 1$, we have $\mathrm{l}\ell(AB) = \lambda_1$ and $\mathrm{l}\ell(AC) = \lambda_2$.

Therefore, the lattice triangle $\triangle ABC$ is $\mathscr{L}$-congruent to the lattice triangle $\triangle A_0B_0C_0$ of the theorem.

3. Extension of ordinary lattice angles. Notion of sums of lattice angles

Throughout this section we work in with an oriented two-dimensional real vector space and a fixed lattice in it. We again fix some (positively oriented) lattice basis and use the system of coordinates OXY corresponding to this basis.

The $\mathscr{L}$-affine transformation is said to be *proper* if it is orientation-preserving (we denote it by $\mathscr{L}_+$-affine transformation).

We say that two sets are *$\mathscr{L}_+$-congruent* to each other if there exist a $\mathscr{L}_+$-affine transformation of R^2 taking the first set to the second.

3.1. On a particular generalization of sails in the sense of Klein

In this subsection we introduce the definition of an oriented broken lines at unit lattice distance from a lattice point. This notion is a direct generalization of the notion of a sail in the sense of Klein (see page 167 for the definition of a sail). We extend the definition of LLS-sequences and continued fractions to the case of these broken lines. We show that extended LLS-sequence for oriented broken lines uniquely identifies the $\mathscr{L}_+$-congruence class of the corresponding broken line. Further, we study the geometrical interpretation of the corresponding continued fraction.

3.1.1. Definition of a lattice signed length-sine sequence. Let us extend the definition of LLS-sequence to the case of certain broken lines.

For the 3-tuples of lattice points A, B, and C we define the function sgn as follows:

$$\operatorname{sgn}(ABC) = \begin{cases} +1, & \text{if the pair of vectors } \overline{BA} \text{ and } \overline{BC} \text{ defines the positive orientation.} \\ 0, & \text{if the points } A,\ B, \text{ and } C \text{ are contained in the same straight line.} \\ -1, & \text{if the pair of vectors } \overline{BA} \text{ and } \overline{BC} \text{ defines the negative orientation.} \end{cases}$$

We also denote by sign : $\mathsf{R} \to \{-1, 0, 1\}$ the sign function over reals.

A segment AB is said to be *at unit distance* from the point C if the lattice vectors of the segment AB, and the vector $\overline{AC}$ generate the lattice.

A union of (ordered) lattice segments $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$ $(n > 0)$ is said to be a *lattice oriented broken line* and denoted by $A_0A_1A_2 \ldots A_n$ if any two consecutive segments are not contained in the same straight line. We also say that the lattice oriented broken line $A_nA_{n-1}A_{n-2} \ldots A_0$ is *inverse* to the lattice oriented broken line $A_0A_1A_2 \ldots A_n$.

DEFINITION 3.1. Consider a lattice oriented broken line and a lattice point V in the complement to this line. The broken line is said to be *at unit distance from the point V (or V-broken line for short)* if all edges of the broken line are at unit distance from V.

Let us now associate to any lattice oriented V-broken line for some lattice point V the following sequence of non-zero elements.

DEFINITION 3.2. Let $A_0A_1 \ldots A_n$ be a lattice oriented V-broken line. The sequence of integers $(a_0, \ldots, a_{2n-2})$ defined as follows:

$$\begin{aligned}
a_0 &= \operatorname{sgn}(A_0VA_1)\, \mathrm{l}\ell(A_0A_1), \\
a_1 &= \operatorname{sgn}(A_0VA_1)\operatorname{sgn}(A_1VA_2)\operatorname{sgn}(A_0A_1A_2)\, \mathrm{lsin}\, \angle A_0A_1A_2, \\
a_2 &= \operatorname{sgn}(A_1VA_2)\, \mathrm{l}\ell(A_1A_2), \\
&\ldots \\
a_{2n-3} &= \operatorname{sgn}(A_{n-2}VA_{n-1})\operatorname{sgn}(A_{n-1}VA_n) \\
&\qquad \operatorname{sgn}(A_{n-2}A_{n-1}A_n)\, \mathrm{lsin}\, \angle A_{n-2}A_{n-1}A_n, \\
a_{2n-2} &= \operatorname{sgn}(A_{n-1}VA_n)\, \mathrm{l}\ell(A_{n-1}A_n),
\end{aligned}$$

is called an *lattice signed length-sine* sequence for the lattice oriented V-broken line. Further we will say *LSLS-sequence* for short.

The element $]a_0, a_1, \ldots, a_{2n-2}[$ of $\overline{\mathsf{Q}}$ is called the *continued fraction for the broken line $A_0A_1 \ldots A_n$*.

If we take LSLS-sequence for some broken line which is a sail, than LSLS-sequence is exactly LLS-sequence for the corresponding angle. So LSLS-sequence is a natural combinatorical-geometrical generalization of LLS-sequences. Note also that if we know the whole LSLS-sequence for some V-broken line and the coordinates of points V, A_0, and A_1 then the coordinates of $A_2, \ldots$ can be restored in the unique way.

Let us show how to identify geometrically the signs of elements of the LSLS-sequence for a lattice oriented V-broken line on Figure 4.

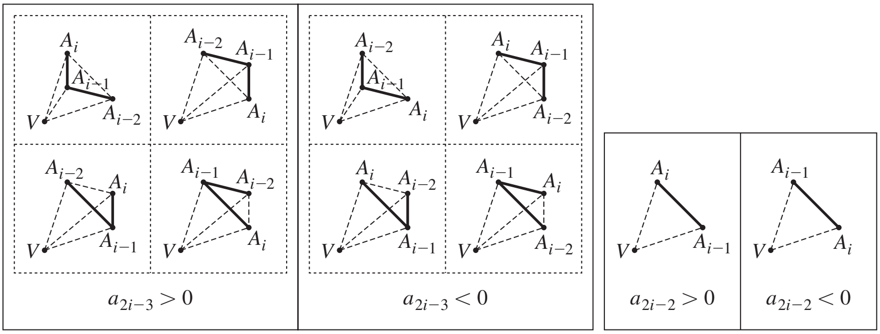


FIGURE 4. All possible (non-degenerate) $\mathscr{L}_+$-affine decompositions for angles and segments of a LSLS-sequence.

On Figure 5 we show an example of lattice oriented V-broken line and the corresponding LSLS-sequence.

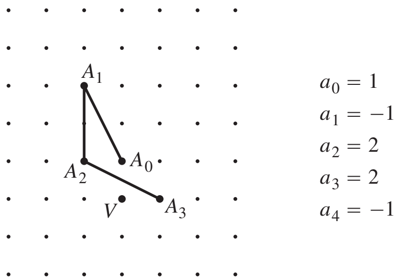


FIGURE 5. A lattice oriented V-broken line and the corresponding LSLS-sequence.

PROPOSITION 3.3. *A LSLS-sequence for the given lattice oriented broken line and the lattice point is invariant under the group action of the $\mathscr{L}_+$-affine transformations.*

3.1.2. On $\mathscr{L}_+$-congruence of lattice oriented V-broken lines. Let us formulate necessary and sufficient conditions for two lattice oriented V-broken lines (for the same lattice point V) to be $\mathscr{L}_+$-congruent.

THEOREM 3.4. *The LSLS-sequences of two lattice oriented V_1-broken and V_2-broken lines (for two lattice points V_1 and V_2) coincide iff there exist a $\mathscr{L}_+$-affine transformation taking the point V_1 to V_2 and one lattice oriented broken line to the other.*

PROOF. The LSLS-sequence for any lattice oriented V-broken line is uniquely defined, and by Proposition 3.3 is invariant under the group action of $\mathscr{L}$-affine orientation preserving transformations. Therefore, the LSLS-sequences for two $\mathscr{L}_+$-congruent lattice oriented broken lines coincide.

Suppose now that two lattice oriented V_1-broken and V_2-broken lines $A_0 \dots A_n$, and $B_0 \dots B_n$ respectively have the same LSLS-sequence $(a_0, a_1, \dots, a_{2n-3}, a_{2n-2})$. Let us prove that these broken lines are $\mathscr{L}_+$-congruent. Without loose of generality we consider the point V_1 at the origin O.

Let ξ be the $\mathscr{L}_+$-affine transformation taking the point V_2 to the point $V_1 = O$, B_0 to A_0, and the lattice straight line containing B_0B_1 to the lattice straight line containing A_0A_1. Let us prove inductively that $\xi(B_i) = A_i$.

Base of induction. Since $a_0 = b_0$, we have

$$\operatorname{sgn}(A_0OA_1)\,\mathrm{l}\ell(A_0A_1) = \operatorname{sgn}(\xi(B_0)O\xi(B_1))\,\mathrm{l}\ell(\xi(B_0)\xi(B_1)).$$

Thus, the lattice segments A_0A_1 and $A_0\xi(B_1)$ are of the same lattice length and of the same direction. Therefore, $\xi(B_1) = A_1$.

Step of induction. Suppose, that $\xi(B_i) = A_i$ holds for any nonnegative integer $i \le k$, where $k \ge 1$. Let us prove, that $\xi(B_{k+1}) = A_{k+1}$. Denote by C_{k+1} the lattice point $\xi(B_{k+1})$. Let $A_k = (q_k, p_k)$. Denote by A'_k the closest lattice point of the segment $A_{k-1}A_k$ to the vertex A_k. Suppose that $A'_k = (q'_k, p'_k)$. We know also

$$a_{2k-1} = \operatorname{sgn}(A_{k-1}OA_k)\operatorname{sgn}(A_kOC_{k+1})\operatorname{sgn}(A_{k-1}A_kC_{k+1})\,\mathrm{lsin}\,\angle A_{k-1}A_kC_{k+1},$$

$$a_{2k} = \operatorname{sgn}(A_kOC_{k+1})\,\mathrm{l}\ell(A_kC_{k+1}).$$

Let the coordinates of C_{k+1} be (x, y). Since $\mathrm{l}\ell(A_kC_{k+1}) = |a_{2k}|$ and the segment A_kC_{k+1} is at unit distance to the origin O, we have $\mathrm{lS}(\triangle OA_kC_{k+1}) = |a_{2k}|$. Since the segment OA_k is of the unit lattice length, the coordinates of C_{k+1} satisfy the following equation:

$$|-p_kx + q_ky| = |a_{2k}|.$$

Since $\operatorname{sgn}(A_kOC_{k+1})\,\mathrm{l}\ell(A_kC_{k+1}) = \operatorname{sign}(a_{2k})$, we have $-p_kx + q_ky = a_{2k}$.

Since $\mathrm{lsin}\,\angle A'_kA_kC_{k+1} = \mathrm{lsin}\,\angle A_{k-1}A_kC_{k+1} = |a_{2k-1}|$, and the lattice lengths of A_kC_{k+1}, and A'_kA_k are $|a_{2k}|$ and 1 respectively, we have $\mathrm{lS}(\triangle A'_kA_kC_{k+1}) = |a_{2k-1}a_{2k}|$. Therefore, the coordinates of C_{k+1} satisfy the following equation:

$$|-(p_k - p'_k)(x - q_k) + (q_k - q'_k)(y - p_k)| = |a_{2k-1}a_{2k}|.$$

Since

$$\begin{cases} \operatorname{sgn}(A_{k-1}OA_k)\operatorname{sgn}(A_kOC_{k+1})\operatorname{sgn}(A_{k-1}A_kC_{k+1}) = \operatorname{sign}(a_{2k-1}) \\ \operatorname{sgn}(A_kOC_{k+1}) = \operatorname{sign}(a_{2k}) \end{cases},$$

we have $(p_k - p_k')(x-q_k) - (q_k - q_k')(y-p_k) = \operatorname{sgn}(A_{k-1}OA_k)a_{2k-1}a_{2k}$.

We obtain the following:

$$\begin{cases} -p_kx + q_ky = a_{2k} \\ (p_k - p_k')(x-q_k) - (q_k - q_k')(y-p_k) = \operatorname{sgn}(A_{k-1}OA_k)a_{2k-1}a_{2k} \end{cases}.$$

Since

$$\left|\det\begin{pmatrix} -p_k & q_k \\ p_k' - p_k & q_k - q_k' \end{pmatrix}\right| = 1,$$

there exist a unique integer solution for the system of equations for x and y. Hence, the points A_{k+1} and C_{k+1} have the same coordinates. Therefore, $\xi(B_{k+1}) = A_{k+1}$. We have proven the step of induction.

The proof of Theorem 3.4 is completed by induction.

3.1.3. Values of continued fractions for lattice oriented broken lines at unit distance from the origin. Now we show the relation between lattice oriented broken lines at unit distance from the origin O and the corresponding continued fractions for them.

THEOREM 3.5. *Let $A_0A_1\ldots A_n$ be a lattice oriented O-broken line. Let also $A_0 = (1,0)$, $A_1 = (1,a_0)$, $A_n = (p,q)$, where $\gcd(p,q) = 1$, and $(a_0, a_1, \ldots, a_{2n-2})$ be the corresponding LSLS-sequence. Then the following holds:*

$$\frac{q}{p} = \,]a_0, a_1, \ldots, a_{2n-2}[.$$

PROOF. To prove this theorem we use an induction on the number of edges of the broken lines.

Base of induction. Suppose that a lattice oriented O-broken line has a unique edge, and the corresponding sequence is (a_0). Then $A_1 = (1, a_0)$ by the assumptions of the theorem. Therefore, we have $\frac{a_0}{1} = \,]a_0[$.

Step of induction. Suppose that the statement of the theorem is correct for any lattice oriented O-broken line with k edges. Let us prove the theorem for the arbitrary lattice oriented O-broken line with $k+1$ edges (and satisfying the conditions of the theorem).

Let $A_0 \ldots A_{k+1}$ be a lattice oriented O-broken line with the following LSLS-sequence $(a_0, a_1, \ldots, a_{2k-1}, a_{2k})$. Let also

$$A_0 = (1,0), \qquad A_1 = (1, a_0), \quad \text{and} \quad A_{k+1} = (p,q).$$

Consider the lattice oriented O-broken line $B_1 \dots B_{k+1}$ with shorter LSLS-sequence for it: $(a_2, a_3, \dots, a_{2k-2}, a_{2k})$. Let also

$$B_1 = (1, 0), \qquad B_2 = (1, a_2), \quad \text{and} \quad B_{k+1} = (p', q').$$

By the induction assumption we have

$$\frac{q'}{p'} =]a_2, a_3, \dots, a_{2k}[.$$

We extend the lattice oriented broken line $B_1 \dots B_{k+1}$ to the lattice oriented O-broken line $B_0 B_1 \dots B_{k+1}$, where $B_0 = (1{+}a_0a_1, -a_0)$. Let the lattice LSLS-sequence for this broken line be $(b_0, b_1, \dots, b_{2k-1}, b_{2k})$. Note that

$$\begin{aligned} b_0 &= \operatorname{sgn}(B_0OB_1)\,\mathrm{l}\ell(B_0B_1) = \operatorname{sign}(a_0)|a_0| = a_0,\\ b_1 &= \operatorname{sgn}(B_0OB_1)\operatorname{sgn}(B_1OB_2)\operatorname{sgn}(B_0B_1B_2)\operatorname{lsin}\angle B_0B_1B_2\\ &= \operatorname{sign} a_0 \operatorname{sign} b_2 \operatorname{sign}(a_0a_1b_2)|a_1| = a_1,\\ b_l &= a_l, \qquad \text{for} \quad l = 2, \dots, 2k. \end{aligned}$$

Consider a $\mathscr{L}_+$-linear transformation ξ that takes the point B_0 to the point $(1, 0)$, and B_1 to $(1, a_0)$. These two conditions uniquely define ξ:

$$\xi = \begin{pmatrix} 1 & a_1 \\ a_0 & 1 + a_0a_1 \end{pmatrix}.$$

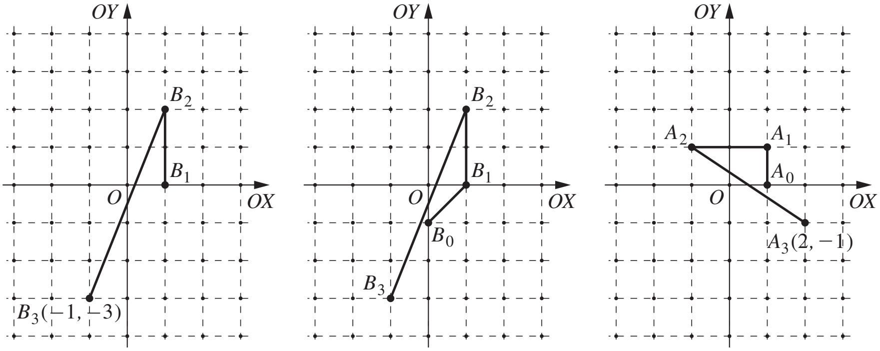
Since $B_{k+1} = (p', q')$, we have $\xi(B_{k+1}) = (p'{+}a_1q', q'a_0{+}p'{+}p'a_0a_1)$.

$$\frac{q'a_1 + p' + p'a_0a_1}{p' + a_1q'} = a_0 + \frac{1}{a_1 + q'/p'} =]a_0, a_1, a_2, a_3, \dots, a_{2n}[.$$

Since, by Theorem 3.4 the lattice oriented broken lines $B_0B_1 \dots B_{k+1}$ and $A_0A_1 \dots A_{k+1}$ are $\mathscr{L}$-linear equivalent, $B_0 = A_0$, and $B_1 = A_1$, these broken lines coincide. Therefore, for the coordinates (p, q) the following hold

$$\frac{q}{p} = \frac{q'a_0 + p' + p'a_0a_1}{p' + a_1q'} =]a_0, a_1, a_2, a_3, \dots, a_{2k}[.$$

On Figure 6 we illustrate the step of induction with an example of lattice oriented O-broken line with the LSLS-sequence: $(1, -1, 2, 2, -1)$. We start (the left picture) with the broken line $B_1B_2B_3$ with the LSLS-sequence: $(2, 2, -1)$. Note that the ratio of the coordinates of the point B_3 is $-3/-1 =]2 : 2; -1[$. Then, (the picture in the middle) we extend the broken line $B_1B_2B_3$ to the broken line $B_0B_1B_2B_3$ with the LSLS-sequence: $(1, -1, 2, 2, -1)$. Finally (the right picture) we apply a corresponding $\mathcal{L}_+$-linear transformation ξ to achieve the resulting broken line $A_0A_1A_2A_3$. Now the ratio of the coordinates of the point A_3 is $-1/2 =]1 : -1; 2; 2; -1[$.



$-3/-1 =]2 : 2; -1[$ $\quad B_0 = (1 + a_0a_1, -a_0)$ $\quad -1/2 =]1 : -1; 2; 2; -1[$

FIGURE 6. The case of lattice oriented O-broken line with LSLS-sequence: $(1, -1, 2, 2, -1)$.

We have proven the step of induction.

The proof of Theorem 3.5 is completed.

REMARK 3.6. Theorem 3.5 immediately implies the statement of Theorem 1.10. One should put the sail of an angle as an oriented-broken line $A_0A_1 \ldots A_n$.

3.2. Extended lattice angles. Sums for ordinary and extended lattice angles

3.2.1. Equivalence classes of lattice oriented broken lines and the corresponding extended angles.

DEFINITION 3.7. Consider a lattice point V. Two lattice oriented V-broken lines l_1 and l_2 are said to be *equivalent* if they have in common the first and the last vertices and the closed broken line generated by l_1 and the inverse of l_2 is homotopy equivalent to the point in $\mathsf{R}^2 \setminus \{V\}$.

An equivalence class of lattice oriented V-broken lines containing the broken line $A_0A_1 \ldots A_n$ is called the *extended lattice angle for the equivalence class of* $A_0A_1 \ldots A_n$ at the *vertex* V (or, for short, *extended angle*) and denoted by $\angle(V, A_0A_1 \ldots A_n)$.

We study the extended angles up to $\mathcal{L}_+$-congruence.

DEFINITION 3.8. Two extended angles Φ_1 and Φ_2 are said to be $\mathcal{L}_+$-*congruent* iff there exist a $\mathcal{L}_+$-affine transformation sending the class of lattice oriented broken lines corresponding to Φ_1 to the class of lattice oriented broken lines corresponding to Φ_2. We denote this by $\Phi_1 \hat{\cong} \Phi_2$.

3.2.2. Revolution numbers for extended angles. Let $r = \{V+\lambda\overline{v} \mid \lambda \geq 0\}$ be the oriented ray for an arbitrary vector $\overline{v}$ with the vertex at V, and AB be an oriented (from A to B) segment not contained in the ray r. Suppose also, that the vertex V of the ray r is not contained in the segment AB. We denote by $\#(r, V, AB)$ the following number:

$$\#(r, V, AB) = \begin{cases} 0, & AB \cap r = \emptyset \\ \frac{1}{2}\operatorname{sgn}\big(A(A-\overline{v})B\big), & AB \cap r \in \{A, B\} \\ \operatorname{sgn}\big(A(A-\overline{v})B\big), & AB \cap r \in AB \setminus \{A, B\}, \end{cases}$$

and call it the *intersection number* of the ray r and the segment AB.

DEFINITION 3.9. Let $A_0A_1 \ldots A_n$ be some lattice oriented broken line, and let r be an oriented ray $\{V+\lambda\overline{v} \mid \lambda \geq 0\}$. Suppose that the ray r does not contain the edges of the broken line, and the broken line does not contain the point V. We call the number

$$\sum_{i=1}^{n} \#(r, V, A_{i-1}A_i)$$

the *intersection number* of the ray r and the lattice oriented broken line $A_0A_1 \ldots A_n$, and denote it by $\#(r, V, A_0A_1 \ldots A_n)$.

DEFINITION 3.10. Consider an arbitrary extended angle $\angle(V, A_0A_1 \ldots A_n)$. Denote the rays $\{V + \lambda\overline{VA_0} \mid \lambda \geq 0\}$ and $\{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}$ by r_+ and r_- respectively. The number

$$\frac{1}{2}\big(\#(r_+, V, A_0A_1 \ldots A_n) + \#(r_-, V, A_0A_1 \ldots A_n)\big)$$

is called the *lattice revolution number* for the extended angle $\angle(V, A_0A_1 \ldots A_n)$, and denoted by $\#(\angle(V, A_0A_1 \ldots A_n))$. We say also that $\#(\angle(V, A_0)) = 0$.

Let us give some examples. Let $O = (0, 0)$, $A = (1, 0)$, $B = (0, 1)$, $C = (-1, -1)$, then

$$\#(\angle(O, A)) = 0, \qquad \#(\angle(O, AB)) = \frac{1}{4},$$
$$\#(\angle(O, ABCA)) = 1, \qquad \#(\angle(O, ACB)) = -\frac{3}{4}.$$

Now we show that the definition of revolution number is correct.

PROPOSITION 3.11. *The revolution number of any extended angle is well-defined.*

PROOF. Consider an arbitrary extended angle $\angle(V, A_0A_1 \dots A_n)$. Let

$$r_+ = \{V + \lambda\overline{VA_0} \mid \lambda \geq 0\} \qquad \text{and} \qquad r_- = \{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}.$$

Since the lattice oriented broken line $A_0A_1 \dots A_n$ is at unit distance from the point V, any segment of this broken line is at unit distance from V. Thus, the broken line does not contain V, and the rays r_+ and r_- do not contain edges of the curve.

Suppose that

$$\angle(V, A_0A_1 \dots A_n) = \angle(V', A'_0A'_1 \dots A'_m).$$

This implies that $V = V'$, $A_0 = A'_0$, $A_n = A'_m$, and the broken line $A_0A_1 \dots A_nA'_{m-1} \dots A'_1A'_0$ is homotopy equivalent to the point in $\mathsf{R}^2 \setminus \{V\}$. Thus,

$$\begin{aligned} &\#(\angle(V, A_0A_1 \dots A_n)) - \#(\angle(V,' A'_0A'_1 \dots A'_m)) \\ &\quad = \frac{1}{2}\big(\#(r_+, V, A_0A_1 \dots A_nA'_{m-1} \dots A'_1A'_0) \\ &\qquad + \#(r_-, V, A_0A_1 \dots A_nA'_{m-1} \dots A'_1A'_0)\big) \\ &\quad = 0 + 0 = 0. \end{aligned}$$

Hence,

$$\#(\angle(V, A_0A_1 \dots A_n)) = \#(\angle(V', A'_0A_1 \dots A'_m)).$$

Therefore, the revolution number of any extended angle is well-defined.

PROPOSITION 3.12. *The revolution number of extended angles is invariant under the group action of the $\mathscr{L}_+$-affine transformations.*

3.2.3. Zero ordinary angles. For the next theorem we will need to define zero ordinary angles and their trigonometric functions. Let A, B, and C be three lattice points of the same lattice straight line. Suppose that B is distinct to A and C and the rays BA and BC coincide. We say that the ordinary angle with the vertex at B and the rays BA and BC is *zero*. Suppose $\angle ABC$ is zero, put by definition

$$\operatorname{lsin}(\angle ABC) = 0, \qquad \operatorname{lcos}(\angle ABC) = 1, \qquad \operatorname{ltan}(\angle ABC) = 0.$$

Denote by $\operatorname{larctan}(0)$ the angle $\angle AOA$ where $A = (1, 0)$, and O is the origin.

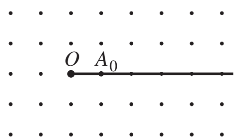
3.2.4. On normal forms of extended angles. Let us formulate and prove a theorem on normal forms of extended angles. We use the following notation: by the sequence

$$\big((a_0, \ldots, a_n) \times k\text{-times}, b_0, \ldots, b_m\big),$$

where $k \geq 0$, we denote the following sequence:

$$(\underbrace{a_0, \ldots, a_n, a_0, \ldots, a_n, \quad \ldots \quad, a_0, \ldots, a_n}_{k\text{-times}}, b_0, \ldots, b_m).$$

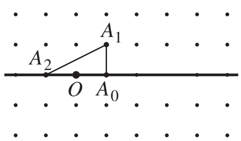
DEFINITION 3.13. I) Suppose O be the origin, A_0 be the point $(1, 0)$. We say that the extended angle $\angle(O, A_0)$ is *of the type* I and denote it by $0\pi + \operatorname{larctan}(0)$ (or 0, for short). The empty sequence is said to be *characteristic* for the angle $0\pi + \operatorname{larctan}(0)$.

Example I: 

type I,
$\angle(O, A_0) = 0\pi + \operatorname{larctan}(0)$,
LSLS-sequence is ().

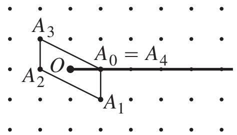
Consider a lattice oriented O-broken line $A_0A_1 \ldots A_s$, where O is the origin. Let also A_0 be the point $(1, 0)$, and the point A_1 be on the straight line $x = 1$. If the LSLS-sequence of the extended angle $\Phi_0 = \angle(O, A_0A_1 \ldots A_s)$ coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):

II$_k$) $\big((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1\big)$, where $k \geq 1$, then we denote the angle Φ_0 by $k\pi + \operatorname{larctan}(0)$ (or $k\pi$, for short) and say that Φ_0 is *of the type* II$_k$;

Exampe II: 

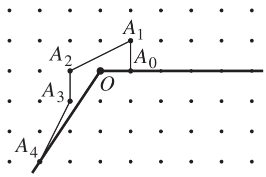
type II$_1$,
$\angle(O, A_0A_1A_2) = \pi + \operatorname{larctan}(0)$,
LSLS-sequence is $(1, -2, 1)$.

III$_k$) $\big((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1\big)$, where $k \geq 1$, then we denote the angle Φ_0 by $-k\pi + \operatorname{larctan}(0)$ (or $-k\pi$, for short) and say that Φ_0 is *of the type* III$_k$;

Example III: 

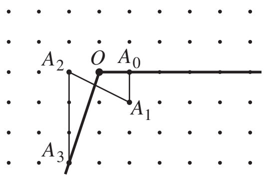
type III_2,
$\angle(O, A_0A_1A_2A_3A_4) = -2\pi + \operatorname{larctan}(0)$,
LSLS-sequence is $(-1, 2, -1, 2, -1, 2, -1)$.

IV_k) $\big((1, -2, 1, -2) \times k\text{-times}, a_0, \ldots, a_{2n}\big)$, where $k \geq 0$, $n \geq 0$, $a_i > 0$, for $i = 0, \ldots, 2n$, then we denote the angle Φ_0 by $k\pi + \operatorname{larctan}([a_0, a_1, \ldots, a_{2n}])$ and say that Φ_0 is *of the type* IV_k;

Example IV: 

type IV_1,
$\angle(O, A_0A_1A_2A_3A_4) = \pi + \operatorname{larctan}\big(\frac{3}{2}\big)$,
LSLS-sequence is $(1, -2, 1, -2, 1, 1, 1)$.

V_k) $\big((-1, 2, -1, 2) \times k\text{-times}, a_0, \ldots, a_{2n}\big)$, where $k > 0$, $n \geq 0$, $a_i > 0$, for $i = 0, \ldots, 2n$, then we denote the angle Φ_0 by $-k\pi + \operatorname{larctan}([a_0, a_1, \ldots, a_{2n}])$ and say that Φ_0 is *of the type* V_k.

Example V: 

type V_1,
$\angle(O, A_0A_1A_2A_3) = -\pi + \operatorname{larctan}(3)$,
LSLS-sequence is $(-1, 2, -1, 2, 3)$.

THEOREM 3.14. *For any extended angle Φ there exist a unique type among the types* I–V *and a unique extended angle Φ_0 of that type such that Φ_0 is $\mathcal{L}_+$-congruent to Φ.*

The extended angle Φ_0 is said to be *the normal form* for the extended angle Φ.

For the proof of Theorem 3.14 we need the following lemma.

LEMMA 3.15. *Let m, $k \geq 1$, and $a_i > 0$ for $i = 0, \ldots, 2n$ be some integers.*

a) *Suppose the LSLS-sequences for the extended angles Φ_1 and Φ_2 are respectively*

$$\big((1, -2, 1, -2) \times (k-1)\textit{-times}, 1, -2, 1, -2, a_0, \ldots, a_{2n}\big)$$

and

$$\big((1, -2, 1, -2) \times (k-1)\textit{-times}, 1, -2, 1, m, a_0, \ldots, a_{2n}\big),$$

then Φ_1 is $\mathcal{L}_+$-congruent to Φ_2.

b) *Suppose the LSLS-sequences for the extended angles* Φ_1 *and* Φ_2 *are respectively*

$$\big((-1,2,-1,2)\times(k{-}1)\textit{-times}, -1,2,-1,m,a_0,\ldots,a_{2n}\big)$$

and

$$\big((-1,2,-1,2)\times(k{-}1)\textit{-times}, -1,2,-1,2,a_0,\ldots,a_{2n}\big),$$

then Φ_1 *is* $\mathscr{L}_+$*-congruent to* Φ_2.

PROOF. We prove the first statement of the lemma. Suppose that m is integer, k is positive integer, and a_i for $i=0,\ldots,2n$ are positive integers.

Let us construct the angle Ψ_1 with vertex at the origin for the lattice oriented broken line $A_0\ldots A_{2k+n+1}$, corresponding to the LSLS-sequence

$$\big((1,-2,1,-2)\times(k{-}1)\text{-times}, 1,-2,1,-2,a_0,\ldots,a_{2n}\big),$$

such that $A_0=(1,0)$, $A_1=(1,1)$. Note that

$$\begin{cases} A_{2l}=((-1)^l,0), & \text{for} \quad l<k-1 \\ A_{2l+1}=((-1)^l,(-1)^l), & \text{for} \quad l<k-1 \\ A_{2k}=((-1)^k,0) & \\ A_{2k+1}=((-1)^k,(-1)^k a_0) & \end{cases}.$$

Let us construct the angle Ψ_2 with vertex at the origin for the lattice oriented broken line $B_0\ldots B_{2k+n+1}$, corresponding to the LSLS-sequence

$$\big((1,-2,1,-2)\times(k{-}1)\text{-times}, 1,-2,1,m,a_0,\ldots,a_{2n}\big).$$

such that $B_0=(1,0)$, $B_1=(-m-1,1)$. Note also that

$$\begin{cases} B_{2l}=((-1)^l,0), & \text{for} \quad l<k-1 \\ B_{2l+1}=((-1)^l(-m-1),(-1)^l), & \text{for} \quad l<k-1 \\ B_{2k}=((-1)^k,0) & \\ B_{2k+1}=((-1)^k,(-1)^k a_0) & \end{cases}.$$

From the above we know, that the points A_{2k} and A_{2k+1} coincide with the points B_{2k} and B_{2k+1} respectively. Since the remaining parts of both LSLS-sequences (i. e. $(a_0,\ldots,a_{2n})$) coincide, the point A_l coincide with the point B_l for $l>2k$.

Since the lattice oriented broken lines $A_0 \ldots A_{2k}$ and $B_0 \ldots B_{2k}$ are of the same equivalence class, and the point A_l coincide with the point B_l for $l > 2k$, we obtain

$$\Psi_1 = \angle(O, A_0 \ldots A_{2k+n+1}) = \angle(O, B_0 \ldots B_{2k+n+1}) = \Psi_2.$$

Therefore, by Theorem 3.4 we have the following:

$$\Phi_1 \hat{\cong} \Psi_1 = \Psi_2 \hat{\cong} \Phi_2.$$

This concludes the proof of Lemma 3.15a.

Since the proof of Lemma 3.15b almost completely repeats the proof of Lemma 3.15a, we omit the proof of Lemma 3.15b here.

PROOF OF THEOREM 3.14. First, we prove that any two distinct extended angles listed in Definition 3.13 are not $\mathscr{L}_+$-congruent. Let us note that the revolution numbers of extended angles distinguish the types of the angles. The revolution number for the extended angle of the type I is 0. The revolution number for the extended angle of the type II_k is $1/2(k+1)$ where $k \geq 0$. The revolution number for the extended angle of the type III_k is $-1/2(k+1)$ where $k \geq 0$. The revolution number for the extended angles of the type IV_k is $1/4 + 1/2k$ where $k \geq 0$. The revolution number for the extended angles of the type V_k is $1/4 - 1/2k$ where $k > 0$.

So we have proven that two extended angles of different types are not $\mathscr{L}_+$-congruent. For the types I, II_k, and III_k the proof is completed, since any such type consists of the unique extended angle.

Let us prove that normal forms of the same type IV_k (or of the same type V_k) are not $\mathscr{L}_+$-congruent for any integer $k \geq 0$ (or $k > 0$). Consider an extended angle $\Phi = k\pi + \text{larctan}([a_0, a_1, \ldots, a_{2n}])$. Suppose that a lattice oriented O-broken line $A_0 A_1 \ldots A_m$, where $m = 2|k| + n + 1$ defines the angle Φ. Let also that the LSLS-sequence for this broken line be characteristic.

Suppose, that k is even, then the ordinary angle $\angle A_0 O A_m$ is $\mathscr{L}_+$-congruent to the ordinary angle $\text{larctan}([a_0, a_1, \ldots, a_{2n}])$. This angle is a $\mathscr{L}_+$-affine invariant for the extended angle Φ. This invariant distinguish the extended angles of type IV_k (or V_k) with even k.

Suppose, that k is odd, then denote $B = O + \overline{A_0 O}$. The ordinary angle $\angle BVA_m$ is $\mathscr{L}_+$-congruent to the ordinary angle $\text{larctan}([a_0, a_1, \ldots, a_{2n}])$. This angle is a $\mathscr{L}_+$-affine invariant for the extended angle Φ. This invariant distinguish the extended angles of type IV_k (or V_k) with odd k.

Therefore, the extended angles listed in Definition 3.13 are not $\mathscr{L}_+$-congruent.

Now we prove that an arbitrary extended angle is $\mathscr{L}_+$-congruent to one of the extended angles of the types I–V.

Consider an arbitrary extended angle $\angle(V, A_0A_1 \dots A_n)$ and denote it by Φ. If $\#(\Phi) = k/2$ for some integer k, then Φ is $\mathscr{L}_+$-congruent to an angle of one of the types I–III. Let $\#(\Phi) = 1/4$, then the extended angle Φ is $\mathscr{L}_+$-congruent to the extended angle defined by the sail of the ordinary angle $\angle A_0VA_n$ of the type IV_0.

Suppose now, that $\#(\Phi) = 1/4 + k/2$ for some positive integer k, then one of its LSLS-sequence is of the following form:

$$\big((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, m, a_0, \dots, a_{2n}\big),$$

where $a_i > 0$, for $i = 0, \dots, 2n$. By Lemma 3.15 the extended angle defined by this sequence is $\mathscr{L}_+$-congruent to an extended angle of the type IV_k defined by the sequence

$$\big((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, -2, a_0, \dots, a_{2n}\big).$$

Finally, let $\#(\Phi) = 1/4 - k/2$ for some positive integer k, then one of its LSLS-sequence is of the following form:

$$\big((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1, m, a_0, \dots, a_{2n}\big),$$

where $a_i > 0$, for $i = 0, \dots, 2n$. By Lemma 3.15 the extended angle defined by this sequence is $\mathscr{L}_+$-congruent to an extended angle of the type V_k defined by the sequence

$$\big((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1, 2, a_0, \dots, a_{2n}\big).$$

This completes the proof of Theorem 3.14.

Let us finally give the definition of trigonometric functions for the extended angles and describe some relations between ordinary and extended angles.

Definition 3.16. Consider an arbitrary extended angle Φ with the normal form $k\pi + \varphi$ for some ordinary (possible zero) angle φ and for an integer k.

a) The ordinary angle φ is said to be *associated* with the extended angle Φ.

b) The numbers $\mathrm{ltan}(\varphi)$, $\mathrm{lsin}(\varphi)$, and $\mathrm{lcos}(\varphi)$ are called the *lattice tangent*, the *lattice sine*, and the *lattice cosine* of the extended angle Φ.

Since all sails for ordinary angles are lattice oriented broken lines, the set of all ordinary angles is naturally embedded into the set of extended angles.

Definition 3.17. For an ordinary angle φ the angle

$$0\pi + \mathrm{larctan}(\mathrm{ltan}\,\varphi)$$

is said to be *corresponding* to the angle φ and denoted by $\overline{\varphi}$.

From Theorem 3.14 it follows that for every ordinary angle φ there exists and unique an extended angle $\overline{\varphi}$ corresponding to φ. Therefore, two ordinary angles φ_1 and φ_2 are $\mathscr{L}$-congruent iff the corresponding lattice angles $\overline{\varphi}_1$ and $\overline{\varphi}_2$ are $\mathscr{L}_+$-congruent.

3.2.5. Opposite extended angles. Sums of extended angles. Sums of ordinary angles. Consider an extended angle Φ with the vertex V for some equivalence class of a given lattice oriented broken line. The extended angle Ψ with the vertex V for the equivalence class of the inverse lattice oriented broken line is called *opposite* to the given one and denoted by $-\Phi$.

PROPOSITION 3.18. *For any extended angle $\Phi \hat{\cong} k\pi+\varphi$ we have:*

$$-\Phi \hat{\cong} (-k-1)\pi + (\pi - \varphi).$$

Let us introduce the definition of sums of ordinary and extended angles.

DEFINITION 3.19. Consider arbitrary extended angles Φ_i, $i = 1, \ldots, l$. Let the characteristic sequences for the normal forms of Φ_i be $(a_{0,i}, a_{1,i}, \ldots, a_{2n_i,i})$ for $i = 1, \ldots, l$. Let $M = (m_1, \ldots, m_{l-1})$ be some $(l-1)$-tuple of integers. The normal form of any extended angle, corresponding to the following LSLS-sequence

$$\big(a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, \ldots, a_{2n_2,2}, m_2, \ldots, m_{l-1}, a_{0,l}, \ldots, a_{2n_l,l}\big),$$

is called the *M-sum of extended angles* Φ_i $(i = 1, \ldots, l)$ and denoted by

$$\sum_{M,i=1}^{l} \Phi_i, \quad \text{or equivalently by} \quad \Phi_1 +_{m_1} \Phi_2 +_{m_2} \cdots +_{m_{l-1}} \Phi_l.$$

PROPOSITION 3.20. *The M-sum of extended angles Φ_i $(i = 1, \ldots, l)$ is well-defined.*

Let us say a few words about properties of M-sums.

Notice that M-sum of extended angles is non-associative. For example, let $\Phi_1 \hat{\cong} \operatorname{larctan} 2$, $\Phi_2 \hat{\cong} \operatorname{larctan}(3/2)$, and $\Phi_3 \hat{\cong} \operatorname{larctan} 5$. Then

$$\begin{aligned} \Phi_1 +_{-1} \Phi_2 +_{-1} \Phi_3 &= \pi + \operatorname{larctan}(4), \\ \Phi_1 +_{-1} (\Phi_2 +_{-1} \Phi_3) &= 2\pi, \\ (\Phi_1 +_{-1} \Phi_2) +_{-1} \Phi_3 &= \operatorname{larctan}(1). \end{aligned}$$

The M-sum of extended angles is non-commutative. For example, let $\Phi_1 \hat{\cong} \operatorname{larctan} 1$, and $\Phi_2 \hat{\cong} \operatorname{larctan} 5/2$. Then

$$\Phi_1 +_1 \Phi_2 = \operatorname{larctan}(12/7) \neq \operatorname{larctan}(13/5) = \Phi_2 +_1 \Phi_1.$$

REMARK 3.21. The M-sum of extended angles is naturally extended to the sum of classes of $\mathscr{L}_+$-congruences of extended angles.

We conclude this section with the definition of sums of ordinary angles.

DEFINITION 3.22. Consider ordinary angles α_i, where $i = 1, \dots, l$. Let $\overline{\alpha}_i$ be the corresponding extended angles for α_i, and $M = (m_1, \dots, m_{l-1})$ be some $(l-1)$-tuple of integers. The ordinary angle φ associated with the extended angle

$$\Phi = \overline{\alpha}_1 +_{m_1} \overline{\alpha}_2 +_{m_2} \cdots +_{m_{l-1}} \overline{\alpha}_l.$$

is called the *M-sum of ordinary angles* α_i $(i = 1, \dots, l)$ and denoted by

$$\sum_{M,i=1}^{l} \alpha_i, \quad \text{or equivalently by} \quad \alpha_1 +_{m_1} \alpha_2 +_{m_2} \cdots +_{m_{l-1}} \alpha_l.$$

REMARK 3.23. Note that the sum of ordinary angles is naturally extended to the classes of $\mathscr{L}$-congruences of lattice angles.

4. Relations between extended and ordinary lattice angles. Proof of the first statement of Theorem 2.2

Throughout this section we again fix some lattice basis and use the system of coordinates OXY corresponding to this basis.

4.1. On relations between continued fractions for lattice oriented broken lines and the lattice tangents of the corresponding extended angles

For a real number r we denote by $\lfloor r \rfloor$ the maximal integer not greater than r.

THEOREM 4.1. *Consider an extended angle $\Phi = \angle(V, A_0A_1 \dots A_n)$. Suppose, that the normal form for Φ is $k\pi + \varphi$ for some integer k and an ordinary angle φ. Let $(a_0, a_1, \dots, a_{2n-2})$ be the LSLS-sequence for the lattice oriented broken line $A_0A_1 \dots A_n$. Suppose that*

$$]a_0, a_1, \dots, a_{2n-2}[= q/p.$$

Then the following holds:

$$\varphi \cong \begin{cases} \operatorname{larctan}(1), & \text{if } q/p = \infty \\ \operatorname{larctan}(q/p), & \text{if } q/p \geq 1 \\ \operatorname{larctan}\big(\frac{|q|}{|p|-\lfloor(|p|-1)/|q|\rfloor|q|}\big), & \text{if } 0 < q/p < 1 \\ 0, & \text{if } q/p = 0 \\ \pi - \operatorname{larctan}\big(\frac{|q|}{|p|-\lfloor(|p|-1)/|q|\rfloor|q|}\big), & \text{if } -1 < q/p < 0 \\ \pi - \operatorname{larctan}(-q/p), & \text{if } q/p \leq -1 \end{cases}.$$

PROOF. Consider the following linear coordinates $(*, *)'$ on the plane, associated with the lattice oriented V-broken line $A_0A_1 \dots A_n$. Let the origin O' be at the vertex V, $(1, 0)' = A_0$, and $(1, 1)' = A_0 + \frac{1}{a_0}\operatorname{sgn}(A_0O'A_1)\overline{A_0A_1}$. The other coordinates are uniquely defined by linearity. We denote this system of coordinates by $O'X'Y'$.

The set of integer points for the coordinate system $O'X'Y'$ coincides with the set of lattice points of the plane. The basis of vectors $(1, 0)'$ and $(0, 1)'$ defines a positive orientation.

Suppose that the new coordinates of the point A_n are $(p', q')'$. Then by Theorem 3.5 we have $q'/p' = q/p$. This directly implies the statement of the theorem for the cases $q' > p' > 0$, $q'/p' = 0$, and $q'/p' = \infty$.

Suppose now that $p' > q' > 0$. Consider the ordinary angle $\varphi = \angle A_0PA_n$. Let $B_0 \dots B_m$ be the sail for it. The direct calculations show that the point

$$D = B_0 + \frac{\overline{B_0B_1}}{\operatorname{l\ell}(B_0B_1)}$$

coincides with the point $(1+\lfloor(p' - 1)/q'\rfloor, 1)$ in the system of coordinates $O'X'Y'$.

Consider the $\mathscr{L}_+$-linear (in the coordinates $O'X'Y'$) transformation ξ that takes the point $A_0 = B_0$ to itself, and the point D to $(1, 1)'$. These conditions uniquely identify ξ.

$$\xi = \begin{pmatrix} 1 & -\lfloor(p' - 1)/q'\rfloor \\ 0 & 1 \end{pmatrix}$$

The transformation ξ takes the point $A_n = B_m$ with the coordinates (p', q') to the point with the coordinates $(p'-\lfloor(p'-1)/q'\rfloor q', q')'$. Since $q'/p' = q/p$, we obtain the following

$$\varphi = \operatorname{larctan}\left(\frac{q'}{p' - \lfloor(p' - 1)/q'\rfloor q'}\right) = \operatorname{larctan}\left(\frac{q}{p - \lfloor(p - 1)/q\rfloor q}\right).$$

The proof for the case $q' > 0$ and $p' < 0$ repeats the described cases after taking to the consideration the adjacent angles.

Finally, the case of $q' < 0$ repeats all previous cases by the central symmetry (centered at the point O') reasons.

This completes the proof of Theorem 4.1.

COROLLARY 4.2. *The revolution number and the continued fraction for a lattice oriented broken line at unit distance from the vertex uniquely define the $\mathscr{L}_+$-congruence class of the corresponding extended angle.*

4.2. Proof of Theorem 2.2a: two preliminary lemmas

We say that the lattice point P is *at lattice distance k* from the lattice segment AB if the lattice vectors of the segment AB and the vector $\overline{AP}$ generate a sublattice of the lattice of index k.

DEFINITION 4.3. Consider a lattice triangle $\triangle ABC$. Denote the number of lattice points at unit lattice distance from the segment AB and contained in the (closed) triangle $\triangle ABC$ by $\mathrm{l}\ell_1(AB;C)$ (see on Figure 7).

Note that all lattice points at lattice unit lattice distance from the segment AB in the (closed) lattice triangle $\triangle ABC$ are contained in one straight line parallel to the straight line AB. Besides, the integer $\mathrm{l}\ell_1(AB;C)$ is positive for any triangle $\triangle ABC$.

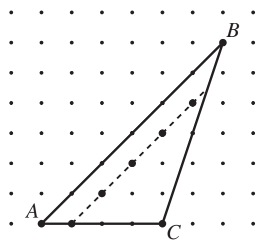


FIGURE 7. For the given triangle $\triangle ABC$ we have $\mathrm{l}\ell_1(AB;C) = 5$.

Now we prove the following lemma.

LEMMA 4.4. *For any lattice triangle $\triangle ABC$ the following holds*

$$\overline{\angle CAB} +_{\mathrm{l}\ell(AB)-\mathrm{l}\ell_1(AB;C)-1} \overline{\angle ABC} +_{\mathrm{l}\ell(BC)-\mathrm{l}\ell_1(BC;A)-1} \overline{\angle BCA} = \pi.$$

PROOF. Consider an arbitrary lattice triangle $\triangle ABC$. Suppose that the pair of vectors $\overline{BA}$ and $\overline{BC}$ defines the positive orientation of the plane (otherwise we apply to the triangle $\triangle ABC$ some $\mathscr{L}$-affine transformation changing the orientation and come to the same position). Denote (see Figure 9 below): $D = A + \overline{BC}$, and $E = A + \overline{AC}$.

Since $CADB$ is a parallelogram, the triangle $\triangle BAD$ is $\mathscr{L}_+$-congruent to the triangle $\triangle ABC$. Thus, the angle $\angle BAD$ is $\mathscr{L}_+$-congruent to the angle $\angle ABC$, and $\mathrm{l}\ell_1(BA; D) = \mathrm{l}\ell_1(AB; C)$. Since $EABD$ is a parallelogram, the triangle $\triangle AED$ is $\mathscr{L}_+$-congruent to the triangle $\triangle BAD$, and hence is $\mathscr{L}_+$-congruent to the triangle $\triangle ABC$. Thus, Thus, $\angle DAE$ is $\mathscr{L}_+$-congruent to $\angle BCA$, and $\mathrm{l}\ell_1(DA; E) = \mathrm{l}\ell_1(BC; A)$.

Let $A_0 \ldots A_n$ be the sail of $\angle CAB$ with the corresponding LLS-sequence $(a_0, \ldots, a_{2n-2})$. Let $B_0B_1 \ldots B_m$ be the sail of $\angle BAD$ (where $B_0 = A_n$) with the corresponding LLS-sequence $(b_0, \ldots, b_{2m-2})$. And let $C_0C_1 \ldots C_l$ be the sail of $\angle DAE$ (where $C_0 = B_m$) with the corresponding LLS-sequence $(c_0, \ldots, c_{2l-2})$.

Consider now the lattice oriented broken line

$$A_0 \ldots A_n B_1 B_2 \ldots B_m C_1 C_2 \ldots C_l.$$

The LSLS-sequence for this broken line is

$$(a_0, \ldots, a_{2n-2}, t, b_0, \ldots, b_{2m-2}, u, c_0, \ldots, c_{2l-2}),$$

where integers t and u are integers defined by the broken line. By definition of the sum of extended angles this sequence defines the extended angle

$$\overline{\angle CAB} +_t \overline{\angle BAD} +_u \overline{\angle DAE}.$$

By Theorem 4.1, we have $\overline{\angle CAB} +_t \overline{\angle BAD} +_u \overline{\angle DAE} = \pi$.

We compute now the integer t. Denote by A'_n the closest lattice point to the point A_n and distinct to A_n in the segment $A_{n-1}A_n$. Consider the set of lattice points at unit lattice distance from the segment AB and lying in the half-plane with the boundary straight line AB and containing the point D. This set coincides with the following set (See Figure 8):

$$\left\{A_{n,k} = A_n + \overline{A'_n A_n} + k\overline{AA_n} \mid k \in \mathbb{Z}\right\}.$$

Since $A_{n,-2} = A + \overline{A'_n A}$, the points $A_{n,k}$ for $k \leq -2$ are in the closed half-plane bounded by the straight line AC and not containing the point B.

Since $A_{n,-1} = A + \overline{A'_n A_n}$, the points $A_{n,k}$ for $k \geq -1$ are in the open half-plane bounded by the straight line AC and containing the point B.

The intersection of the parallelogram $AEDB$ and the open half-plane bounded by the straight line AC and containing the point B contains exactly $\mathrm{l}\ell(AB)$ points of the described set: only the points $A_{n,k}$ with $-1 \leq k \leq \mathrm{l}\ell(AB) - 2$.

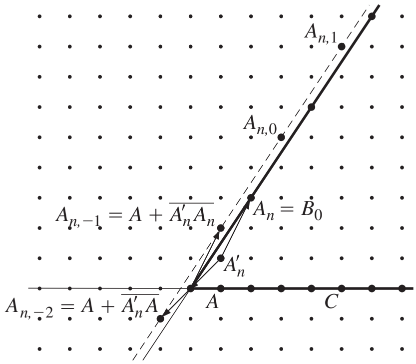


FIGURE 8. Lattice points $A_{n,t}$.

Since the triangle $\triangle BAD$ is $\mathscr{L}_+$-congruent to $\triangle ABC$, the number of points $A_{n,k}$ in the closed triangle $\triangle BAD$ is $\mathrm{l}\ell_1(AB; C)$: the points $A_{n,k}$ for

$$\mathrm{l}\ell(AB) - \mathrm{l}\ell_1(AB; C) - 1 \leq k \leq \mathrm{l}\ell(AB) - 2.$$

Denote the integer $\mathrm{l}\ell(AB) - \mathrm{l}\ell_1(AB; C) - 1$ by k_0.

The point A_{n,k_0} is contained in the segment B_0B_1 of the sail for the ordinary angle $\angle BAD$ (see Figure 9). Since the angles $\angle BAD$ and $\angle ABC$ are $\mathscr{L}_+$-congruent, we have

$$\begin{aligned} t &= \operatorname{sgn}(A_{n-1}AA_n)\operatorname{sgn}(A_nAB_1)\operatorname{sgn}(A_{n-1}A_nB_1)\operatorname{lsin}\angle A_{n-1}A_nB_1 \\ &= 1 \cdot 1 \cdot \operatorname{sgn}(A_{n-1}A_nA_{n,k_0})\operatorname{lsin}\angle A_{n-1}A_nA_{n,k_0} \\ &= \operatorname{sign}(k_0)|k_0| = k_0 = \mathrm{l}\ell(AB) - \mathrm{l}\ell_1(AB; C) - 1. \end{aligned}$$

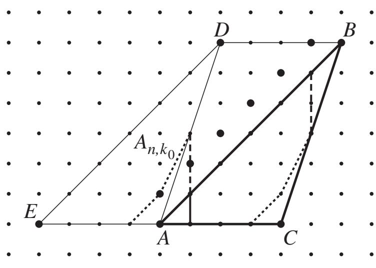


FIGURE 9. The point A_{n,k_0}.

Exactly by the same reasons,

$$u = \mathrm{l}\ell(DA) - \mathrm{l}\ell_1(DA; E) - 1 = \mathrm{l}\ell(BC) - \mathrm{l}\ell_1(BC; A) - 1.$$

Therefore, $\overline{\angle CAB} +_{\mathrm{l}\ell(AB)-\mathrm{l}\ell_1(AB;C)-1} \overline{\angle ABC} +_{\mathrm{l}\ell(BC)-\mathrm{l}\ell_1(BC;A)-1} \overline{\angle BCA} = \pi$.

LEMMA 4.5. *Let α, β, and γ be nonzero ordinary angles. Suppose that $\overline{\alpha} +_u \overline{\beta} +_v \overline{\gamma} = \pi$, then there exist a triangle with three consecutive ordinary angles $\mathscr{L}$-congruent to α, β, and γ.*

PROOF. Denote by O the point $(0,0)$, by A the point $(1,0)$, and by D the point $(-1,0)$ in the fixed system of coordinates OXY.

Let us choose the points $B=(p_1,q_1)$ and $C=(p_2,q_2)$ with integers p_1, p_2 and positive integers q_1, q_2 such that

$$\angle AOB = \operatorname{larctan}(\operatorname{ltan}\alpha), \qquad \text{and} \qquad \angle AOC = \overline{\angle AOB} +_u \overline{\beta}.$$

Thus the vectors $\overline{OB}$ and $\overline{OC}$ defines the positive orientation, and $\angle BOC \cong \beta$. Since

$$\overline{\alpha} +_u \overline{\beta} +_v \overline{\gamma} = \pi \qquad \text{and} \qquad \overline{\alpha} +_u \overline{\beta} \,\hat{\cong}\, \angle AOC,$$

the ordinary angle $\angle COD$ is $\mathscr{L}$-congruent to γ.

Denote by B' the point (p_1q_2, q_1q_2), and by C' the point (p_2q_1, q_1q_2) and consider the triangle $B'OC'$. Since the ordinary angle $\angle B'OC'$ coincides with the ordinary angle $\angle BOC$, we obtain

$$\angle B'OC' \cong \beta.$$

Since the ordinary angle β is nonzero, the points B' and C' are distinct and the straight line $B'C'$ does not coincide with the straight line OA. Since the second coordinate of the both points B' and C' equal q_1q_2, the straight line $B'C'$ is parallel to the straight line OA. Thus, by Proposition 1.13 it follows that

$$\angle C'B'O \cong \angle AOB' = \angle AOB \cong \alpha, \quad \text{and} \quad \angle OC'B' \cong \angle C'OD = \angle COD \cong \gamma.$$

So, we have constructed the triangle $\triangle B'OC'$ with three consecutive ordinary angles $\mathscr{L}$-congruent to α, β, and γ.

4.3. Proof of Theorem 2.2a: conclusion of the proof

Now we return to the proof of the first statement of the theorem on sums of lattice tangents for ordinary angles in lattice triangles.

PROOF OF THEOREM 2.2A. Let α, β, and γ be nonzero ordinary angles satisfying the conditions i) and ii) of Theorem 2.2a.

The second condition: $]\operatorname{ltan}(\alpha), -1, \operatorname{ltan}(\beta), -1, \operatorname{ltan}(\gamma)[= 0$ implies that

$$\overline{\alpha} +_{-1} \overline{\beta} +_{-1} \overline{\gamma} = k\pi.$$

Since all three tangents are positive, we have $k=1$, or $k=2$.

Consider the first condition: $]\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta[$ is either negative or greater than $\mathrm{ltan}\,\alpha$. It implies that $\overline{\alpha} +_{-1} \overline{\beta} = 0\pi + \varphi$, for some ordinary angle φ, and hence $k = 1$.

Therefore, by Lemma 4.5 there exist a triangle with three consecutive ordinary angles $\mathscr{L}$-congruent to α, β, and γ.

Let us prove the converse. We prove that condition ii) of Theorem 2.2a holds by reductio ad absurdum. Suppose, that there exist a triangle $\triangle ABC$ with consecutive ordinary angles $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$, such that

$$\begin{cases}]\mathrm{ltan}(\alpha), -1, \mathrm{ltan}(\beta), -1, \mathrm{ltan}(\gamma)[\neq 0 \\ ]\mathrm{ltan}(\beta), -1, \mathrm{ltan}(\gamma), -1, \mathrm{ltan}(\alpha)[\neq 0 \\ ]\mathrm{ltan}(\gamma), -1, \mathrm{ltan}(\alpha), -1, \mathrm{ltan}(\beta)[\neq 0 \end{cases}.$$

These inequalities and Lemma 4.4 imply that at least two of the integers

$$\mathrm{l}\ell(AB) - \mathrm{l}\ell_1(AB; C) - 1,$$

$$\mathrm{l}\ell(BC) - \mathrm{l}\ell_1(BC; A) - 1,$$

and

$$\mathrm{l}\ell(CA) - \mathrm{l}\ell_1(CA; B) - 1$$

are nonnegative.

Without losses of generality we suppose that

$$\begin{cases} \mathrm{l}\ell(AB) - \mathrm{l}\ell_1(AB; C) - 1 \geq 0 \\ \mathrm{l}\ell(BC) - \mathrm{l}\ell_1(BC; A) - 1 \geq 0 \end{cases}.$$

Since all integers of the continued fraction

$$r =]\mathrm{ltan}(\alpha), \mathrm{l}\ell(AB) - \mathrm{l}\ell_1(AB; C) - 1, \\ \mathrm{ltan}(\beta), \mathrm{l}\ell(BC) - \mathrm{l}\ell_1(BC; A) - 1, \mathrm{ltan}(\gamma)[$$

are non-negative and the last one is positive, we obtain that $r > 0$ (or $r = \infty$). From the other hand, by Lemma 4.4 and by Theorem 4.1 we have that $r = 0/{-1} = 0$. We come to the contradiction.

Now we prove that condition i) of Theorem 2.2a holds. Suppose that there exist a triangle $\triangle ABC$ with consecutive ordinary angles $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$, such that

$$]\mathrm{ltan}(\alpha), -1, \mathrm{ltan}(\beta), -1, \mathrm{ltan}(\gamma)[= 0.$$

Since $\overline{\alpha} +_{-1} \overline{\beta} +_{-1} \overline{\gamma} = \pi$, we have $\overline{\alpha} +_{-1} \overline{\beta} = 0\pi + \varphi$ for some ordinary angle φ. Therefore, the first condition of the theorem holds.

This concludes the proof of Theorem 2.2.

4.4. *Theorem on sum of lattice tangents for ordinary lattice angles of convex polygons*

A satisfactory description for $\mathcal{L}$-congruence classes of lattice convex polygons has not been yet found. It is only known that the number of convex polygons with lattice area bounded from above by n growths exponentially in $n^{1/3}$, while n tends to infinity (see [2] and [3]). We conclude this section with the following theorem on necessary and sufficient condition for the lattice angles to be the angles of some convex lattice polygon.

THEOREM 4.6. *Let $\alpha_1, \ldots, \alpha_n$ be an arbitrary ordered n-tuple of ordinary non-zero (lattice) angles. Then the following two conditions are equivalent:*

- *there exist a convex n-vertex polygon with consecutive ordinary angles $\mathcal{L}$-congruent to the ordinary angles α_i for $i = 1, \ldots, n$;*
- *there exist a set of integers $M = \{m_1, \ldots, m_{n-1}\}$ such that*

$$\sum_{M,i=1}^{n} \overline{\pi - \alpha_i} = 2\pi.$$

PROOF. Consider an arbitrary n-tuple of ordinary angles α_i, here $i = 1, \ldots, n$.

Suppose that there exist a convex polygon $A_1A_2\ldots A_n$ with consecutive angles α_i for $i = 1, \ldots, n$. Let also the pair of vectors $\overline{A_2A_3}$ and $\overline{A_2A_1}$ defines the positive orientation of the plane (otherwise we apply to the polygon $A_1A_2\ldots A_n$ some $\mathcal{L}$-affine transformation changing the orientation and come to the initial position).

Let $B_1 = O + \overline{A_nA_1}$, and $B_i = O + \overline{A_{i-1}A_i}$ for $i = 2, \ldots, n$. We put by definition

$$\beta_i = \begin{cases} \angle B_iOB_{i+1}, & \text{if } i = 1, \ldots, n-1 \\ \angle B_nOB_1, & \text{if } i = n \end{cases}.$$

Consider the union of the sails for all β_i. This lattice oriented broken line is of the class of the extended angle with the normal form $2\pi + 0$. The LSLS-sequence for this broken line contains exactly $n-1$ elements that are not contained in the LLS-sequences for the sails of β_i. Denote these numbers by $m_1, \ldots, m_{n-1}$, and the set $\{m_1, \ldots, m_{n-1}\}$ by M. Then

$$\sum_{M,i=1}^{n} \overline{\beta_i} = 2\pi.$$

From the definition of β_i for $i = 1, \dots, n$ it follows that $\beta_i \cong \pi - \alpha_i$. Therefore,

$$\sum_{M,i=1}^{n} \overline{\pi - \alpha_i} = 2\pi.$$

The proof of the first part of the statement is completed.

Suppose now, that there exist a set of integers $M = \{m_1, \dots, m_{n-1}\}$ such that

$$\sum_{M,i=1}^{n} \overline{\pi - \alpha_i} = 2\pi.$$

This implies that there exist lattice points $B_1 = (1, 0)$, $B_i = (x_i, y_i)$, for $i = 2, \dots n-1$, and $B_n = (-1, 0)$ such that

$$\angle B_i O B_{i-1} \cong \pi - \alpha_{i-1}, \quad \text{for } i = 2, \dots, n, \quad \text{and} \quad \angle B_1 O B_n \cong \pi - \alpha_n.$$

Denote by M the lattice point

$$O + \sum_{i=1}^{n} \overline{OB_i}.$$

Since all α_i are non-zero, the angles $\pi - \alpha_i$ are ordinary. Hence, the origin O is an interior point of the convex hull of the points B_i for $i = 1, \dots, k$. This implies that there exist two consecutive lattice points B_s and B_{s+1} (or B_n and B_1), such that the lattice triangle $\triangle B_s M B_{s+1}$ contains O and the edge $B_s B_{s+1}$ does not contain O. Therefore,

$$O = \lambda_1 \overline{OM} + \lambda_2 \overline{OB_i} + \lambda_3 \overline{OB_{i+1}},$$

where λ_1 is a positive integer, and λ_2 and λ_3 are nonnegative integers. So there exist positive integers a_i, where $i = 1, \dots, n$, such that

$$O = O + \sum_{i=1}^{n} (a_i \overline{OB_i}).$$

Put by definition $A_0 = O$, and $A_i = A_{i-1} + a_i \overline{OB_i}$ for $i = 2, \dots, n$. The broken line $A_0 A_1 \dots A_n$ is lattice and by the above it is closed (i. e. $A_0 = A_n$). By construction, the ordinary angle at the vertex A_i of the closed lattice broken line is $\mathscr{L}_+$-congruent to α_i ($i = 1, \dots n$). Since the integers a_i are positive for $i = 1, \dots, n$ and the vectors $\overline{OB_i}$ are all in the counterclockwise order, the broken line is a convex polygon.

The proof of Theorem 4.6 is completed.

REMARK 4.7. Theorem 4.6 generalizes the statement of Theorem 2.2a. Note that the direct generalization of Theorem 2.2b is false: the ordinary angles do not uniquely determine the $\mathscr{L}_+$-affine homothety types of convex polygons. See an example on Figure 10.

FIGURE 10. An example of different types of polygons with the $\mathscr{L}_+$-congruent ordinary angles.

Appendix A. On global relations on algebraic singularities of complex projective toric varieties corresponding to integer-lattice triangles

In this appendix we describe an application of theorems on sums of lattice tangents for the angles of lattice triangles and lattice convex polygons to theory of complex projective toric varieties. We refer the reader to the general definitions of theory of toric varieties to the works of V. I. Danilov [4], G. Ewald [5], W. Fulton [6], and T. Oda [18].

Let us briefly recall the definition of complex projective toric varieties associated to lattice convex polygons. Consider a lattice convex polygon P with vertices $A_0, A_1, \ldots, A_n$. Let the intersection of this (closed) polygon with the lattice consists of the points $B_i = (x_i, y_i)$ for $i = 0, \ldots, m$. Let also $B_i = A_i$ for $i = 0, \ldots, n$. Denote by Ω the following set in $\mathbb{C}P^m$:

$$\big\{\big(t_1^{x_1}t_2^{y_1}t_3^{-x_1-y_1} : t_1^{x_2}t_2^{y_2}t_3^{-x_2-y_2} : \ldots : t_1^{x_m}t_2^{y_m}t_3^{-x_m-y_m}\big) \bigm| t_1, t_2, t_3 \in \mathbb{C} \setminus \{0\}\big\}.$$

The closure of the set Ω in the natural topology of $\mathbb{C}P^m$ is called the *complex toric variety associated with the polygon* P and denoted by X_P.

For any $i = 0, \ldots, m$ we denote by $\tilde{A}_i$ the point $(0 : \ldots : 0 : 1 : 0 : \ldots : 0)$ where 1 stands on the $(i+1)$-th place.

From general theory it follows that:

a) the set X_P is a complex projective complex-two-dimensional variety with isolated algebraic singularities;
b) the complex toric projective variety contains the points $\tilde{A}_i$ for $i = 0, \ldots, n$ (where $n+1$ is the number of vertices of convex polygon);
c) the points of $X_P \setminus \{\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_n\}$ are non-singular;
d) the point $\tilde{A}_i$ for any integer i satisfying $0 \le i \le n$ is singular iff the corresponding ordinary angle α_i at the vertex A_i of the polygon P is not $\mathscr{L}$-congruent to $\operatorname{larctan}(1)$;

e) the algebraic singularity at $\tilde{A}_i$ for any integer i satisfying $0 \leq i \leq n$ is uniquely determined by the $\mathscr{L}$-affine type of the non-oriented sail of the lattice angle α_i.

The algebraic singularity is said to be *toric* if there exists a projective toric variety with the given algebraic singularity.

Note that the $\mathscr{L}$-affine classes of non-oriented sails for angles α and β coincide iff $\beta \cong \alpha$, or $\beta \cong \alpha^t$. This allows us to associate to any complex-two-dimensional toric algebraic singularity, corresponding to the sail of the angle α, the unordered pair of rationals (a, b), where $a = \operatorname{ltan}\alpha$ and $b = \operatorname{ltan}\alpha^t$.

REMARK A.1. Note that the continued fraction for the sail α is slightly different to the Hirzebruch-Jung continued fractions for toric singularities (see the works [9] by H. W. E. Jung, and [8] by F. Hirzebruch). The relations between these continued fractions is described in the paper [19] by P. Popescu-Pampu.

COROLLARY A.2. *Suppose, that we are given by three complex-two-dimensional toric singularities defined by pairs of rationals (a_i, b_i) for $i = 1, 2, 3$. There exist a complex toric variety associated with some triangle with these three singularities iff there exist a permutation $\sigma \in S_3$ and the rationals c_i from the sets $\{a_i, b_i\}$ for $i = 1, 2, 3$, such that the following conditions hold:*

i) *the continued fraction* $]c_{\sigma(1)}, -1, c_{\sigma(2)}[$ *is either negative, or greater than* $c_{\sigma(1)}$*, or equals* ∞*;*

ii) $]c_{\sigma(1)}, -1, c_{\sigma(2)}, -1, c_{\sigma(3)}[= 0$.

We note again that we use odd continued fractions for c_1, c_2, and c_3 in the statement of the above proposition (see Subsection 2.1 for the notation of continued fractions).

PROOF. The proposition follows directly from Theorem 2.2a.

PROPOSITION A.3. *For any collection (with multiplicities) of complex-two-dimensional toric algebraic singularities there exist a complex-two-dimensional toric projective variety with exactly the given collection of toric singularities.*

For the proof of Proposition A.3 we need the following lemma.

LEMMA A.4. *For any collection of ordinary angles α_i $(i = 1, \ldots, n)$, there exist an integer $k \geq n-1$ and a k-tuple of integers $M = (m_1, \ldots, m_k)$, such that*

$$\overline{\alpha_1} +_{m_1} \cdots +_{m_{n-1}} \overline{\alpha_n} +_{m_n} \operatorname{larctan}(1) +_{m_{n+1}} \cdots +_{m_k} \operatorname{larctan}(1) = 2\pi.$$

PROOF. Consider any collection of ordinary angles α_i $(i = 1, \ldots, n)$ and denote

$$\Phi = \overline{\alpha_1} +_1 \overline{\alpha_2} +_1 \cdots +_1 \overline{\alpha_n}.$$

There exist an oriented lattice broken line for the angle Φ with the LSLS-sequence with positive elements. Hence, $\Phi \mathrel{\hat{\cong}} \varphi + 0\pi$.

If $\varphi \cong \operatorname{larctan}(1)$, we have

$$\Phi +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) = 2\pi.$$

Then $k = n + 2$, and $M = (1, \dots, 1, -2, -2, -2)$.

Suppose now $\varphi \not\cong \operatorname{larctan}(1)$, then the following holds

$$\overline{\varphi} +_{-1} \overline{\pi - \varphi} +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) = 2\pi.$$

Consider the sail for the angle $\pi - \varphi$. Suppose the sequence of all its lattice points (not only vertices) is $B_0, \dots, B_s$ (with the order coinciding with the order of the sail). Then we have

$$\angle B_i O B_{i+1} \mathrel{\hat{\cong}} \operatorname{larctan}(1) \qquad \text{for any} \quad i = 1, \dots, s.$$

Denote by b_i the values of $\operatorname{lsin} \angle B_i O B_{i+1}$ for $i = 1, \dots, s$. Then we have

$$\begin{aligned}
&\overline{\varphi} +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) \\
&\quad = \overline{\alpha_1} +_1 \overline{\alpha_2} +_1 \cdots +_1 \overline{\alpha_n} +_{-1} \operatorname{larctan}(1) +_{b_1} \operatorname{larctan}(1) +_{b_2} \\
&\qquad \cdots +_{b_s} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) +_{-2} \operatorname{larctan}(1) \\
&\quad = 2\pi.
\end{aligned}$$

Therefore, $k = n + s + 3$, and

$$M = (\underbrace{1, 1, \dots, 1, 1}_{(n-1)\text{-times}}, -1, b_1, \dots, b_s, -2, -2, -2).$$

The proof of Lemma A.4 is completed.

PROOF OF THE STATEMENT OF THE PROPOSITION A.3. Consider an arbitrary collection of two-dimensional toric algebraic singularities. Suppose that they are represented by ordinary angles α_i ($i = 1, \dots, n$). By Lemma A.4 there exist an integer $k \geq n-1$ and a k-tuple of integers $M = (m_1, \dots, m_k)$, such that

$$\overline{(\pi - \alpha_1)} +_{m_1} \cdots +_{m_{n-1}} \overline{(\pi - \alpha_n)} +_{m_n} \operatorname{larctan}(1) +_{m_{n+1}} \cdots +_{m_k} \operatorname{larctan}(1) = 2\pi.$$

By Theorem 4.6 there exist a convex polygon $P = A_0 \dots A_k$ with angles $\mathscr{L}_+$-congruent to the ordinary angles α_i ($i = 1, \dots, n$), and $k - n + 1$ angles $\operatorname{larctan}(1)$.

By the above, the toric variety X_P is nonsingular at points of $P_X \setminus \{\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_k\}$. It is also nonsingular at the points $\tilde{A}_i$ with the corresponding ordinary angles $\mathscr{L}$-congruent to $\operatorname{larctan}(1)$. The collection of the toric singularities at the remaining points coincide with the given collection.

This concludes the proof of Proposition A.3.

On Figure 11 we show an example of the polygon for a projective toric variety with the unique toric singularity, represented by the sail of $\operatorname{larctan}(7/5)$. The ordinary angle α on the figure is $\mathscr{L}_+$-congruent to $\operatorname{larctan}(7/5)$, the angles β and γ are $\mathscr{L}_+$-congruent to $\operatorname{larctan}(1)$.

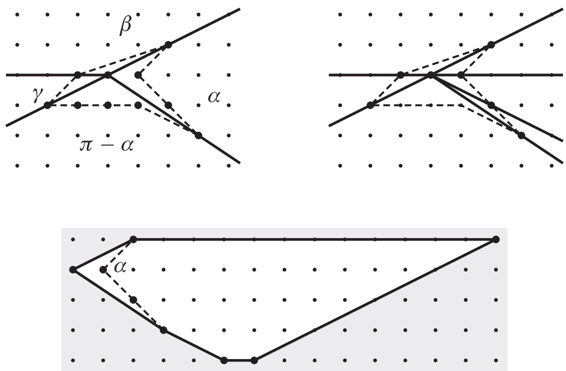


FIGURE 11. Constructing a polygon with all angles $\mathscr{L}_+$-congruent to $\operatorname{larctan}(1)$ except one angle that is $\mathscr{L}_+$-congruent to $\operatorname{larctan}(7/5)$.

Appendix B. On $\mathscr{L}$-congruence criterions for lattice triangles

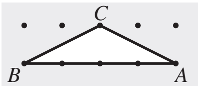
Here we discuss the $\mathscr{L}$-congruence criterions for lattice lattice triangles. By the first criterion of $\mathscr{L}$-congruence for lattice triangles we obtain that the number of $\mathscr{L}$-congruence classes for lattice triangles with bounded lattice area is finite. We write down the numbers of $\mathscr{L}$-congruence classes for triangles with lattice area less then or equal to 20.

On criterions of lattice triangle $\mathscr{L}$-congruence. We start with the study of lattice analogs for the first, the second, and the third Euclidean criterions of triangle congruence.

STATEMENT B.1 (The first criterion of lattice triangle $\mathscr{L}$-congruence). *Consider two lattice triangles $\triangle ABC$ and $\triangle A'B'C'$. Suppose that the edge AB is $\mathscr{L}$-congruent to the edge $A'B'$, the edge AC is $\mathscr{L}$-congruent to the edge $A'C'$, and the ordinary angle $\angle CAB$ is $\mathscr{L}$-congruent to the ordinary angle $\angle C'A'B'$, then the triangle $\triangle A'B'C'$ is $\mathscr{L}$-congruent to the $\triangle ABC$.*

It turns out that the second and the third criterions taken from Euclidean geometry do not hold. The following two examples illustrate these phenomena.

EXAMPLE B.2. The second criterion of triangle $\mathscr{L}$-congruence does not hold in lattice geometry. On Figure 12 we show two lattice triangles $\triangle ABC$ and $\triangle A'B'C'$. The edge AB is $\mathscr{L}$-congruent to the edge $A'B'$ (here $\mathrm{l}\ell(A'B') = \mathrm{l}\ell(AB) = 4$). The ordinary angle $\angle ABC$ is $\mathscr{L}$-congruent to the ordinary angle $\angle A'B'C'$ (since $\angle ABC \cong \angle A'B'C' \cong \mathrm{larctan}(1)$), and the ordinary angle $\angle CAB$ is $\mathscr{L}$-congruent to the ordinary angle $\angle C'A'B'$ (since $\angle CAB \cong \angle C'A'B' \cong \mathrm{larctan}(1)$), The triangle $\triangle A'B'C'$ is not $\mathscr{L}$-congruent to the triangle $\triangle ABC$, since $\mathrm{lS}(\triangle ABC) = 4$ and $\mathrm{lS}(\triangle A'B'C') = 8$.



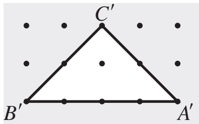
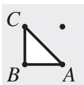


FIGURE 12. The second criterion of triangle $\mathscr{L}$-congruence does not hold.

EXAMPLE B.3. The third criterion of triangle $\mathscr{L}$-congruence does not hold in lattice geometry. On Figure 13 we show two lattice triangles $\triangle ABC$ and $\triangle A'B'C'$. All edges of both triangles are $\mathscr{L}$-congruent (of length one), but the triangles are not $\mathscr{L}$-congruent, since $\mathrm{lS}(\triangle ABC) = 1$ and $\mathrm{lS}(\triangle A'B'C') = 3$.



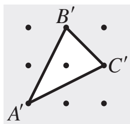


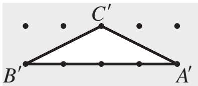
FIGURE 13. The third criterion of triangle $\mathscr{L}$-congruence does not hold.

Instead of the second and the third criterions there exists the following additional criterion of lattice triangles $\mathscr{L}$-congruence.

STATEMENT B.4 (An additional criterion of lattice triangle integer-congruence). *Consider two lattice triangles $\triangle ABC$ and $\triangle A'B'C'$ of the same lattice area. Suppose that the ordinary angle $\angle ABC$ is $\mathscr{L}$-congruent to the ordinary angle $\angle A'B'C'$, the ordinary angle $\angle CAB$ is $\mathscr{L}$-congruent to the ordinary angle $\angle C'A'B'$, the ordinary angle $\angle BCA$ is $\mathscr{L}$-congruent to the ordinary angle $\angle B'C'A'$, then the triangle $\triangle A'B'C'$ is $\mathscr{L}$-congruent to the triangle $\triangle ABC$.*

In the following example we show that the additional criterion of lattice triangle $\mathscr{L}$-congruence is not improvable.

EXAMPLE B.5. On Figure 14 we show an example of two lattice non-equivalent triangles $\triangle ABC$ and $\triangle A'B'C'$ of the same lattice area equals 4 and the same ordinary angles $\angle ABC$, $\angle CAB$, and $\angle A'B'C'$, $\angle C'A'B'$ all $\mathcal{L}$-equivalent to the angle larctan(1), but $\triangle ABC \not\cong \triangle A'B'C'$.



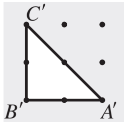


FIGURE 14. The additional criterion of lattice triangle $\mathcal{L}$-congruence is not improvable.

Lattice triangles of small area. The above criterions allows to enumerate all lattice triangles of small lattice area up to the lattice equivalence. In the following table we write down the numbers $N(d)$ of nonequivalent lattice triangles of lattice area d for $d \leq 20$.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$N(d)$	1	1	2	3	2	4	4	5	5	6	4	10	6	8	8	11	6	13	8	14

As it is easy to show, we always have $d/3 \leq N(d) \leq d$. The asymptotic behaviour of $N(d)$ and even of the average of $N(d)$ (if they exist) is unknown to the author.

Appendix C. Some unsolved question on lattice trigonometry

We conclude this paper with a small collection of unsolved questions.

Let us start with some questions on elementary definitions of lattice trigonometry. In this paper we do not show any geometrical meaning of lattice cosine. Here arise the following question.

PROBLEM 1. Find a natural description of lattice cosine for ordinary angles in terms of lattice invariants of the corresponding sublattices.

This problem seems to be close to the following one.

PROBLEM 2. Does there exist a lattice analog of the cosine formula for the angles of triangles in Euclidean geometry?

Let us continue with questions on lattice analogs of classical trigonometric formulas for trigonometric functions of angles of triangles in Euclidean geometry.

PROBLEM 3. a) Knowing the lattice trigonometric functions for lattice angles α, β and integer n, find the explicit formula for the lattice trigonometric functions of the extended angle $\overline{\alpha} +_n \overline{\beta}$.

b) Knowing the lattice trigonometric functions for a lattice angle α, an integer m, and positive integer m, find the explicit formula for the lattice trigonometric functions of the extended angle

$$\sum_{M,i=1}^{l} \overline{\alpha},$$

where $M = (m, \ldots, m)$ is an n-tuple.

Now we formulate a problem on generalization of the statement of Theorem 2.2b to the case of n ordinary angles. Such generalization is important in toric geometry and theory of multidimensional continued fractions.

PROBLEM 4. Find a necessary and sufficient conditions for the existence of an n-gon with the given ordered sequence of ordinary angles $(\alpha_1, \ldots, \alpha_n)$ and the consistent sequence of lattice lengths of the edges $(l_1, \ldots, l_n)$ in terms of continued fractions for $n \geq 4$.

We conclude this paper with the following problem. We remind that $(N(d))$ is the numbers of nonequivalent lattice triangles having the lattice area being equal to d (see Appendix B).

PROBLEM 5. Find an explicit formula for the numbers $N(d)$.

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On irrational lattice angles

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Abstract The aim of this paper is to generalize the notions of ordinary and expanded lattice angles and their sums studied in the work [7] by the author to the case of angles with lattice vertices but not necessarily with lattice rays. We find normal forms and extend the definition of lattice sums to a certain special case of such angles.

Keywords Lattices · Continued fractions · Convex hulls

Mathematics Subject Classification (2000) 11H06

Introduction

The aim of this paper is to generalize the notions of ordinary and expanded lattice angles and their sums studied in the work [7] by the author to the case of angles with lattice vertices but not necessarily with lattice rays. We find normal forms and extend the definition of lattice sums to a certain special case of such angles. The sum of angles described in the paper seems to be a natural notion of ordinary continued fractions "addition".

The study of lattice angles is an imprescriptible part of modern lattice geometry. Invariants of lattice angles are used in the study of lattice convex polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the works of F. Klein [9], V. I. Arnold [1], E. Korkina [11], M. Kontsevich and Yu. Suhov [10], G. Lachaud [12], and the author [5]). Lattice polygons and polytopes of the lattice geometry are in the limelight of complex projective toric varieties (for more information, see the works of V. I. Danilov [2], G. Ewald [3], T. Oda [13], and W. Fulton [4]).

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The studies of lattice angles and measures related to them were started by A. G. Khovanskii and A. V. Pukhlikov in [15] and [16] in 1992. They introduced and investigated special additive polynomial measure for the expanded notion of polytopes. The relations between sum-formulas of lattice trigonometric functions and lattice angles in the Khovanskii–Pukhlikov sense are unknown to the author.

In the work [7], the trigonometry of rational angles and their relation to the triangles was studied in details. Some properties of rational trigonometric functions follows from the statements of the work [14].

This paper is organized as follows. In Sect. 1 we remind the definition and main properties of ordinary continued fractions, and give definitions of ordinary lattice angles. The aim of Sect. 2 is to introduce trigonometric functions of ordinary lattice angles. Further in Sect. 3 we denote and study expanded irrational angles. These angles are necessary for the definition of sum of lattice angles. We study equivalence classes (with respect to the group of affine lattice preserving transformations) of expanded lattice angles and show a normal form for such classes. Finally, in Sect. 4 we give definitions of sums of lattice angles. We conclude the paper in Sect. 5 with related questions and problems.

1 Basic definitions

1.1 Ordinary continued fractions

With any finite sequence $(a_0, a_1, \dots, a_n)$ where the elements $a_1, \dots, a_n$ are positive integers and a_0 is an arbitrary integer, we associate the following rational q:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \quad \vdots \atop \qquad a_{n-1} + \cfrac{1}{a_n}}}.$$

This representation of the rational q is called an *ordinary continued fraction* for q and denoted by $[a_0, a_1, \dots, a_n]$. An ordinary continued fraction $[a_0, a_1, \dots, a_n]$ is said to be *odd* if the number of the elements of the sequence (i.e., $n+1$) is odd, and *even* if the number is even.

Theorem 1.1 *For any rational there exist exactly one odd ordinary continued fraction and exactly one even ordinary continued fraction.* □

We continue with the standard definition of infinite ordinary continued fraction.

Theorem 1.2 *Consider a sequence $(a_0, a_1, \dots, a_n, \dots)$ of positive integers. There exists the following limit*: $r = \lim_{k\to\infty}\big([a_0, a_1, \dots, a_k]\big)$. □

This representation of r is called an (*infinite*) *ordinary continued fraction* for r and denoted by $[a_0, a_1, \dots, a_n, \dots]$.

Theorem 1.3 *For any irrational there exists a unique infinite ordinary continued fraction. Any rational has no infinite ordinary continued fractions.* □

For the proofs of these theorems, we refer to the book [8] by A. Ya. Khinchin.

1.2 Lattice ordinary angles

A linear (affine) lattice preserving transformation is said to be *lattice*.

Let A, B, and C do not lie in the same straight line. Suppose also that B is lattice. We denote the angle with the vertex at B and the rays BA and BC by $\angle ABC$. If both open rays BA and BC contain lattice points, then we say that the angle $\angle ABC$ is an *ordinary rational* angle. If the open ray BA (the open ray BC) contains lattice points, and the remaining open ray of the angle does not contain lattice points, then we say that the angle $\angle ABC$ is an *ordinary R-irrational* (*L-irrational*) angle. If the union of open rays BA and BC does not contain lattice points, then we say that the angle $\angle ABC$ is an ordinary lattice *LR-irrational* angle.

Definition 1.4 Two ordinary lattice angles $\angle AOB$ and $\angle A'O'B'$ are said to be $\mathcal{L}$*-congruent* if there exist a lattice-affine transformation which takes the vertex O to the vertex O' and the rays OA and OB to the rays $O'A'$ and $O'B'$ respectively. We denote this as follows: $\angle AOB \cong \angle A'O'B'$.

2 Some properties of ordinary lattice angles

2.1 A few $\mathcal{L}$-congruence invariants

We start this section with definitions of some important invariants of the group of lattice-affine transformations.

For a lattice segment AB (i.e., a segment with lattice endpoints) we define its *lattice lengths* to be equal to the number of lattice inner points plus one and denote it by $\mathrm{l}\ell(AB)$.

The *lattice area* of a parallelogram $ABCD$ with lattice points $A,\ B,\ C,\ D$ is the index of the sublattice generated by the vectors $\overline{AB}$ and $\overline{AC}$ in the whole lattice. We denote the lattice area by $\mathrm{lS}(ABCD)$.

Consider an arbitrary rational angle $\angle ABC$. Let $D = C + \overline{BA}$. The *lattice sine* of $\angle ABC$ is the following positive integer:

$$\frac{\mathrm{lS}(ABCD)}{\mathrm{l}\ell(BA)\,\mathrm{l}\ell(BC)};$$

we denote it by $\mathrm{lsin}\,\angle ABC$.

Suppose some points A, B, and V are not in the same straight line. The *integer distance* from the lattice segment AB to the lattice point V is an index of sublattice generated by lattice vectors contained in AB and a vector $\overline{AV}$ in the whole lattice.

For the triples of lattice points A, B, and C, we define the function sgn as follows:

$$\mathrm{sgn}(ABC) = \begin{cases} +1, & \text{if the ordered pair of vectors } \overline{BA} \text{ and } \overline{BC} \\ & \text{defines the positive orientation,} \\ 0, & \text{if the points } A,\ B, \text{ and } C \text{ are contained in the same straight line,} \\ -1, & \text{if the ordered pair of vectors } \overline{BA} \text{ and } \overline{BC} \\ & \text{defines the negative orientation.} \end{cases}$$

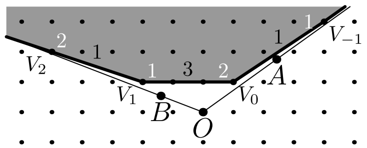


Fig. 1 LR-irrational angle $\angle AOB$, its sail and LLS-sequence.

2.2 LLS-sequences for ordinary angles

Consider an ordinary angle $\angle AOB$. Let also the vectors $\overline{OA}$ and $\overline{OB}$ be linearly independent.

Denote the closed convex solid cone for the ordinary irrational angle $\angle AOB$ by $C(AOB)$. The boundary of the convex hull of all lattice points of the cone $C(AOB)$ except the origin is homeomorphic to a straight line. The closure in the plane of the intersection of this boundary with the open cone AOB is called the *sail* for the cone $C(AOB)$. A lattice point of the sail is said to be a *vertex* of the sail if there is no segment of the sail containing this point in the interior. The sail of the cone $C(AOB)$ is a broken line with a finite or infinite number of vertices and without self-intersections. We orient the sail in the direction from $\overline{OA}$ to $\overline{OB}$. (For the definition of the sail and its higher dimensional generalization, see, for instance, the works [1], [11], and [5].)

In the case of ordinary R-irrational and rational angle we denote the vertices of the sail by V_i, for $i \geq 0$, according to the orientation of the sail (such that V_0 is contained in the ray OA). In the case of ordinary L-irrational angle we denote the vertices of the sail by V_{-i}, for $i \geq 0$, according to the orientation of the sail (such that V_0 is contained in the ray OB). In the case of ordinary LR-irrational angle we denote the vertices of the sail by V_{-i}, for $i \in \mathbb{Z}$, according to the orientation of the sail (such that V_0 is an arbitrary vertex of the sail).

Definition 2.1 Suppose that the vectors $\overline{OA}$ and $\overline{OB}$ of an ordinary angle $\angle AOB$ are linearly independent. Let V_i be the vertices of the corresponding sail. The sequence of lattice lengths and sines

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \mathrm{l}\ell(V_1V_2), \mathrm{lsin}\,\angle V_1V_2V_3, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n)),$$

$$\text{or} \qquad (\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \mathrm{l}\ell(V_1V_2), \mathrm{lsin}\,\angle V_1V_2V_3, \ldots),$$

$$\text{or} \qquad (\ldots, \mathrm{lsin}\,\angle V_{-3}V_{-2}V_{-1}, \mathrm{l}\ell(V_{-2}V_{-1}), \mathrm{lsin}\,\angle V_{-2}V_{-1}V_0, \mathrm{l}\ell(V_{-1}V_0)),$$

$$\text{or} \qquad (\ldots, \mathrm{lsin}\,\angle V_{-2}V_{-1}V_0, \mathrm{l}\ell(V_{-1}V_0), \mathrm{lsin}\,\angle V_{-1}V_0V_1, \mathrm{l}\ell(V_0V_1), \ldots)$$

is called the *LLS-sequence* for the ordinary angle $\angle AOB$, if this angle is rational, R-irrational, L-irrational, or LR-irrational respectively.

In Fig. 1 we show an example of an LR-irrational ordinary angle $\angle AOB$. The convex hull of all lattice points inside is colored with gray, its boundary is the sail of the angle. The lattice lengths of the segments are in black and the lattice sines of the angles are in white respectively. The LLS-sequence of the angle is $(\ldots, 1, 1, 2, 3, 1, 1, 2, \ldots)$.

Proposition 2.2 **a)** *The elements of the LLS-sequence for any ordinary rational/irrational angle are positive integers.* **b)** *The LLS-sequences of $\mathcal{L}$-congruent ordinary rational/irrational angles coincide.* □

2.3 Lattice tangents for ordinary rational and R-irrational angles

Let us give the definitions of lattice tangents for ordinary rational angles.

Definition 2.3 Let the vectors $\overline{OA}$ and $\overline{OB}$ of an ordinary rational angle $\angle AOB$ be linearly independent. Suppose that V_i are the vertices of the corresponding sail. Let

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n))$$

be the LLS-sequence for the angle $\angle AOB$. The *lattice tangent* of the ordinary angle $\angle AOB$ is the following rational:

$$[\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n)].$$

We denote it by $\mathrm{ltan}\,\angle AOB$.

Definition 2.4 Let the vectors $\overline{OA}$ and $\overline{OB}$ of an ordinary R-irrational angle $\angle AOB$ be linearly independent. Suppose that V_i are the vertices of the corresponding sail. Let

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n), \ldots)$$

be the LLS-sequence for the angle $\angle AOB$. The *lattice tangent* of the ordinary R-irrational angle $\angle AOB$ is the following irrational:

$$[\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n), \ldots].$$

We denote it by $\mathrm{ltan}\,\angle AOB$.

Let A, B, O, V_{-1}, V_1 be as in Fig. 1, then

$$\mathrm{ltan}\,\angle V_{-1}OV_1 = [1, 2, 3] = \frac{10}{7}, \qquad \mathrm{ltan}\,\angle V_1OV_{-1} = [3, 2, 1] = \frac{10}{3},$$

$$\mathrm{ltan}\,\angle V_{-1}OB = [1, 2, 3, 1, 1, 2, \ldots], \qquad \mathrm{ltan}\,\angle V_1OA = [3, 2, 1, 1, \ldots].$$

Proposition 2.5 **a)** *For any ordinary rational/R-irrational angle $\angle AOB$ with linearly independent vectors $\overline{OA}$ and $\overline{OB}$ the rational/irrational* $\mathrm{ltan}\,\angle AOB$ *is greater than or equivalent to* 1. **b)** *The values of the function* ltan *at two $\mathcal{L}$-congruent ordinary angles coincide.* □

2.4 Lattice arctangent for ordinary rational and R-irrational angles

Consider a coordinate system OXY in the space $\mathbb{R}^2$ with coordinates (x, y) and origin O. We work with the integer lattice of OXY.

For any reals p_1 and p_2, we denote by α_{p_1,p_2} the angle with the vertex at the origin and two edges $\{(x, p_i x) \mid x > 0\}$, where $i = 1, 2$.

Definition 2.6 For any real $s \geq 1$, the ordinary angle $\angle AOB$ with the vertex O at the origin, $A = (1, 0)$, and $B = (1, s)$, is called the *lattice arctangent* of s and denoted by $\mathrm{larctan}\, s$.

The following theorem shows that ltan and larctan are actually inverse to each other.

Theorem 2.7 **a)** *For any real* s, *such that* $s \geq 1$,

$$\operatorname{ltan}(\operatorname{larctan} s) = s.$$

b) *For any ordinary rational or R-irrational angle* α *the following holds*:

$$\operatorname{larctan}(\operatorname{ltan} \alpha) \cong \alpha.$$

Proof Both statements of the theorem for rational angles were proven in the paper [6].

Let us prove the first statement of Theorem 2.7 for the irrational case. Let $s > 1$ be some irrational real. Suppose that the sail of the angle $\operatorname{larctan} s$ is an infinite broken line $A_0A_1\ldots$ and the corresponding ordinary continued fraction is $[a_0, a_1, a_2, \ldots]$. Let also the coordinates of A_i be (x_i, y_i).

We consider the ordinary angles α_i, corresponding to the broken lines $A_0 \ldots A_i$, for $i > 0$. Then,

$$\lim_{i\to\infty} (y_i/x_i) = s/1.$$

By the statement of the theorem for rational angles, for any positive integer i the ordinary angle α_i coincides with $\operatorname{larctan}([a_0, a_1, \ldots, a_{2i-2}])$, and hence the coordinates (x_i, y_i) of A_i satisfy

$$y_i/x_i = [a_0, a_1, \ldots, a_{2i-2}].$$

Therefore,

$$\lim_{i\to\infty} ([a_0, a_1, \ldots, a_{2i-2}]) = s.$$

So, we obtain the first statement of the theorem:

$$\operatorname{ltan}(\operatorname{larctan} s) = s.$$

Now we prove the second statement. Consider an ordinary lattice R-irrational angle α. Suppose that the sail of the angle α is the infinite broken line $A_0A_1\ldots$.

For any positive integer i, we consider the ordinary angle α_i, corresponding to the broken lines $A_0 \ldots A_i$.

For an ordinary angle β, denote by $C(\beta)$ the cone, corresponding to β. Note that $C(\beta')$ and $C(\beta'')$ are $\mathcal{L}$-congruent iff $\beta \cong \beta'$.

By the statement of the theorem for rational angles we have:

$$\operatorname{larctan}(\operatorname{ltan} \alpha_i) \cong \alpha_i.$$

Since for any positive integer n the following is true:

$$\bigcup_{i=1}^{n} C(\alpha_i) \cong \bigcup_{i=1}^{n} C(\operatorname{larctan}(\operatorname{ltan} \alpha_i)),$$

we obtain

$$C(\alpha) \cong \bigcup_{i=1}^{\infty} C(\alpha_i) \cong \bigcup_{i=1}^{\infty} C(\operatorname{larctan}(\operatorname{ltan} \alpha_i)) \cong C(\operatorname{larctan}(\operatorname{ltan} \alpha)).$$

Therefore,

$$\operatorname{larctan}(\operatorname{ltan}\alpha) \cong \alpha.$$

This concludes the proof of Theorem 2.7. □

Now we give the following description of ordinary rational and R-irrational angles.

Theorem 2.8 (Description of ordinary rational and R-irrational angles)
a) *For any finite/infinite sequence of positive integers* $(a_0, a_1, a_2, \dots)$ *there exists some ordinary rational/R-irrational angle* α *such that* $\operatorname{ltan}\alpha = [a_0, a_1, a_2, \dots]$.
b) *Two ordinary lattice rational/R-irrational angles are* $\mathcal{L}$*-congruent iff they have equivalent lattice tangents.*

Proof Theorem 2.7a implies the first statement of the theorem.

Let us prove the second statement. Suppose that the ordinary rational/R-irrational angles α and β are $\mathcal{L}$-congruent, then their sails are also $\mathcal{L}$-congruent. Thus their LLS-sequences coincide. Therefore, $\operatorname{ltan}\alpha = \operatorname{ltan}\beta$.

Suppose now that the lattice tangents for two ordinary rational/R-irrational angles α and β are equivalent. Now we apply Theorem 2.7b and obtain

$$\alpha \cong \operatorname{larctan}(\operatorname{ltan}\alpha) = \operatorname{larctan}(\operatorname{ltan}\beta) \cong \beta.$$

Therefore, the angles α and β are $\mathcal{L}$-congruent. □

Corollary 2.9 (Description of ordinary L-irrational and LR-irrational angles)
a) *For any sequence of positive integers* $(\dots, a_{-2}, a_{-1}, a_0)$ *(or* $(\dots a_{-1}, a_0, a_1, \dots)$*) there exists an ordinary L-irrational (LR-irrational) angle with the LLS-sequence equivalent to the given one.*
b) *Two ordinary L-irrational (LR-irrational) angles are* $\mathcal{L}$*-congruent iff they have the same LLS-sequences.*

Proof The statement on L-irrational angles follows immediately from Theorem 2.8.

Let us construct an LR-angle with a given LLS-sequence $(\dots a_{-1}, a_0, a_1, \dots)$. First we construct

$$\alpha_1 = \operatorname{larctan}([a_0, a_1, a_2, \dots]).$$

Denote the points $(1, 0)$ and $(1, a_0)$ by A_0 and A_1, and construct the angle α_2 that is $\mathcal{L}$-congruent to the angle

$$\operatorname{larctan}([a_0, a_{-1}, a_{-2}, \dots]),$$

and that has the first two vertices A_1 and A_0 respectively. Now the angle obtained by the rays of α_1 and α_2 that do not contain lattice points is the LR-angle with the given LLS-sequence.

Suppose now we have two LR-angles β_1 and β_2 with the same LLS-sequences. Consider a lattice transformation taking the vertex of β_2 to the vertex of β_1, and one of the segments of β_2 to the segment B_0B_1 of β_1 with the appropriate order in LLS-sequence. Denote this angle by β_2'. Consider the R-angles $\overline{\beta_1}$ and $\overline{\beta_2'}$ corresponding to the sequences of vertices of β_1 and β_2' starting from V_0 in the direction towards V_1. These two angles are $\mathcal{L}$-congruent by Theorem 2.8, therefore $\overline{\beta_1}$ and $\overline{\beta_2'}$ coincide. So the angles β_1 and β_2' have a common ray. By the same reason the second ray of β_1 coincides with the second ray of β_2'. Therefore α_1 coincides with β_2' and is $\mathcal{L}$-congruent to β_2. □

Remark on zero ordinary angles Further we use zero ordinary angles and their trigonometric functions. Let A, B, and C be three lattice points of the same lattice straight line. Suppose that B is distinct from both A and C, and the rays BA and BC coincide. We say that the ordinary lattice angle with the vertex at B and the rays BA and BC is *zero*. Suppose $\angle ABC$ is zero, put by definition

$$\operatorname{lsin}(\angle ABC) = 0, \quad \operatorname{lcos}(\angle ABC) = 1, \quad \operatorname{ltan}(\angle ABC) = 0.$$

Denote by $\operatorname{larctan}(0)$ the angle $\angle AOA$ where $A = (1, 0)$, and O is the origin.

3 Lattice expanded angles

3.1 Signed LLS-sequences

In this subsection we work in the oriented two-dimensional real vector space with a fixed lattice. As previously, we fix a coordinate system OXY in this space.

A finite (infinite to the right, to the left, or in both directions) union of ordered lattice segments $\ldots, A_{i-1}A_i, A_iA_{i+1}, A_{i+1}A_{i+2}, \ldots$ is said to be a *lattice oriented finite* (*R-infinite, L-infinite, or LR-infinite*) *broken line*, if any segment of the broken line is not of zero length, and any two consecutive segments are not contained in the same straight line. We denote this broken line by $\ldots A_{i-1}A_iA_{i+1}A_{i+2}\ldots$. We also say that the lattice oriented broken line $\ldots A_{i+2}A_{i+1}A_iA_{i-1}\ldots$ is *inverse* to the broken line $\ldots A_{i-1}A_iA_{i+1}A_{i+2}\ldots$.

Definition 3.1 Consider a lattice infinite oriented broken line and a point not in this line. The broken line is said to be *at the unit distance* from the point if all edges of the broken line are at the unit lattice distance from the given point.

Now, let us associate with any lattice oriented broken line at the unit distance from some point the following sequence of nonzero elements.

Definition 3.2 Let $\ldots A_{i-1}A_iA_{i+1}A_{i+2}\ldots$ be a lattice oriented broken line at the unit distance from some lattice point V. Let

$$a_{2i-3} = \operatorname{sgn}(A_{i-2}VA_{i-1})\operatorname{sgn}(A_{i-1}VA_i)\operatorname{sgn}(A_{i-2}A_{i-1}A_i)\operatorname{lsin}\angle A_{i-2}A_{i-1}A_i,$$
$$a_{2i-2} = \operatorname{sgn}(A_{i-1}VA_i)\,\operatorname{l}\ell(A_{i-1}A_i)$$

for all possible indices i. The sequence $(\ldots a_{2i-3}, a_{2i-2}, a_{2i-1}\ldots)$ is called a *signed LLS-sequence* for the lattice oriented finite/infinite broken line at the unit distance from V (or for simplicity just LSLS-sequence).

In Fig. 2 we identify geometrically the signs of elements of the LSLS-sequence for a lattice oriented V-broken line.

Proposition 3.3 *An LSLS-sequence for the given lattice oriented broken line and the point is invariant under the group action of orientation preserving lattice-affine transformations.*

Proof The statement holds, since the functions sgn, $\operatorname{l}\ell$, and lsin are invariant under the group action of orientation preserving lattice-affine transformations. □

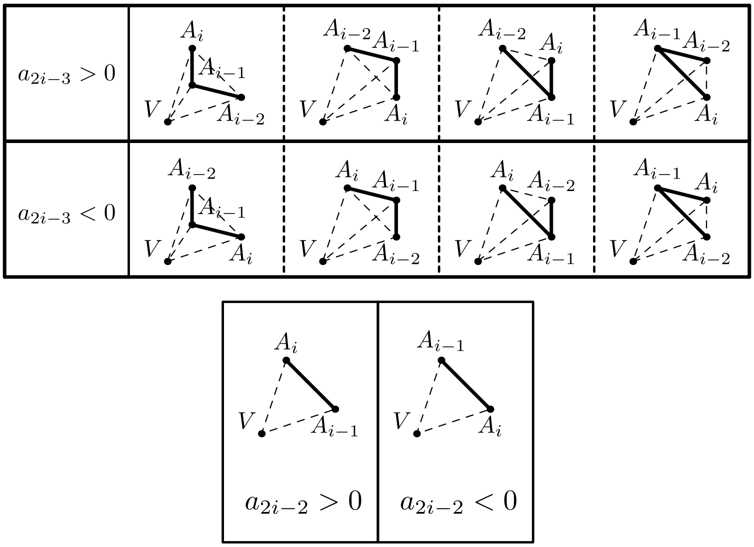


Fig. 2 All possible different combinatorial cases for angles and segments of an LSLS-sequence.

3.2 $\mathcal{L}_+$-congruence of lattice oriented broken lines at the unit distance from the lattice points

Two lattice oriented broken lines at the unit distance from lattice points V_1 and V_2 are said to be $\mathcal{L}_+$*-congruent* iff there exists an orientation preserving lattice-affine transformation taking V_1 to V_2 and the first broken line to the second.

Let us formulate a necessary and sufficient conditions for two lattice oriented broken lines at the unit distance from two lattice points to be $\mathcal{L}_+$-congruent.

Theorem 3.4 *The LSLS-sequences of two lattice finite or infinite oriented broken lines at the unit distance from lattice points V_1 and V_2 respectively coincide, iff there exists an orientation preserving lattice-affine transformation taking the point V_1 to V_2 and one oriented broken line to the other.*

Proof The case of finite broken lines was studied in [7], we skip the proof here.

The LSLS-sequence for any lattice infinite oriented broken line at the unit distance is invariant under the group action of orientation preserving lattice-affine transformations, since the functions sgn, lℓ, lsin are invariant. Therefore, the LSLS-sequences for two $\mathcal{L}_+$-congruent broken lines coincide.

Suppose now that we have two lattice oriented infinite broken lines $\ldots A_{i-1}A_iA_{i+1}\ldots$ and $\ldots B_{i-1}B_iB_{i+1}\ldots$ at the unit distance from the points V_1 and V_2, and with the same LSLS-sequences. Consider the lattice-affine transformation ξ that takes the point V_1 to V_2, A_i to B_i, and A_{i+1} to B_{i+1} for some integer i. Since $\operatorname{sgn}(A_iVA_{i+1}) = \operatorname{sgn}(B_iVB_{i+1})$, the lattice-affine transformation ξ is orientation preserving. By Theorem 3.4 for the finite case the transformation ξ takes any finite oriented broken line $A_sA_{s+1}\ldots A_t$ containing the segment A_iA_{i+1} to the oriented broken line $B_sB_{s+1}\ldots B_t$. Therefore, the transformation ξ takes the lattice oriented infinite broken line $\ldots A_{i-1}A_iA_{i+1}\ldots$ to the oriented broken line $\ldots B_{i-1}B_iB_{i+1}\ldots$ and the lattice point V_1 to the lattice point V_2.

This concludes the proof of Theorem 3.4 for the infinite broken lines. □

3.3 Equivalence classes of almost positive lattice infinite oriented broken lines and corresponding expanded infinite angles

We start this section with the following general definition.

Definition 3.5 We say that the lattice infinite oriented broken line at the unit distance from some lattice point is *almost positive* if the elements of the corresponding LSLS-sequence are all positive, except for a finite number of elements.

Let l be the lattice (finite or infinite) oriented broken line $\ldots A_{n-1}A_n \ldots A_m A_{m+1} \ldots$. Denote by $l(-\infty, A_n)$ the broken line $\ldots A_{n-1}A_n$. Denote by $l(A_m, +\infty)$ the broken line $A_m A_{m+1} \ldots$. Denote by $l(A_n, A_m)$ the broken line $A_n \ldots A_m$.

Definition 3.6 Two lattice oriented infinite broken lines l_1 and l_2 at the unit distance from V are said to be *equivalent* if there exist two vertices W_{11} and W_{12} of the broken line l_1 and two vertices W_{21} and W_{22} of the broken line l_2 such that the following three conditions are satisfied:

i) the broken line $l_1(W_{12}, +\infty)$ coincides (edge by edge) with the broken line $l_2(W_{22}, +\infty)$;
ii) the broken line $l_1(-\infty, W_{11})$ coincides with the broken line $l_2(-\infty, W_{21})$;
iii) the closed broken line generated by $l_1(W_{11}, W_{12})$ and the inverse of $l_2(W_{21}, W_{22})$ is homotopy equivalent to a point in $\mathbb{R}^2 \setminus \{V\}$.

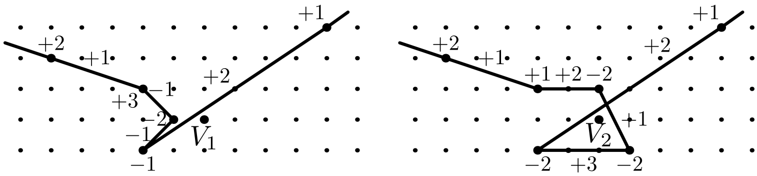
Now we give the definition of expanded angles.

Definition 3.7 An equivalence class of lattice finite (R/L/LR-infinite) oriented broken lines at the unit distance from V containing the broken line l is called the *expanded finite (R/L/LR-infinite) angle for the equivalence class of* l at the *vertex* V and denoted by $\angle(V; l)$ (or, for short, *expanded R/L/LR-infinite angle*).

Remark 1 Since all the sails for ordinary angles are lattice oriented broken lines, the set of all ordinary irrational angles is naturally embedded into the set of expanded irrational angles. An ordinary angle with a sail S corresponds to the expanded angle with the equivalence class of the broken line S.

Definition 3.8 Two expanded angles Φ_1 and Φ_2 are said to be $\mathcal{L}_+$*-congruent* iff there exists an orientation preserving lattice-affine transformation sending the class of lattice oriented broken lines corresponding to Φ_1 to the class of lattice oriented broken lines corresponding to Φ_2. We denote it by $\Phi_1 \hat{\cong} \Phi_2$.

In Fig. 3 we show two LR-infinite broken lines, we also indicate their LSLS-sequences. (We suppose that outside of the pictures the broken lines are the same.) These broken lines define two non-equivalent expanded LR-infinite angles. The broken line of Fig. 3a and the sail of Fig. 1a define equivalent expanded angles.



(a) $(\ldots,1,2,-1,-1,-2,-1,3,1,2,\ldots)$. (b) $(\ldots,1,2,-2,3,-2,1,-2,2,1,1,2,\ldots)$.

Fig. 3 Examples of expanded angles for two particular LSLS-sequences.

3.4 Revolution number for expanded rational, L- and R-irrational angles

First we define the revolution number for the case of finite broken lines.

Let $r=\{V+\lambda\overline{v}\mid\lambda\ge 0\}$ be the oriented ray for an arbitrary vector $\overline{v}$ with the vertex at V, and AB be an oriented (from A to B) segment not contained in the ray r. Suppose also, that the vertex V of the ray r is not contained in the segment AB. We denote by $\#(r,V,AB)$ the following number:

$$\#(r,V,AB)=\begin{cases}0, & \text{if the segment } AB \text{ does not intersect the ray } r,\\ \frac{1}{2}\operatorname{sgn}\big(A(A+\overline{v})B\big), & \text{if the segment } AB \text{ intersects the ray } r\\ & \text{at } A \text{ or at } B,\\ \operatorname{sgn}\big(A(A+\overline{v})B\big), & \text{if the segment } AB \text{ intersects the ray } r\\ & \text{at an interior point of } AB,\end{cases}$$

and call it the *intersection number* of the ray r and the segment AB.

Definition 3.9 Let $A_0A_1\ldots A_n$ be some lattice oriented broken line, and let r be an oriented ray $\{V+\lambda\overline{v}\mid\lambda\ge 0\}$. Suppose that the ray r does not contain the edges of the broken line, and the broken line does not contain the point V. We call the number

$$\sum_{i=1}^{n}\#(r,V,A_{i-1}A_i)$$

the *intersection number* of the ray r and the lattice oriented broken line $A_0A_1\ldots A_n$, and denote it by $\#(r,V,A_0A_1\ldots A_n)$.

Definition 3.10 Consider an arbitrary expanded angle $\angle(V,A_0A_1\ldots A_n)$. Denote the rays $\{V+\lambda\overline{VA_0}\mid\lambda\ge 0\}$ and $\{V-\lambda\overline{VA_0}\mid\lambda\ge 0\}$ by r_+ and r_- respectively. The number

$$\frac{1}{2}\big(\#(r_+,V,A_0A_1\ldots A_n)+\#(r_-,V,A_0A_1\ldots A_n)\big)$$

is called the *lattice revolution number* for the expanded angle $\angle(V,A_0A_1\ldots A_n)$, and denoted by $\#(\angle(V,A_0A_1\ldots A_n))$.

The revolution number of any expanded angle is well-defined and is invariant under the group action of the orientation preserving lattice-affine transformations (see [7] for more details).

Let now us extend the revolution number to the case of almost positive infinite oriented broken lines.

Definition 3.11 Let $\ldots A_{i-1}A_iA_{i+1}\ldots$ be some lattice R-, L- or LR-infinite almost positive oriented broken line, and $r = \{V + \lambda\overline{v} \mid \lambda \geq 0\}$ be the oriented ray for an arbitrary vector $\overline{v}$ with the vertex at V. Suppose that all straight lines containing the edges of the broken line do not pass through the vertex V. We call the number

$$\lim_{n\to+\infty} \#(r, V, A_0A_1\ldots A_n) \qquad \text{if the broken line is R-infinite,}$$

$$\lim_{n\to+\infty} \#(r, V, A_{-n}\ldots A_{-1}A_0) \qquad \text{if the broken line is L-infinite,}$$

$$\lim_{n\to+\infty} \#(r, V, A_{-n}A_{-n+1}\ldots A_n) \qquad \text{if the broken line is LR-infinite}$$

the *intersection number* of the ray r and the lattice almost positive infinite oriented broken line broken line $\ldots A_{i-1}A_iA_{i+1}\ldots$ and denote it by $\#(r, V, \ldots A_{i-1}A_iA_{i+1}\ldots)$.

Proposition 3.12 *The intersection number of a ray r and an almost positive lattice infinite oriented broken line is well-defined.*

Proof Consider an almost positive lattice infinite oriented broken line l. Let us show that the broken line l intersects the ray r only finitely many times.

By Definition 3.5 there exist vertices W_1 and W_2 of this broken line such that the LSLS-sequence for the lattice oriented broken line $l(-\infty, W_1)$ contains only positive elements, and the LSLS-sequence for the oriented broken line $l(W_2, +\infty)$ also contains only positive elements.

The positivity of the LLS-sequences implies that the lattice oriented broken lines $l(-\infty, W_1)$, and $l(W_2, +\infty)$ are the sails for some angles with the vertex V. Thus, these two broken lines intersect the ray r at most once each. Therefore, the broken line l intersects the ray r at most once in the part $l(-\infty, W_1)$, only finitely many times in the part $l(W_1, W_2)$, and at most once in the part $l(W_2, +\infty)$.

So, the lattice infinite oriented broken line l intersects the ray r only finitely many times, and, therefore, the corresponding intersection number is well-defined. □

Now we give a definition of the lattice revolution number for expanded R-irrational and L-irrational angles.

Definition 3.13 Consider an arbitrary R-infinite (or L-infinite) expanded angle $\angle(V, l)$, where V is some lattice point, and l is a lattice infinite oriented almost-positive broken line. Let A_0 be the first (the last) vertex of l. Denote the rays $\{V + \lambda\overline{VA_0} \mid \lambda \geq 0\}$ and $\{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}$ by r_+ and r_- respectively. The number

$$\frac{1}{2}\big(\#(r_+, V, l) + \#(r_-, V, l)\big)$$

is called the *lattice revolution number* for the expanded irrational angle $\angle(V, l)$, and denoted by $\#(\angle(V, l))$.

The revolution numbers for the angles defined by the broken lines of Fig. 3a and 3b are respectively: $1/4$ and $5/4$.

Proposition 3.14 *The revolution number of an R-irrational (or L-irrational) expanded angle is well-defined.*

Proof Consider an arbitrary expanded R-irrational angle $\angle(V, A_0A_1\ldots)$. Let

$$r_+ = \{V + \lambda\overline{VA_0} \mid \lambda \geq 0\} \quad \text{and} \quad r_- = \{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}.$$

Suppose that

$$\angle V, A_0A_1A_2\ldots = \angle V', A'_0A'_1A'_2\ldots$$

This implies that $V = V'$, $A_0 = A'_0$, $A_{n+k} = A'_{m+k}$ for some integers n and m and any non-negative integer k, and the broken line $A_0A_1\ldots A_nA'_{m-1}\ldots A'_1A'_0$ is homotopy equivalent to a point in $\mathbb{R}^2 \setminus \{V\}$. Thus,

$$\begin{aligned}
&\#(\angle V, A_0A_1\ldots) - \#(\angle V', A'_0A'_1\ldots)\\
&\quad= \frac{1}{2}\big(\#(r_+, A_0A_1\ldots A_nA'_{m-1}\ldots A'_1A'_0) + \#(r_-, A_0A_1\ldots A_nA'_{m-1}\ldots A'_1A'_0)\big)\\
&\quad= 0 + 0 = 0,
\end{aligned}$$

and hence

$$\#(\angle V, A_0A_1A_2\ldots) = \#(\angle V', A'_0A'_1A'_2\ldots).$$

Therefore, the revolution number of any expanded R-irrational angle is well-defined.

The proof for L-irrational angles repeats the proof for R-irrational angles and is omitted here. □

Proposition 3.15 *The revolution number of expanded R/L-irrational angles is invariant under the group action of the orientation preserving lattice-affine transformations.* □

Let us finally give the definition of trigonometric functions for the expanded angles and describe some relations between ordinary and expanded angles.

Definition 3.16 Consider an arbitrary expanded angle Φ with the normal form $k\pi + \varphi$ for some ordinary (possible zero) angle φ and for an integer k.
a) The ordinary angle φ is said to be *associated* with the expanded angle Φ.
b) The numbers $\mathrm{ltan}(\varphi)$, $\mathrm{lsin}(\varphi)$, and $\mathrm{lcos}(\varphi)$ are called the *lattice tangent*, the *lattice sine*, and the *lattice cosine* of the expanded angle Φ.

3.5 Normal forms of expanded rational angles

In this section we list the results of [7] in rational case.

We use the following notation. By the sequence

$$\big((a_0, \ldots, a_n) \times k\text{-times}, b_0, \ldots, b_m\big),$$

where $k \geq 0$, we denote the following sequence:

$$(\underbrace{a_0, \ldots, a_n, \quad a_0, \ldots, a_n, \quad \ldots, \quad a_0, \ldots, a_n,}_{k \text{ times}} \quad b_0, \ldots, b_m).$$

Definition 3.17 **I**) Let O be the origin, A_0 be the point $(1,0)$. We say that the expanded angle $\angle(O, A_0)$ is *of the type* **I** and denote it by $0\pi + \text{larctan}(0)$ (or 0, for short). The empty sequence is said to be *characteristic* for the angle $0\pi + \text{larctan}(0)$.

Consider a lattice oriented broken line $A_0A_1 \ldots A_s$ at the unit distance from the origin O. Let also A_0 be the point $(1,0)$, and the point A_1 be on the straight line $x=1$. If the LSLS-sequence of the expanded angle $\Phi_0 = \angle(O, A_0A_1\ldots A_s)$ coincides with the following sequence (we call it the *characteristic sequence* for the corresponding angle):
II$_k$) $\big((1,-2,1,-2)\times(k-1)\text{-times}, 1,-2,1\big)$, where $k \geq 1$, then we denote the angle Φ_0 by $k\pi + \text{larctan}(0)$ (or $k\pi$, for short) and say that Φ_0 is *of the type* **II**$_k$;
III$_k$) $\big((-1,2,-1,2)\times(k-1)\text{-times}, -1,2,-1\big)$, where $k \geq 1$, then we denote the angle Φ_0 by $-k\pi + \text{larctan}(0)$ (or $-k\pi$, for short) and say that Φ_0 is *of the type* **III**$_k$;
IV$_k$) $\big((1,-2,1,-2)\times k\text{-times}, a_0,\ldots,a_{2n}\big)$, where $k \geq 0$, $n \geq 0$, $a_i > 0$, for $i=0,\ldots,2n$, then we denote the angle Φ_0 by $k\pi + \text{larctan}([a_0,a_1,\ldots,a_{2n}])$ and say that Φ_0 is *of the type* **IV**$_k$;
V$_k$) $\big((-1,2,-1,2)\times k\text{-times}, a_0,\ldots,a_{2n}\big)$, where $k > 0$, $n \geq 0$, $a_i > 0$, for $i=0,\ldots,2n$, then we denote the angle Φ_0 by $-k\pi + \text{larctan}([a_0,a_1,\ldots,a_{2n}])$ and say that Φ_0 is *of the type* **V**$_k$.

Theorem 3.18 *For any expanded rational angle Φ there exist a unique type among the types* **I**–**V** *and a unique rational expanded angle Φ_0 of that type such that Φ_0 is $\mathcal{L}_+$-congruent to Φ*. The expanded angle Φ_0 is said to be the *normal form* for the expanded angle Φ. □

Further we use the following lemma of [7].

Lemma 3.19 *Let $m, k \geq 1$, and $a_i > 0$ for $i=0,\ldots,2n$ be some integers.*
a) *Suppose the LSLS-sequences for the expanded angles Φ_1 and Φ_2 are respectively*

$$\big((1,-2,1,-2)\times(k-1)\textit{-times}, 1,-2,1,-2,a_0,\ldots,a_{2n}\big) \qquad \textit{and}$$
$$\big((1,-2,1,-2)\times(k-1)\textit{-times}, 1,-2,1,m,a_0,\ldots,a_{2n}\big),$$

then Φ_1 is $\mathcal{L}_+$-congruent to Φ_2.
b) *Suppose the LSLS-sequences for the expanded angles Φ_1 and Φ_2 are respectively*

$$\big((-1,2,-1,2)\times(k-1)\textit{-times}, -1,2,-1,m,a_0,\ldots,a_{2n}\big) \qquad \textit{and}$$
$$\big((-1,2,-1,2)\times(k-1)\textit{-times}, -1,2,-1,2,a_0,\ldots,a_{2n}\big),$$

then Φ_1 is $\mathcal{L}_+$-congruent to Φ_2. □

3.6 Normal forms of expanded R- and L-irrational angles

In this section we formulate and prove a theorem on normal forms of expanded lattice R-irrational and L-irrational angles.

For the theorems of this section we introduce the following notation. By the sequence

$$\big((a_0,\ldots,a_n)\times k\text{-times}, b_0, b_1, \ldots\big),$$

where $k \geq 0$, we denote the sequence:

$$(\underbrace{a_0,\ldots,a_n,\quad a_0,\ldots,a_n,\quad \ldots,\quad a_0,\ldots,a_n,}_{k\text{ times}}\quad b_0,b_1,\ldots).$$

By the sequence

$$\big(\ldots, b_{-2}, b_{-1}, b_0, (a_0, \ldots, a_n) \times k\text{-times}\big),$$

where $k \geq 0$, we denote the sequence:

$$(\ldots, b_{-2}, b_{-1}, b_0, \quad \underbrace{a_0, \ldots, a_n, \quad a_0, \ldots, a_n, \quad \ldots, \quad a_0, \ldots, a_n}_{k \text{ times}}).$$

We start with the case of expanded R-irrational angles.

Definition 3.20 Consider a lattice R-infinite oriented broken line $A_0A_1\ldots$ at the unit distance from the origin O. Let also A_0 be the point $(1, 0)$, and the point A_1 be on the line $x = 1$. If the LSLS-sequence of the expanded R-irrational angle $\Phi_0 = \angle(O, A_0A_1\ldots)$ coincides with the following sequence (we call it the *characteristic sequence* for the corresponding angle):
IV$_k$) $\big((1, -2, 1, -2) \times k\text{-times}, a_0, a_1, \ldots\big)$, where $k \geq 0$, $a_i > 0$, for $i \geq 0$, then we denote the angle Φ_0 by $k\pi + \operatorname{larctan}([a_0, a_1, \ldots])$ and say that Φ_0 is *of the type* **IV**$_k$;
V$_k$) $\big((-1, 2, -1, 2) \times k\text{-times}, a_0, a_1, \ldots\big)$, where $k > 0$, $a_i > 0$, for $i \geq 0$, then we denote the angle Φ_0 by $-k\pi + \operatorname{larctan}([a_0, a_1, \ldots])$ and say that Φ_0 is *of the type* **V**$_k$.

Theorem 3.21 *For any expanded R-irrational angle Φ there exist a unique type among the types* **IV**-**V** *and a unique expanded R-irrational angle Φ of that type such that Φ_0 is $\mathcal{L}_+$-congruent to Φ_0*. The expanded R-irrational angle Φ_0 is said to be the *normal form* for the expanded R-irrational angle Φ.

Proof First, we prove that any two distinct expanded R-irrational angles listed in Definition 3.20 are not $\mathcal{L}_+$-congruent. Let us note that the revolution numbers of expanded angles distinguish the types of the angles. The revolution number for the expanded angles of the type **IV**$_k$ is $1/4 + 1/2k$ where $k \geq 0$. The revolution number for the expanded angles of the type **V**$_k$ is $1/4 - 1/2k$ where $k > 0$.

We now prove that the normal forms of the same type **IV**$_k$ (or **V**$_k$) are not $\mathcal{L}_+$-congruent for any integer k. Consider the expanded R-infinite angle $\Phi = k\pi + \operatorname{larctan}([a_0, a_1, \ldots])$. Suppose that a lattice oriented broken line $A_0A_1A_2\ldots$ at the unit distance from O defines the angle Φ. Let also the LSLS-sequence for this broken line be characteristic.

If k is even, then the ordinary R-irrational angle with the sail $A_{2k}A_{2k+1}\ldots$ is $\mathcal{L}_+$-congruent to $\operatorname{larctan}([a_0, a_1, \ldots])$. This angle is a proper lattice-affine invariant for the expanded R-irrational angle Φ (since $A_{2k} = A_0$). This invariant distinguish the expanded R-irrational angles of type **IV**$_k$ (or **V**$_k$) for even k.

If k is odd, then denote $B_i = V + \overline{A_iV}$. The ordinary R-irrational angle with the sail $B_{2k}B_{2k+1}\ldots$ is $\mathcal{L}_+$-congruent to the angle $\operatorname{larctan}([a_0, a_1, \ldots])$. This angle is a proper lattice-affine invariant of the expanded R-irrational angle Φ (since $B_{2k} = V + \overline{A_0V}$). This invariant distinguish the expanded R-irrational angles **IV**$_k$ (or **V**$_k$) for odd k.

Therefore, the expanded angles listed in Definition 3.20 are not $\mathcal{L}_+$-congruent.

Secondly, we prove that an arbitrary expanded R-irrational angle is $\mathcal{L}_+$-congruent to some of the expanded angles listed in Definition 3.20.

Consider an arbitrary expanded R-irrational angle $\Phi = \angle(V, A_0A_1\ldots)$. Suppose that $\#(\Phi) = 1/4 + k/2$ for some nonnegative integer k. By Proposition 3.12, there exist an integer positive number n_0 such that the lattice oriented broken line $A_{n_0}A_{n_0+1}\ldots$ does not intersect the rays $r_+ = \{V + \lambda\overline{VA_0} \mid \lambda \geq 0\}$ and $r_- = \{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}$, and the LSLS-sequence

$(a_{2n_0-2}, a_{2n_0-1}, \ldots)$ for the oriented broken line $A_{n_0}A_{n_0+1}\ldots$ does not contain nonpositive elements.

By Theorem 3.18, there exist integers k and m, and a lattice oriented broken line

$$A_0B_1B_2\ldots B_{2k}B_{2k+1}\ldots B_{2k+m}A_{n_0}$$

with LLS-sequence of the form

$$\big((1,-2,1,-2)\times k\text{-times}, b_0, b_1, \ldots, b_{2m-2}\big),$$

where all b_i are positives.

Consider now the lattice oriented infinite broken line $A_0B_1B_2\ldots B_{2k+m-1}A_{n_0}A_{n_0+1}\ldots$. The LLS-sequence for this broken line is as follows:

$$\big((1,-2,1,-2)\times k\text{-times}, b_0, b_1, \ldots, b_{2m-2}, v, a_{2n_0-2}a_{2n_0-1}, \ldots\big),$$

where v is (not necessarily positive) integer.

Note that the lattice oriented broken line $A_0B_1B_2\ldots B_{2k+m}A_{n_0}$ is a sail for the angle $\angle A_0VA_{n_0}$ and the broken line $A_{n_0}A_{n_0+1}\ldots$ is a sail for some R-irrational angle (we denote it by α). Let H_1 be the convex hull of all lattice points of the angle $\angle A_0VA_{n_0}$ except the origin, and H_2 be the convex hull of all lattice points of the angle α except the origin. Note that H_1 intersects H_2 in the ray with the vertex at A_{n_0}.

The lattice oriented infinite broken line $B_{2k}B_{2k+2}\ldots B_{2k+m}A_{n_0}A_{n_0+1}\ldots$ intersects the ray r_+ in the unique point B_{2k} and does not intersect the ray r_-. Hence there exists a straight line l intersecting both boundaries of H_1 and H_2, such that the open half-plane with the boundary straight line l containing the origin does not intersect the sets H_1 and H_2.

Denote $B_0 = A_0$ and $B_{2k+m+1} = A_{n_0}$. The intersection of the straight line l with H_1 is either a point B_s (for $2k \le s \le 2k+m+1$), or a boundary segment B_sB_{s+1} for some integer s satisfying $2k \le s \le 2k+m$. The intersection of l with H_2 is either a point A_t for some integer $t \ge n_0$, or a boundary segment $A_{t-1}A_t$ for some integer $t > n_0$.

Since the triangle $\triangle VA_tB_s$ does not contain interior points of H_1 and H_2, the lattice points of $\triangle VA_tB_s$ distinct from B are on the segment A_tB_s. Hence, the segment A_tB_s is at the unit lattice distance from the vertex V. Therefore, the lattice infinite oriented broken line

$$A_0B_1B_2\ldots B_sA_tA_{t+1}\ldots$$

is at the lattice unit distance.

Since the lattice oriented broken line $B_k\ldots B_sA_tA_{t+1}\ldots$ is convex, it is a sail for some R-irrational angle. (Actually, the case $B_s = A_t = A_{n_0}$ is also possible, then delete one of the copies of A_{n_0} from the sequence.) We denote this broken like by $C_{2k+1}C_{2k+2}\ldots$. The corresponding LSLS-sequence is $(c_{4k}, c_{4k+1}, c_{4k+2}, \ldots)$, where $c_i > 0$ for $i \ge 4k$. Thus the LSLS-sequence for the lattice ordered broken line $A_0B_1B_2\ldots B_{2k}C_{2k+1}C_{2k+2}\ldots$ is

$$\big((1,-2,1,-2)\times(k-1)\text{-times}, 1,-2,1, w, c_{4k}, c_{4k+1}, c_{4k+2}, \ldots\big),$$

where w is an integer that is not necessarily equivalent to -2.

Consider an expanded angle $\angle(V, A_0B_1B_2\ldots B_{2k}C_{2k+1})$. By Lemma 3.19, there exists a lattice oriented broken line $C_0\ldots C_{2k+1}$ with the vertices $C_0 = A_0$ and C_{2k+1} of the same equivalence class, such that $C_{2k} = B_{2k}$, and the LSLS-sequence for it is

$$\big((1,-2,1,-2)\times k\text{-times}, c_{4k}, c_{4k+1}\big).$$

Therefore, the lattice oriented R-infinite broken line $C_0C_1\ldots$ for the angle $\angle(V, A_0A_1\ldots)$ has the LSLS-sequence coinciding with the characteristic sequence for the following angle $k\pi + \operatorname{larctan}([c_{4k}, c_{4k+1}, \ldots])$. Therefore,

$$\Phi \hat{\cong} k\pi + \operatorname{larctan}([c_{4k}, c_{4k+1}, \ldots]).$$

This concludes the proof of the theorem for the case of nonnegative integer k.

The proof for the case of negative k repeats the proof for the nonnegative case and is omitted here. □

Let us give the definition of trigonometric functions for expanded R-irrational angles.

Definition 3.22 Consider an arbitrary expanded R-irrational angle Φ with the normal form $k\pi + \varphi$ for some integer k.
a) The ordinary R-irrational angle φ is said to be *associated* with the expanded R-irrational angle Φ.
b) The number $\operatorname{ltan}(\varphi)$ is called the lattice *tangent* of the expanded R-irrational angle Φ.

We continue now with the case of expanded L-irrational angles.

Definition 3.23 The expanded irrational angle $\angle(V, \ldots A_{i+2}A_{i+1}A_i\ldots)$ is said to be transpose to the expanded irrational angle $\angle(V, \ldots A_iA_{i+1}A_{i+2}\ldots)$ and denoted by $\big(\angle(V, \ldots A_iA_{i+1}A_{i+2}\ldots)\big)^t$.

Definition 3.24 Consider a lattice L-infinite oriented broken line $\ldots A_{-1}A_0$ at the unit distance from the origin O. Let also A_0 be the point $(1, 0)$, and the point A_{-1} be on the straight line $x = 1$. If the LSLS-sequence of the expanded L-irrational angle $\Phi_0 = \angle(O, \ldots A_{-1}A_0)$ coincides with the following sequence (we call it the *characteristic sequence* for the corresponding angle):
IV$_k$) $\big(\ldots, a_{-1}, a_0, (-2, 1, -2, 1) \times k\text{-times}\big)$, where $k \geq 0$, $a_i > 0$, for $i \leq 0$, then we denote the angle Φ_0 by $k\pi + \operatorname{larctan}^t([a_0, a_{-1}, \ldots])$ and say that Φ_0 is *of the type* **IV**$_k$;
V$_k$) $\big(\ldots, a_{-1}, a_0, (2, -1, 2, -1) \times k\text{-times}\big)$, where $k > 0$, $a_i > 0$, for $i \leq 0$, then we denote the angle Φ_0 by $-k\pi + \operatorname{larctan}^t([a_0, a_{-1}, \ldots])$ and say that Φ_0 is *of the type* **V**$_k$.

Theorem 3.25 *For any expanded L-irrational angle Φ there exist a unique type among the types* **IV**-**V** *and a unique expanded L-irrational angle Φ_0 of that type such that Φ is $\mathcal{L}_+$-congruent to Φ_0*. The expanded L-irrational angle Φ_0 is said to be the *normal form* for the expanded L-irrational angle Φ.

Proof After transposing the set of all angles and change of the orientation of the plane the statement of Theorem 3.25 coincides with the statement of Theorem 3.21. □

4 Sums of expanded angles and expanded irrational angles

Now we can give definitions of sums of ordinary angles, and ordinary R-irrational or/and L-irrational angles.

Definition 4.1 Consider expanded angles Φ_i, where $i = 1, \ldots, t$, an expanded R-irrational angle Φ_r, and an expanded L-irrational angle Φ_l. Let the characteristic LSLS-sequences for the normal forms of the angles Φ_i be $(a_{0,i}, a_{1,i}, \ldots, a_{2n_i,i})$; of Φ_r — be $(a_{0,r}, a_{1,r}, \ldots)$, and of Φ_l — be $(\ldots, a_{-1,l}, a_{0,l})$.

Let $M = (m_1, \ldots, m_{t-1})$ be some $(t-1)$-tuple of integers. The normal form of any expanded angle, corresponding to the LSLS-sequence

$$\big(a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t}\big)$$

is called the *M-sum of expanded angles Φ_i $(i = 1, \ldots, t)$*.

Let $M_R = (m_1, \ldots, m_{t-1}, m_r)$ be some t-tuple of integers. The normal form of any expanded angle, corresponding to the LSLS-sequence

$$\big(a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots$$
$$\ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t} m_r, a_{0,r}, a_{1,r}, \ldots\big)$$

is called the *M_R-sum of expanded angles Φ_i $(i = 1, \ldots, t)$ and Φ_r*.

Let $M_L = (m_l, m_1, \ldots, m_{t-1})$ be some t-tuple of integers. The normal form for any expanded angle, corresponding to the LSLS-sequence

$$\big(\ldots, a_{-1,l}, a_{0,l}, m_l, a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots$$
$$\ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t}\big)$$

is called the *M_L-sum of expanded angles Φ_l, and Φ_i $(i = 1, \ldots, t)$*.

Let $M_{LR} = (m_l, m_1, \ldots, m_{t-1}, m_r)$ be some $(t+1)$-tuple of integers. Any expanded LR-irrational angle, corresponding to the LSLS-sequence

$$\big(\ldots, a_{-1,l}, a_{0,l}, m_l, a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots$$
$$\ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t} m_r, a_{0,r}, a_{1,r}, \ldots\big)$$

is called a *M_{LR}-sum of expanded angles Φ_l, Φ_i $(i = 1, \ldots, t)$ and Φ_r*.

We denote the $(a_1, a_2, \ldots, a_{n-1})$-sum of angles $\Phi_1, \ldots, \Phi_n$ by

$$\Phi_1 +_{a_1} \Phi_2 +_{a_2} \ldots +_{a_{n-1}} \Phi_n.$$

Finally, we give a few examples of sums.

$$\operatorname{larctan} 1 +_n \operatorname{larctan} 1 \hat{\cong} \operatorname{larctan} \frac{n+2}{n+1} \quad (n > 0),$$

$$\operatorname{larctan} 3 +_{-1} \operatorname{larctan} \frac{5}{3} +_{-2} \operatorname{larctan} 1 \hat{\cong} 2\pi,$$

$$\operatorname{larctan} \frac{3}{2} +_1 \operatorname{larctan} \frac{1+\sqrt{5}}{2} \hat{\cong} \operatorname{larctan} \frac{1+\sqrt{5}}{2}.$$

5 Related questions and problems

We conclude the paper with the following questions and problems.

Problem 1 **a)** Find a natural definition of lattice tangents for L-irrational angles, and LR-irrational angles.
b) Find a natural definition of lattice sines and cosines for irrational angles (see also in [7]).

Problem 2 Does there exist a natural definition of the sums of
a) any expanded LR-irrational angle and any expanded angle;
b) any expanded R-irrational angle and any expanded angle;
c) any expanded angle and any expanded L-irrational angle?

Problem 3 Find an effective algorithm to verify whether or not two given almost-positive LSLS-sequences define $\mathcal{L}$-congruent expanded irrational angles.

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On existence and uniqueness conditions for an integer triangle with given angles

O. N. Karpenkov

The problem of describing integer convex polygons in integer-invariant terms is still open. At present it is known only that the number of integer convex polygons with integer area bounded above by n grows exponentially with respect to $n^{1/3}$ (see the papers [1] and [2]). In this note we give a complete description for the case of integer triangles. The author is grateful to V. I. Arnold, I. Bárány, and A. G. Khovanskii for their attention to this work and for useful remarks.

General definitions. We consider a two-dimensional oriented real affine plane and fix some system of coordinates OXY in this plane. A point of the plane is called an *integer* point if all its coordinates are integers. The convex hull of a finite number of integer points not all contained in one line is called an *integer convex polygon*. Consider a minimal set of points defining a given polygon. The points of this set are called the *vertices* of the polygon. Since all the vertices are on the boundary of the convex hull, they can be ordered in a cyclic counterclockwise or clockwise way: $A_1, \dots, A_n$. We call such a polygon *positively* or *negatively oriented*, respectively, and denote it by $A_1 \dots A_n$.

By an *angle* we mean an ordered set of two closed rays with common vertex and not contained in the same line. The rays are called *sides* of the angle, and their common vertex the *vertex* of the angle. An angle is called an *integer* angle if its vertex is an integer point and both sides contain integer points distinct from the vertex. An angle $\angle ABC$ of an oriented integer polygon with consecutive vertices A, B, and C is the integer angle with integer vertex B and sides BA and BC.

An affine transformation of the plane is said to be *integer-affine* if it preserves the set of all integer points. Two polygons $A_1 \dots A_n$ and $B_1 \dots B_n$ (two angles $\angle A_1A_2A_3$ and $\angle B_1B_2B_3$) are said to be *integer-equivalent* if there exists an integer-affine transformation of the plane taking the points A_i to the points B_i, for $i = 1, \dots, n$ (respectively, taking the rays A_2A_1 and A_2A_3 to the rays B_2B_1 and B_2B_3).

For any positive integer n and a point $A(x, y)$ we denote by nA the point with coordinates (nx, ny). The polygon $nA_0 \dots nA_k$ is said to be *n-homothetic* to the polygon $P = A_0 \dots A_k$ and is denoted by nP. Two polygons P_1 and P_2 are said to be *integer-homothetic* if there exist positive integers m_1 and m_2 such that m_1P_1 is integer-equivalent to m_2P_2.

Finite continued fractions. Let us complete the set of rationals with the operations $+$ and $1/*$ by the element ∞ and denote this completion by $\overline{\mathbb{Q}}$. We set $q \pm \infty = \infty$, $1/0 = \infty$, and $1/\infty = 0$ (the expressions $\infty \pm \infty$ are not defined).

To any finite sequence $(a_0, a_1, \dots, a_n)$ of integers we assign the element

$$a_0 + 1/(a_1 + 1/(a_2 + \dots + 1/(a_{n-1} + 1/a_n) \cdots))$$

of $\overline{\mathbb{Q}}$ and denote it by $q =]a_0, a_1, \dots, a_n[$. If the terms of a sequence $a_1, \dots, a_n$ are positive, then the expression for q is called an *ordinary continued fraction*.

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Proposition. *For any rational number there exists a unique ordinary continued fraction with an odd number of elements.*

For $q_i \in \mathbb{Q}$, $i = 1, \dots, k$, we consider the corresponding ordinary continued fractions $q_i =]a_{i,0}, a_{i,1}, \dots, a_{i,2n_i}[$ with an odd number of elements. Denote by $]q_1, q_2, \dots, q_k[$ the element

$$]a_{1,0}, a_{1,1}, \dots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \dots, a_{2,2n_2}, \dots, a_{k,0}, a_{k,1}, \dots, a_{k,2n_k}[\in \overline{\mathbb{Q}}.$$

Integer tangents. The *integer length* of a segment AB (denoted by $\mathrm{l}\ell(AB)$) is the ratio of its Euclidean length to the minimal Euclidean length of integer vectors with vertices in AB. The *integer (non-oriented) area* of a polygon P (denoted by $\mathrm{lS}(P)$) is twice the Euclidean area of the polygon.

Consider an arbitrary integer angle $\angle ABC$. The boundary of the convex hull of the set of all integer points except B in the convex hull of $\angle ABC$ is called the *sail* of the angle. The sail of the angle is a finite broken line with the first and the last vertices on different sides of the angle. Let us orient the broken line in the direction from the ray BA to the ray BC and denote its vertices by $A_0, \dots, A_{m+1}$. Let $a_i = \mathrm{l}\ell(A_iA_{i+1})$ for $i = 0, \dots, m$, and let $b_i = \mathrm{lS}(A_{i-1}A_iA_{i+1})$ for $i = 1, \dots, m$. The following rational number is called the *integer tangent of the angle* $\angle ABC$:

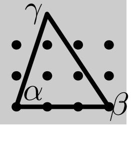
$$]a_0, b_1, a_1, b_2, a_2, \dots, b_m, a_m[\qquad (\text{notation: } \operatorname{ltan} \angle ABC).$$

Statement of the theorem. In plane Euclidean geometry an existence condition for a triangle with given angles can be written using the tangents of the angles in the following way. There exists a triangle with angles α, β, and γ if and only if $\tan(\alpha+\beta+\gamma) = 0$ and $\tan(\alpha+\beta) \notin [0, \tan\alpha]$ (without loss of generality, we suppose here that α is acute). We present an integer analogue of the latter statement.

Theorem. a) *Let $\alpha_0, \alpha_1,$ and α_2 be an ordered triple of integer angles. There exists an oriented integer triangle with consecutive angles integer-equivalent to the angles α_0, α_1, and α_2 if and only if there exists $j \in \{1, 2, 3\}$ such that the angles $\alpha = \alpha_j$, $\beta = \alpha_{j+1\,(\mathrm{mod}\,3)}$, and $\gamma = \alpha_{j+2\,(\mathrm{mod}\,3)}$ satisfy the following conditions*: i) $]\operatorname{ltan}\alpha, -1, \operatorname{ltan}\beta, -1, \operatorname{ltan}\gamma[= 0$; ii) $]\operatorname{ltan}\alpha, -1, \operatorname{ltan}\beta[\notin [0, \operatorname{ltan}\alpha]$.

b) *Two integer triangles with the same sequence of integer tangents are integer-homothetic.*

Note that in the hypothesis of the theorem we always take ordinary continued fractions with an odd number of elements for the tangents of the angles. We illustrate the theorem with the following particular example:



$\operatorname{ltan}\alpha = 3 =]3[$;
$\operatorname{ltan}\beta = 9/7 =]1, 3, 2[$;
$\operatorname{ltan}\gamma = 3/2 =]1, 1, 1[$;

i) $]3, -1, 1, 3, 2, -1, 1, 1, 1[= 0$;
ii) $]3, -1, 1, 3, 2[= -3/2 \notin [0, 3]$.

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CLASSIFICATION OF LATTICE-REGULAR LATTICE CONVEX POLYTOPES.

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Abstract. In this paper for any dimension n we give a complete description of lattice convex polytopes in $\mathbb{R}^n$ that are regular with respect to the group of affine transformations preserving the lattice.

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Introduction.

Consider an n-dimensional real vector space. Let us fix a full-rank lattice in it. A *convex polytope* is a convex hull of a finite number of points. A hyperplane π is said to be *supporting* for a (closed) convex polytope P, if the intersections of P and π is not empty, and the whole polytope P is contained in one of the closed half-spaces bounded by π. An intersection of any polytope P with any its supporting hyperplane is called a *face* of the polytope. Zero- and one-dimensional faces are called *vertices* and *faces*.

Consider an arbitrary n-dimensional convex polytope P. An arbitrary unordered $(n+1)$-tuple of faces containing the whole polytope P, some its hyperface, some hyperface of this hyperface, and so on (up to a vertex of P) is called a *face-flag* for the polytope P.

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A convex polytope is said to be lattice if all its vertices are lattice points. An affine transformation is called *lattice-affine* if it preserves the lattice. Two convex lattice polytopes are said to be *lattice-congruent* if there exist a lattice-affine transformation taking one polytope to the other. A lattice polytope is called *lattice-regular* if for any two its face-flags there exist a lattice-affine transformation preserving the polytope and taking one face-flag to the other.

In this paper we give a complete description of lattice-regular convex lattice polytopes in $\mathbb{R}^n$ for an arbitrary n (Theorem 2.2 in Section 2).

The study of convex lattice polytopes is actual in different branches of mathematics, such as lattice geometry(see, for example [3], [4], [8], [18]), geometry of toric varieties (see [7], [10], [17]) and multidimensional continued fractions (see [1], [11], [9], [12], [16]). Mostly, it is naturally to study such polytopes with respect to the lattice-congruence equivalence relation.

Now we formulate two classical examples of unsolved problems on convex lattice polytopes. The first one comes from the geometry of toric varieties.

Problem 1. Find a complete invariant of lattice-congruence classes of convex lattice (two-dimensional) polygons.

Only some estimates are known at this moment (see for example [2] and [4]).

The second problem comes from lattice geometry and theory of multi-dimensional continued fractions. A lattice symplex is called *empty* if the intersection of this (solid) symplex with the lattice coincides with the set of its vertices.

Problem 2. Find a description of lattice-congruence classes of empty symplices.

The answer to the second problem in the two-dimensional case is simple. All empty triangles are lattice-congruent. Tree-dimensional case is much more complicated. The key to the description gives White's theorem (1964) shown in [20] (for more information see [18], [16], and [9]).

The problems similar to the shown above are complicated and seem not to be solved in the nearest future. Nevertheless, specialists of algebraic geometry or theory of multidimensional continued fractions usually do not need the complete classifications but just some special examples.

In the present paper we make the first steps in the study of the lattice polytopes with non-trivial group of *lattice-symmetries* (i.e. the group of lattice-affine transformations, preserving the polytope). We describe the "maximally" possible lattice-symmetric polytopes: the lattice-regular polytopes.

Let us formulate statement for the second step in the study of the lattice polytopes with non-trivial group of lattice-symmetries. A convex lattice pyramid P with the base B is said to be *lattice-regular* if B is a lattice-regular polytope, and the group of lattice-symmetries of the base B (in the hyperplane containing B) is expandable to the group of lattice-symmetries of the whole pyramid P.

Problem 3. Find a description of lattice-regular convex lattice pyramids.

This paper is organized as follows. We give a well-known classical description of Euclidean and abstract regular polytopes in terms of Schläfli symbols in Section 1. In Section 2 we give necessary definitions of lattice geometry and formulate a new theorem on lattice-affine classification of lattice-regular convex polytopes. Further in Section 2 we prove this theorem for the two-dimensional case. We study the cases (in any dimension) of lattice-regular symplices, cubes, and generalized octahedra in Sections 4, 5, and 6 respectively. Finally in Section 7 we investigate the remaining cases of low-dimensional polytopes and conclude the proof of the main theorem.

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1. Euclidean and abstract regular polytopes.

For the proof of the main theorem on lattice-regular polytopes we use the classification of abstract convex polytopes. We start this section with the description of Euclidean regular polytopes and Schläfli symbols for them, and then continue with the case of abstract regular polytopes.

1.1. **Euclidean regular polytopes.** Consider an arbitrary n-dimensional Euclidean regular polytope P. Let $(F_n{=}P, F_{n-1}, \ldots, F_1, F_0)$ be one of its flags. Denote by O_i the mass center of the face F_i considered as a homogeneous solid body (for $i = 0, \ldots, n$). The n-dimensional tetrahedron $O_0O_1 \ldots O_{n-1}O_n$ is called the *chamber* of a regular polytope P corresponding to the given flag. Denote by r_i (for $i = 0, \ldots, n-1$) the reflection about the $(n{-}1)$-dimensional plane spanning the points $F_n, \ldots, F_{i+1}$, $F_{i-1}, \ldots, F_0$. See Figure 1. These reflections are sometimes called *basic*.

The following classical statement holds.

Statement 1.1. *The reflections $r_0, r_1, \ldots, r_{n-1}$ generate the group of Euclidean symmetries of the Euclidean regular polytope P.*
For $i = 1, \ldots, n{-}1$ the angle between the fixed hyperplanes of the symmetries r_{i-1} and r_i equals π/a_i, where a_i is an integer greater than or equivalent to 3.

The symbol $\{a_1, \ldots, a_{n-1}\}$ is said to be the *Schläfli symbol* for the polygon P. Traditionally, the string $a, a, \ldots, a$ of the length s in Schläfli symbol is replaced by the symbol a^s.

Since all face-flags of any regular polytope are congruent, the Schläfli symbol is well-defined.

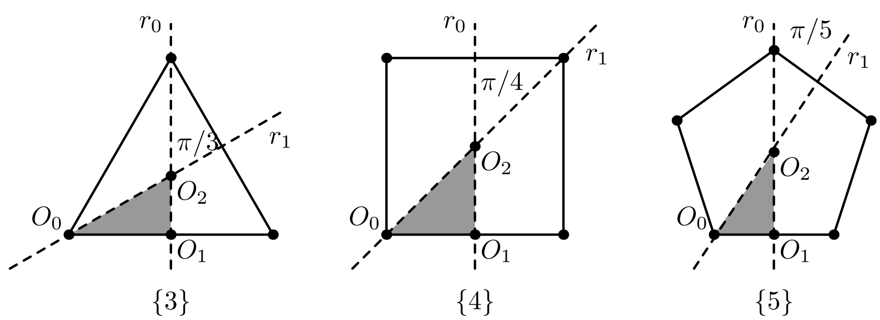

Figure 1. Basic reflections and Schläfli symbols for some regular polygons.

Theorem A. On classification of regular Euclidean polytopes. *Any regular convex Euclidean polytope is homothetic to some polytope of the following list.*

List of regular Euclidean polytopes.

Dimension 1: *a segment with Schläfli symbol* $\{\}$.
Dimension 2: *a regular polygon with m vertices (for any $m \geq 3$) with Schläfli symbol $\{m\}$.*
Dimension 3: *a regular tetrahedron ($\{3,3\}$), a regular octahedron ($\{3,4\}$), a regular cube ($\{4,3\}$), a regular icosahedron ($\{3,5\}$), a regular dodecahedron ($\{5,3\}$).*
Dimension 4: *a regular symplex ($\{3,3,3\}$), a regular cube ($\{4,3,3\}$), a regular generalized octahedron (or cross polytope, or hyperoctahedron; with Schläfli symbol $\{3,3,4\}$), a regular 24-cell (or hyperdiamond, or icositetrachoron; with Schläfli symbol $\{3,4,3\}$), a regular 600-cell (or hypericosahedron, or hexacosichoron; with Schläfli symbol $\{3,3,5\}$), a regular 120-cell (or hyperdodecahedron, or hecatonicosachoronor; with Schläfli symbol $\{5,3,3\}$).* **Dimension n (n>4):** *a regular symplex ($\{3^{n-1}\}$), a regular cube ($\{4,3^{n-2}\}$), a regular generalized octahedra ($\{3^{n-2},4\}$).*

Remark 1.2. The cases of dimension one, two, and three were already known to the ancient mathematicians. The cases of higher dimensions were studied by Schläfli (see in [19]).

1.2. **Abstract regular polytopes.** In this subsection we consider arbitrary convex polytopes. Consider two n-dimensional polytopes. A homeomorphism of two n-dimensional polytopes is said to be *combinatorical* if it takes any face of one polytope to some face of the same dimension of the other polytope. Two polytopes are called *combinatorically isomorphic* if there exist a combinatorical homeomorphism between them.

A convex polytope is called *combinatorical regular* if for any two its face-flags there exist a combinatorical homeomorphism taking the polytope to itself and one face-flag to the other.

Theorem B. (McMullen [13]**.)** *A polytope is combinatorical regular iff it is combinatorically isomorphic to a regular polytope.*

The proof of this statement essentially uses the work of Coxeter [5].

Remark 1.3. Theorem B implies the classification of real affine and projective polytopes (see [14]). Both classifications coincide with the classification of Euclidean regular polytopes. For further investigations of abstract polytopes see for example the work of L. Danzer and E. Schulte [6] and the book on abstract regular polytopes by P. McMullen and E. Schulte [15].

2. Definitions and formulation of the main result.

Let us fix some basis of lattice vectors $\overline{e}_i$ for $i = 1, \ldots, n$ generating the lattice in $\mathbb{R}^n$. Denote by O the origin in $\mathbb{R}^n$.

Consider arbitrary non-zero integers $n_1, \ldots, n_k$ for $k \geq 2$. By $\gcd(n_1, \ldots, n_k)$ we denote the greater common divisor of the integers n_i, where $i = 1, \ldots, k$. We write that $a \equiv b (\operatorname{mod} c)$ if the reminders of a and b modulo c coincide.

2.1. **Some definitions of lattice geometry.** Let Q be an arbitrary lattice polytope with the vertices $A_i = O + \overline{v}_i$ (where $\overline{v}_i$ — lattice vectors) for $i = 1, \ldots, m$, and t be an arbitrary positive integer. The polygon P with the vertices $B_i = O + t\overline{v}_i$ for $i = 1, \ldots, m$ is said to be the *t-multiple* of the polygon Q.

Definition 2.1. A lattice polytope P is said to be *elementary* if for any integer $t > 1$ and any lattice polytope Q the polytope P is not lattice-congruent to the t-multiple of the lattice polytope Q.

2.2. **Notation for particular lattice polytopes.** We will use the following notation.

Symplices. For any $n > 1$ we denote by $\{3^{n-1}\}_p^L$ the n-dimensional symplex with the vertices:

$$V_0 = O, \qquad V_i = O + \overline{e}_i, \text{ for } i = 1, \ldots, n-1, \quad \text{and} \quad V_n = (p-1)\sum_{k=1}^{n-1} \overline{e}_k + p\overline{e}_n.$$

Cubes. Any lattice cube is generated by some lattice point P and a n-tuple of linearly independent lattice vectors $\overline{v}_i$:

$$\Big\{ P + \sum_{i=1}^{n} \alpha_i \overline{v}_i \Big| 0 \leq \alpha_i \leq 1, i = 1, \ldots, n \Big\}$$

We denote by $\{4, 3^{n-2}\}_1^L$ for any $n \geq 2$ the lattice cube with a vertex at the origin and generated by all basis vectors.
By $\{4, 3^{n-2}\}_2^L$ for any $n \geq 2$ we denote the lattice cube with a vertex at the origin and generated by the first $n-1$ basis vectors and the vector $\overline{e}_1 + \overline{e}_2 + \ldots + \overline{e}_{n-1} + 2\overline{e}_n$.
By $\{4, 3^{n-2}\}_3^L$ for any $n \geq 3$ we denote the lattice cube with a vertex at the origin and generated by the vectors: $\overline{e}_1$, and $\overline{e}_1 + 2\overline{e}_i$ for $i = 2, \ldots, n$.

Generalized octahedra. We denote by $\{3^{n-2}, 4\}_1^L$ for any $n \geq 2$ the lattice generalized octahedron with the vertices $O \pm \overline{e}_i$ for $i = 1, \ldots, n$.
By $\{3^{n-2}, 4\}_2^L$ for any positive n we denote the lattice generalized octahedron with the vertices

$O \pm \overline{e}_i$ for $i = 1, \ldots, n-1$, and $O \pm \big(\overline{e}_1 + \overline{e}_2 + \ldots + \overline{e}_{n-1} + 2\overline{e}_n\big)$.
By $\{3^{n-2}, 4\}_3^L$ for any positive n we denote the lattice generalized octahedron with the vertices O, $O - \overline{e}_1$, $O - \overline{e}_1 - \overline{e}_i$ for $i = 2, \ldots, n$, and e_i for $i = 2, \ldots, n$.

A segment, octagons, and 24-sells. Denote by $\{\}^L$ the lattice segment with the vertices O and $O + \overline{e}_1$.
By $\{6\}_1^L$ we denote the hexagon with the vertices $O \pm \overline{e}_1$, $O \pm \overline{e}_2$, $O \pm (\overline{e}_1 - \overline{e}_2)$.
By $\{6\}_2^L$ we denote the hexagon with the vertices $O \pm (2\overline{e}_1 + \overline{e}_2)$, $O \pm (\overline{e}_1 + 2\overline{e}_2)$, $O \pm (\overline{e}_1 - \overline{e}_2)$.
By $\{3, 4, 3\}_1^L$ we denote the 24-sell with 8 vertices of the form

$$O \pm 2(\overline{e}_2 + \overline{e}_3 + \overline{e}_4), \quad O \pm 2(\overline{e}_1 + \overline{e}_2 + \overline{e}_4), \quad O \pm 2(\overline{e}_1 + \overline{e}_3 + \overline{e}_4), \quad O \pm 2\overline{e}_4,$$

and 16 vertices of the form

$$O \pm (\overline{e}_2 + \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_2 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_3 + \overline{e}_4) \pm \overline{e}_4.$$

By $\{3, 4, 3\}_2^L$ we denote the 24-sell with 8 vertices of the form

$$\begin{array}{ll} O \pm 2(\overline{e}_1 + \overline{e}_2 + \overline{e}_3 + \overline{e}_4), & O \pm 2(\overline{e}_1 - \overline{e}_2 + \overline{e}_3 + \overline{e}_4), \\ O \pm 2(\overline{e}_1 + \overline{e}_2 - \overline{e}_3 + \overline{e}_4), & O \pm 2(\overline{e}_1 + \overline{e}_2 + \overline{e}_3 - \overline{e}_4), \end{array}$$

and 16 vertices of the form

$$O \pm (\overline{e}_1 + \overline{e}_2 + \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 - \overline{e}_2 + \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_2 - \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_2 + \overline{e}_3 - \overline{e}_4).$$

2.3. **Theorem on enumeration of convex elementary lattice-regular lattice polytopes.** Now we formulate the main statement of the work.

Theorem 2.2. *Any elementary lattice-regular convex lattice polytope is lattice-congruent to some polytope of the following list.*

List of the polygons.
Dimension 1: *the segment* $\{\}^L$.
Dimension 2: *the triangles* $\{3\}_1^L$ *and* $\{3\}_2^L$*;*
the squares $\{4\}_1^L$ *and* $\{4\}_2^L$*;*
the octagons $\{6\}_1^L$ *and* $\{6\}_2^L$.
Dimension 3: *the tetrahedra* $\{3, 3\}_i^L$, *for* $i = 1, 2, 4$*;*
the octahedra $\{3, 4\}_i^L$, *for* $i = 1, 2, 3$*;*
the cubes $\{4, 3\}_i^L$, *for* $i = 1, 2, 3$.
Dimension 4: *the symplices* $\{3, 3, 3\}_1^L$ *and* $\{3, 3, 3\}_5^L$*;*
the generalized octahedra $\{3, 3, 4\}_i^L$, *for* $i = 1, 2, 3$*;*
the 24-sells $\{3, 4, 3\}_1^L$ *and* $\{3, 4, 3\}_2^L$*;*
the cubes $\{4, 3, 3\}_i^L$, *for* $i = 1, 2, 3$.
Dimension n (n>4): *the symplices* $\{3^{n-1}\}_i^L$ *where positive integers* i *are divisors of* $n+1$*;*
the generalized octahedra $\{3^{n-2}, 4\}_i^L$, *for* $i = 1, 2, 3$*;*
the cubes $\{4, 3^{n-2}\}_i^L$, *for* $i = 1, 2, 3$.

All polytopes of this list are lattice-regular. Any two polytopes of the list are not lattice-congruent to each other.

On Figure 2 we show the adjacency diagram for the elementary lattice-regular convex lattice polygons of dimension not exceeding 7. Lattice-regular lattice polygons of different (six) types are shown on Figure 3 in the next section. Lattice-regular lattice three-dimensional polygons of different (nine) types are shown on Figures 4, 5, and 6 further in Sections 4, 5, and 6 respectively.

Further in the proofs we will use the following definition. Consider a k-dimensional lattice polytope P. Let its Euclidean volume equal V. Denote the Euclidean volume of the minimal k-dimensional symplex in the k-dimensional plane of the polytope by V_0. The ratio V/V_0 is said to be the *lattice volume* of the given polytope (if $k = 1$, or 2 — the *lattice length* of the segment, or the *lattice area* of the polygon respectively).

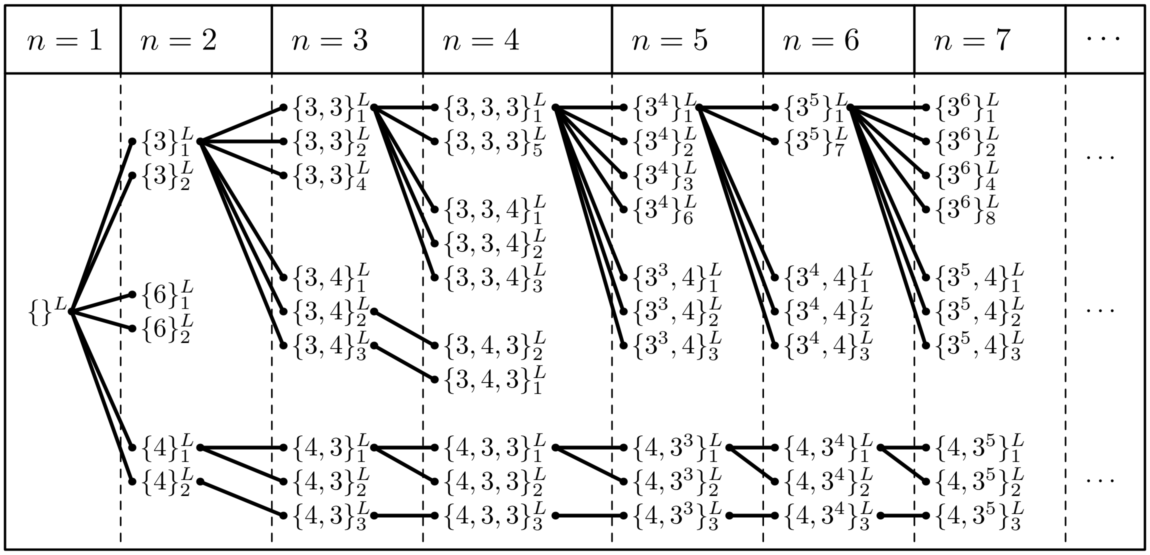

FIGURE 2. The adjacency diagram for the elementary lattice-regular convex lattice polytopes.

3. TWO-DIMENSIONAL CASE.

In this section we prove Theorem 2.2 for the two-dimensional case.

Proposition 3.1. *Any elementary lattice-regular (two-dimensional) lattice convex polygon is lattice-congruent to one of the following polygons (see Figure 3):*
1) $\{3\}_1^{\mathbb{Z}}$; **2)** $\{3\}_2^{\mathbb{Z}}$; **3)** $\{4\}_1^{\mathbb{Z}}$; **4)** $\{4\}_2^{\mathbb{Z}}$; **5)** $\{6\}_1^{\mathbb{Z}}$; **6)** $\{6\}_2^{\mathbb{Z}}$.

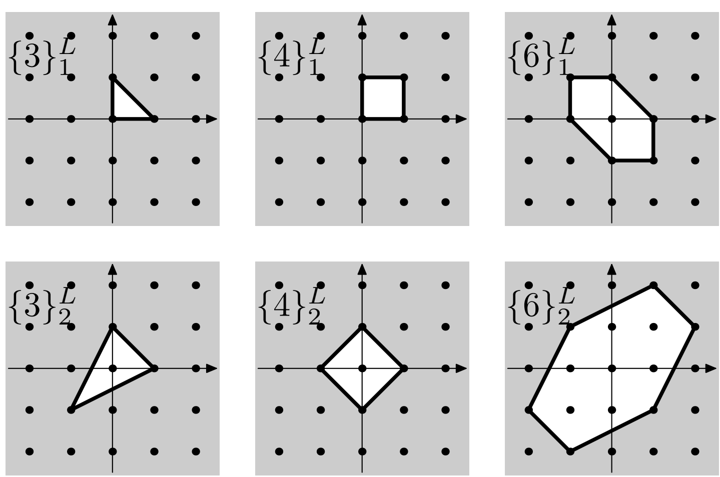

FIGURE 3. The lattice-regular polygons with edges of unit length.

Proof. Suppose that the lattice polygon $A_1A_2\ldots A_n$, where $n \geq 3$, is primitive and lattice-regular.

Let us prove that all edges of $A_1A_2\ldots A_n$ are of unite lattice lengths. Since the polygon is lattice-regular, all its edges are lattice-congruent, and hence they are of the same lattice length. Suppose that the lattice lengths of all edges equal some positive integer k. Then our polygon is lattice-congruent to the k-tuple of the polygon $A'_1A'_2\ldots A'_n$, where $A'_1 = A_1$, and $A'_l = A_l{+}1/k(\overline{A_lA_{l+1}})$ for $l = 2,\ldots,n{-}1$. Therefore, $k = 1$.

Denote by B_i the midpoint of the edge A_iA_{i+1} for $i = 1, \ldots, n-1$, and by B_n the midpoint of the edge A_nA_1. Suppose that n is even ($n = 2\tilde{n}$). Denote by M the midpoint of the segment $A_1A_{\tilde{n}+1}$. Note that the point M is the common intersection of the segments $A_iA_{\tilde{n}+i}$ for $i = 1, \ldots, \tilde{n}$. Suppose that n is odd ($n = 2\tilde{n}+1$). Denote by M the intersection point of the segments $A_1B_{\tilde{n}}$ and $A_2B_{\tilde{n}+1}$. Note that the point M is the common intersection of the segments $A_iB_{\tilde{n}+i}$, $A_{\tilde{n}+i+1}B_i$ for $i = 1, \ldots, \tilde{n}$, and the segment $A_{\tilde{n}}B_{2\tilde{n}+1}$.

For any integer i such that $1 \leq i \leq n$ the following holds. The transformation that preserves the points M and B_i, and taking the point A_i to the point A_{i+1} (or A_n to A_1 in the case of $i = n$) is lattice-affine and preserve the polygon $A_1A_2 \ldots A_n$.

Suppose that the polygon $A_1A_2 \ldots A_n$ contains some lattice point not contained in the union of its vertices and segments MB_i for $i = 1, \ldots, n$. Then by symmetry reasons the triangle A_1MB_1 contains at least one lattice point, that is not contained in the edges A_1B_1 and MB_1. Denote one of such points by P. Let Q be the point symmetric to the point P about the line MB_1. The segment PQ is parallel to the segment, and hence the lattice point $A_1 + \overline{PQ}$ is contained in the interior of the segment A_1A_2. Then the lattice length of the edge A_1A_2 is not unit. We come to the contradiction with the above.

Therefore, all inner lattice points of the polygon $A_1A_2 \ldots A_n$ are contained in the union of the segments MB_i for $i = 1, \ldots, n$ and vertices. Now we study all different cases of configurations of lattice points on the segment MB_1.

Case 1. Suppose that MB_1 does not contain lattice points. Then by symmetry reasons the polygon $A_1A_2 \ldots A_n$ does not contain lattice points different to its vertices. Hence the vectors $\overline{A_2A_1}$ and $\overline{A_2A_3}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, A_2, and A_3 have the coordinates $(0, 1)$, $(0, 0)$, and $(1, 0)$ in it respectively.

If $n = 3$, then the triangle $A_1A_2A_3$ is lattice-congruent to the triangle $\{3\}_1^L$.

Let $n > 3$. Since the vectors $\overline{A_1A_2}$ and $\overline{A_2A_3}$ generate the lattice, and the vectors $\overline{A_2A_3}$ and $\overline{A_3A_4}$ generate the lattice, the point A_4 has the coordinates $(a, 1)$ for some integer a. Since the segment A_1A_4 does not contain lattice points distinct to the endpoints, $A_4 = (1, 1)$. By the same reasons $A_n = (1, 1)$. Therefore, $n = 4$, and the lattice polygon $A_1A_2A_3A_4$ is lattice-congruent to the lattice-regular quadrangle $\{4\}_1^L$.

Case 2. Suppose that the point M is lattice and the segment MB_1 does not contain lattice points distinct to M. Then the vectors $\overline{MA_1}$ and $\overline{MA_2}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, M, and A_2 have the coordinates $(0, 1)$, $(0, 0)$, and $(1, 0)$ in it. Since the vectors $\overline{A_1M}$ and $\overline{MA_2}$ generate the lattice, and the vectors $\overline{A_2M}$ and $\overline{MA_3}$ generate the lattice, the point A_3 has the coordinates $(-1, a)$ for some integer a.

If $a \geq 2$, then the polygon is not convex or it contains straight angles.

If $a = 1$, then the vectors $\overline{A_1A_2}$ and $\overline{A_2A_3}$ generate the lattice. Since the vectors $\overline{A_3M}$ and $\overline{MA_4}$ generate the lattice, and the vectors $\overline{A_2A_3}$ and $\overline{A_3A_4}$ generate the lattice, the new coordinates of the point A_4 are $(0, -1)$. Since $A_4 = A_1 + 2\overline{A_1M}$, we have

$$A_5 = A_2 + 2\overline{A_2M} = (0, -1), \qquad A_6 = A_3 + 2\overline{A_3M} = (1, -1), \quad \text{and} \quad n = 6.$$

Therefore, the lattice-regular polygon $A_1A_2A_3A_4A_5A_6$ is lattice-congruent to the lattice-regular hexagon $\{6\}_1^L$.

If $a = 2$, then $A_3 = A_1 + 2\overline{A_1M}$. Hence $A_3 = A_2 + 2\overline{A_2M} = (0, -1)$, and $n = 4$. Therefore, the polygon $A_1A_2A_3A_4$ is lattice-congruent to the lattice-regular quadrangle $\{4\}_2^L$.

If $a = 3$, then A_3 is contained in the line MB_1. Hence $n = 3$. Therefore, the lattice triangle $\triangle A_1A_2A_3$ is lattice-congruent to the lattice-regular triangle $\{3\}_2$.

Since $A_n = (a, -1)$, the edges A_nA_1 and A_1A_2 intersect for the case of $a > 3$.

Case 3. Suppose that the segment MB_1 contains the unique lattice point P distinct from the endpoints of the segment MB_1. Then the vectors $\overline{PA_1}$ and $\overline{PA_2}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, P, and A_2 have the coordinates $(0,1)$, $(0,0)$, and $(1,0)$ in it. Since the polygon is lattice-regular, the point $M+\overline{PM}$ is also a lattice point, and hence $M=(-1/2,-1/2)$. Denote the point (-1,-2) by M'. (Note that the point M is the midpoint of the segment $M'B$).

The vectors $\overline{A_1M'}$ and $\overline{M'A_2}$ generate a sublattice of index 3. The vectors $\overline{A_2M'}$ and $\overline{M'A_3}$ generate a sublattice of index 3. The segment A_2A_3 is of unit lattice length. Therefore, the point A_3 has the coordinates $(2a{-}1, 6a{+}2)$ for some integer a.

If $a \ge 0$, then the polygon is not convex, or it contains straight angles. Since $A_n=(6a{+}2, 2a{-}1)$, the edges A_nA_1 and A_1A_2 intersect for the case of $a<0$.

Case 4. Suppose that the point M is lattice and the segment MB_1 contains the unique interior lattice point P. Then the vectors $\overline{PA_1}$ and $\overline{PA_2}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, P, and A_2 have the coordinates $(0,1)$, $(0,0)$, and $(1,0)$ in it. Since the polygon is lattice-regular, the point $M+\overline{PM}$ is also lattice, and hence $M=(-1,-1)$.

The vectors $\overline{A_1M'}$ and $\overline{M'A_2}$ generate a sublattice of index 3. The vectors $\overline{A_2M'}$ and $\overline{M'A_3}$ generate a sublattice of index 3. The segment A_2A_3 is of unit lattice length. Therefore, the point A_3 has the coordinates $(a{-}1, 2a{+}2)$ for some integer a, such that $a \not\equiv 1 (\operatorname{mod} 3)$.

If $a \ge 0$, then the polygon is not convex, or it contains straight angles, but this is impossible.

If $a=-1$, then the vectors $\overline{A_1A_2}$ and $\overline{A_2A_3}$ generate a sublattice of index 3. Since the vectors $\overline{A_3M'}$ and $\overline{M'A_4}$ generate a sublattice of index 3, and the vectors $\overline{A_2A_3}$ and $\overline{A_3A_4}$ generate a sublattice of index 3, the point $A_4=(-3,-2)$. Since $A_4=A_1{+}2\overline{A_1M}$, we have

$$A_5 = A_2 + 2\overline{A_2M} = (-2,-3), \qquad A_6 = A_3 + 2\overline{A_3M} = (0,-2), \quad \text{and} \quad n=6.$$

Therefore, the lattice polygon $A_1A_2A_3A_4A_5A_6$ is lattice-congruent to the lattice-regular hexagon $\{6\}_2^L$.

Since $A_n=(2a{+}2, a{-}1)$, the edges A_nA_1 and A_1A_2 are intersecting for the case of $a<-1$.

The remaining cases. Suppose that the segment MB_1 contains at least two interior lattice points. Let P_1 and P_2 be two distinct interior lattice points of the segment MB_1. Let also the segment MP_2 contains the point P_1.

Consider an lattice-affine transformation ξ taking the point M to itself, and the segment A_1A_2 to the segment A_2A_3. The points $Q_1=\xi(P_1)$ and $Q_2=\xi(P_2)$ are contained in the segment MB_2. Since the lines P_1Q_1 and P_2Q_2 are parallel and the triangle P_2MQ_2 contains the segment P_1Q_1, the lattice point $S=P_2{+}\overline{Q_2P_2}$ is contained in the interior of the segment P_1Q_1. Hence the lattice point S of the polygon $A_1A_2\ldots A_n$ is not contained in the union of segments MB_i for $i=1,\ldots,n$. We come to the contradiction.

We have studied all possible cases of configurations of lattice points contained inside lattice polygons. The proof of Proposition 3.1 is completed. □

4. Lattice-regular lattice symplices.

In this section we study all lattice-regular lattice symplices for all integer dimensions.

Let us fix some basis of lattice vectors $\overline{e}_i$, for $i=1,\ldots,n$ generating the lattice in $\mathbb{R}^n$ and the corresponding coordinate system. Denote by O the origin in $\mathbb{R}^n$.

Proposition 4.1. Sym$_1$. *All elementary lattice-regular one-dimensional lattice symplices are lattice segments of unit lattice length.*

Sym$_n$ (for $n>1$). *i) The symplex $\{3^{n-1}\}_n^L$ where p is a positive divisor of $n{+}1$ is elementary and lattice-regular;*
ii) any two symplices listed in (i) are not lattice-congruent to each other;
iii) any elementary lattice-regular n-dimensional lattice symplex is lattice-congruent to one of the symplices listed in (i).

The three-dimensional tetrahedra $\{3,3\}_1^L$, $\{3,3\}_2^L$, and $\{3,3\}_4^L$ are shown on Figure 4.

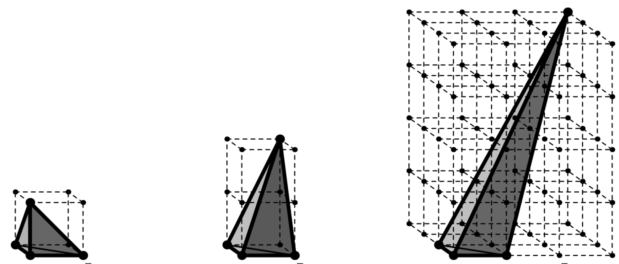

FIGURE 4. Three-dimensional elementary lattice-regular convex lattice tetrahedra.

Proof. We start the proof of studying of some low-dimensional cases. The one-dimensional case is trivial and is omitted here. The two-dimensional case was described in Proposition 3.1. Let us study the three-dimensional case.

Three-dimensional case. Consider an arbitrary elementary lattice-regular three-dimensional lattice tetrahedron S. Since its faces are lattice-regular, by Proposition 3.1 the faces are lattice-congruent either to $\{3\}_1^L$ or to $\{3\}_3^L$.

Suppose that the faces of S are lattice-congruent to $\{3\}_1^L$. Then there exist a positive integer b, nonnegative integers a_1, a_2 less than b, and a lattice-affine transformation taking the the tetrahedron S to the tetrahedron S' with the vertices

$$V_0 = O, \quad V_1 = O + \overline{e}_1, \quad V_2 = O + \overline{e}_2, \quad V_3 = O + a_1\overline{e}_1 + a_2\overline{e}_2 + b\overline{e}_3.$$

Since S' is also a lattice-regular tetrahedron, the group of its symmetries is isomorphic to the group of permutations of order 4. This group is generated by the following transpositions of vertices: V_1 and V_2, V_2 and V_3, and V_0 and V_2. The first two transpositions are linear, and the third one is linear after shifting by the vector $-\overline{e}_1$. Direct calculations shows, that the matrices of the corresponding linear transformations are the following:

$$\begin{pmatrix} 0 & 1 & \frac{a_1-a_2}{b} \\ 1 & 0 & \frac{a_2-a_1}{b} \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a_1 & -\frac{a_1(a_2+1)}{b} \\ 0 & a_2 & \frac{1-a_2^2}{b} \\ 0 & b & -a_2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & \frac{2a_2+a_1-1}{b} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the listed transformations are lattice-linear we have only the following possibilities, all these matrices are integer. Therefore, $a_1 \equiv a_2 \equiv b{-}1 (\mathrm{mod}\, b)$. Since, positives a_1 and a_2 was chosen to be smaller than b, we have $a_1 = a_2 = b{-}1$. Since the matrix A_3 is integer, the coefficient $3{-}4/b$ is also integer. So we have to check only the following cases for a_1, a_2, and b: $b = 1$, and $a_1 = a_2 = 0$; $b = 2$, and $a_1 = a_2 = 1$; $b = 4$, and $a_1 = a_2 = 3$. These cases corresponds to the tetrahedra $\{3,3\}_1^L$, $\{3,3\}_2^L$, and $\{3,3\}_4^L$ respectively. Since the lattice volume of $\{3,3\}_p^L$ equals p, the above tetrahedra are not lattice-congruent.

Let us prove that the faces of S are not lattice-congruent to $\{3\}_3^L$ by reductio ad absurdum. Suppose it is so. Let V_0, V_1, V_2, and V_3 be the vertices of S. Since the faces $V_0V_2V_3$ and $V_1V_2V_3$ are congruent to $\{3\}_3^L$, the face $V_0V_2V_3$ contains a unique lattice point in its interior (we denote it by P_1), and the face $V_1V_2V_3$ contains a unique lattice point in its interior (we denote it by P_2). Consider a lattice-symmetry of S permuting V_0 and V_1 and preserving V_2 and V_3. This symmetry takes the face $V_0V_2V_3$ to the face $V_1V_2V_3$, and hence it maps the point P_1 to P_2. Thus, the lattice vector $\overline{P_1P_2}$ is parallel to the vector V_0V_1. Hence, the point $V_0 + \overline{P_1P_2}$ is interior lattice point of the segment V_0V_1. Therefore, the segment V_0V_1 is not of unit lattice length and the face $V_0V_1V_2$ is not lattice-congruent to $\{3\}_3^L$. We come to the contradiction.

This completes the proof of Proposition 4.1 for the three-dimensional case.

In the higher dimensional case we first study two certain families of symplices.

The first type of n-dimensional symplices. Consider any symplex (for $n>3$) with the vertices

$$V_0=O, \qquad V_k=O+\overline{e}_i, \text{ for } k=1,\ldots,n{-}1, \quad \text{and} \quad V_n=O+\sum_{i=1}^{n}a_i\overline{e}_i,$$

we denote it by $S^n(a_1,\ldots,a_n)$. We suppose that all a_i are nonnegative integers satisfying $a_k<a_n$ for $k=1,\ldots,n{-}1$. Let us find the conditions on a_i for the symplex to be lattice-regular.

If $S^n(p;a_1,\ldots,a_n)$ is a lattice-regular symplex then the group of its symmetries is isomorphic to the group of permutations of order $n{+}1$. This group is generated by the following transpositions of vertices: the transposition exchanging V_k and V_{k+1} for $k=1,\ldots,n{-}1$, and the transposition exchanging V_0 and V_2. The first $n{-}1$ transpositions are linear (let their matrices be A_k for $k=1,\ldots,n{-}1$), and the last one is linear after shifting by the vector $-\overline{e}_1$ (denote the corresponding matrix by A_n). Let us describe the matrices of these transformations explicitly.

The matrix A_k for $k=1,\ldots n{-}2$ coincides with the matrix transposing the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$, except the last column. The n-th column contains the coordinates of the vector

$$\tfrac{a_k-a_{k+1}}{a_n}(\overline{e}_k-\overline{e}_{k+1})+\overline{e}_n.$$

The matrix A_{n-1} coincides with the matrix of identity transformation except the last two columns. These columns contain the coefficients of the following two vectors respectively:

$$\sum_{j=1}^{n}a_j\overline{e}_j, \quad \text{and} \quad \sum_{j=1}^{n-2}\Big(\tfrac{-a_j(a_{n-1}+1)}{a_n}\overline{e}_j\Big)+\tfrac{1-a_{n-1}^2}{a_n}\overline{e}_{n-1}-a_{n-1}\overline{e}_n.$$

The matrix A_n coincides with the unit matrix except the second row. This row is as follows: $(-1,\ldots,-1,\frac{a_1+\ldots+a_{n-1}-1+a_2}{a_n})$.

The determinants of all such matrices equal -1. So the corresponding affine transformations are lattice iff all the coefficients of all the matrices are lattice. The matrices A_k for $k\le n{-}2$ are lattice iff

$$a_1\equiv a_2\ldots\equiv a_{n-1}(\operatorname{mod} a_n).$$

Since $a_k<a_n$, we have the equalities. Suppose $a_1=\ldots=a_{n-1}=p{-}1$ for some positive integer p. The matrix A_{n-1} is integer, iff $1{-}p^2\equiv p(p{+}1)\equiv 0(\operatorname{mod} a_n)$. Therefore, $r{+}1$ is divisible by a_n, and hence $a_n=p$. The matrix A_n is integer, iff $n(p{-}1){-}1$ is divisible by p, or equivalently $n{+}1$ is divisible by p.

So, we have already obtained that the symplex $S^n(a_1,\ldots,a_n)$ where $n>3$ and $0\le a_k<a_n$ for $k=1,\ldots,n{-}1$ is lattice-regular iff it is coincides with some $\{3^{n-1}\}_p^L$ for some p dividing n. Since the lattice volume of S_p^n equals p, the above symplices are not lattice-congruent.

The second type of n-dimensional symplices. Here we study symplices (for $n>3$) with the vertices

$$V_0=O, \qquad V_k=O+\overline{e}_i, \text{ for } k=1,\ldots,n{-}2,$$
$$V_{n-1}=O+(p-1)\sum_{i=1}^{n-2}\overline{e}_i+p\overline{e}_{n-1}, \quad \text{and} \quad V_n=O+\sum_{i=1}^{n}a_i\overline{e}_i,$$

denote such symplices by $S^n(p;a_1,\ldots,a_n)$. We also suppose, that all a_i are nonnegative integers satisfying $a_k<a_n$ for $k=1,\ldots,n{-}1$, and $p\ge 2$. Let us show that all these symplices are not lattice-regular. Consider an arbitrary symplex $S^n(p;a_1,\ldots,a_n)$, satisfying the above conditions.

Consider the symmetry exchanging V_{n-1} and V_n. This transformation is linear. Its matrix coincide with the matrix of identity transformation, except for the last two columns. These columns contains the coefficients of the following two vectors respectively:

$$\sum_{j=1}^{n-2}\Big(\frac{a_j+1}{p}-1\Big)\overline{e}_j+\frac{a_{n-1}}{p}\overline{e}_{n-1}+\frac{a_n}{p}\overline{e}_n,\quad \text{and}$$
$$\sum_{j=1}^{n-2}\Big(\frac{p(a_{n-1}-a_j+p-1)-a_{n-1}-a_ja_{n-1}}{pa_n}\overline{e}_j\Big)+\frac{p^2-a_{n-1}^2}{pa_n}\overline{e}_{n-1}-\frac{a_{n-1}}{p}\overline{e}_n.$$

If this transformation is lattice-linear, then a_2+1 is divisible by p.

Consider the symmetry exchanging V_1 and V_n. This transformation is linear. Its matrix coincide with the matrix of identity transformation, except for the first column and the last two columns. These columns contains the coefficients of the following three vectors respectively:

$$\sum_{j=1}^{n}a_j\overline{e}_j,\qquad -\sum_{j=1}^{n}\Big(\frac{a_j(p-1)}{p}\overline{e}_j\Big)+\frac{p-1}{p}\overline{e}_1+\overline{e}_{n-1},\quad\text{and}$$
$$\frac{a_{n-1}p-pa_1-a_{n-1}-p}{pa_n}\sum_{j=1}^{n}a_j\overline{e}_j+\frac{p+a_1}{pa_n}\overline{e}_1+\overline{e}_n.$$

If this transformation is lattice-linear then a_2 is divisible by r.

Since a_2 and a_2+1 are divisible by r and $r\geq 2$, the symplex $S^n(p;a_1,\ldots,a_n)$ is not lattice-regular.

Conclusion of the proof.

Statement (i) is already proven. Since p is a lattice volume of S_p^n, Statement (ii) holds. We prove Statement (iii) of the proposition by the induction on the dimension n. For $n=1,2,3$ the statement is already proven. Suppose that it is true for an arbitrary $n\geq 3$. Let us prove the statement for $n+1$.

Consider any lattice-regular $(n+1)$-dimensional lattice symplex S. Since it is lattice-regular, all its faces are lattice-regular. By the induction assumption there exist a positive integer p dividing $n+1$ such that the faces of S are lattice-congruent to $\{3^{n-1}\}_p^L$. Therefore, S is lattice-affine equivalent to the symplex $S^{n+1}(p;a_1,\ldots,a_{n+1})$, where $p\geq 1$, and $a_k<a_n$ for $k=1,\ldots,n-1$. By the above cases the lattice-regularity implies, that $p=1$, and that there exist a positive integer p' dividing $n+2$ such that the symplex $S^{n+1}(1;a_1,\ldots,a_{n+1})$ coincides with $\{3^n\}_{p'}^L$. This concludes the proof of the Statement (iii) for the arbitrary dimension.

Proposition 4.1 is proven. □

5. Lattice-regular lattice cubes.

In this section we describe all lattice-regular lattice cubes for all integer dimensions.

Proposition 5.1. Cube$_1$. *All elementary lattice-regular one-dimensional lattice cubes are lattice segments of unit lattice length.*

Cube$_2$. *All elementary lattice-regular two-dimensional lattice cubes are lattice-congruent to $\{4\}_1^L$, or to $\{4\}_2^L$. The cubes $\{4\}_1^L$ and $\{4\}_2^L$ are not lattice-congruent.*

Cube$_n$ (for $n>2$). *All elementary lattice-regular n-dimensional lattice cubes are lattice-congruent to $\{4,3^{n-2}\}_1^L$, $\{4,3^{n-2}\}_2^L$, or $\{4,3^{n-2}\}_3^L$. The cubes $\{4,3^{n-2}\}_1^L$, $\{4,3^{n-2}\}_2^L$, and $\{4,3^{n-2}\}_3^L$ are not lattice-congruent to each other.*

The three-dimensional cubes $\{4,3^{n-2}\}_1^L$, $\{4,3^{n-2}\}_2^L$, and $\{4,3^{n-2}\}_3^L$ are shown on Figure 5.

We use the following two facts.

Lemma 5.2. *Any elementary lattice-regular n-dimensional lattice cube contains at most one lattice point in its interior. If the cube contains an interior lattice point, this point coincides with the intersection point of the diagonals of the cube.*

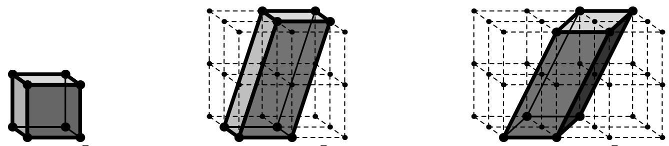

FIGURE 5. Three-dimensional elementary lattice-regular convex lattice cubes.

Proof. Suppose the converse is true. There exist a primitive n-dimensional cube with an interior lattice point A distinct to the intersections of the diagonals. Then there exist a lattice-linear reflection ξ of the cube above some $n-1$ dimensional plane, that do not preserve A. So the line of vector $\overline{A\xi(A)}$ coincide with the line containing one of the generating vectors of the cube. Hence, the one-dimensional faces of the cube are not elementary. Therefore, the cube is not elementary. We come to the contradiction. □

Let us give the following important definition. Consider some hyperplane containing a sublattice of the lattice of corank 1 and a lattice point in the complement to this hyperplane. Let the Euclidean distance from the given point to the given hyperplane equals l. The minimal value of nonzero Euclidean distances from the points of the lattice to the hyperplane is denoted by l_0. The ratio l/l_0 is said to be the *lattice distance* from the given lattice point to the given lattice hyperplane.

Corollary 5.3. *The lattice distances from any vertex of any elementary lattice-regular n-dimensional lattice cube to any its $(n-1)$-dimensional face (that does not containing the given vertex) equals either 1, or 2.* □

Proof of Proposition 5.1. The one-dimensional case is trivial. The two-dimensional case was described in Proposition 3.1. Let us study higher-dimensional cases.

The lattice cubes $\{4,3^{n-2}\}_1^L$, $\{4,3^{n-2}\}_2^L$, and $\{4,3^{n-2}\}_3^L$. First, let us study the cases of the polytopes $\{4,3^{n-2}\}_1^L$, $\{4,3^{n-2}\}_2^L$, and $\{4,3^{n-2}\}_3^L$ for any $n\geq 3$. Since lattice volumes of $\{4,3^{n-2}\}_1^L$, $\{4,3^{n-2}\}_2^L$, and $\{4,3^{n-2}\}_3^L$ are $n!$, $2n!$, and $2^{n-1}n!$ respectively, the listed cubes are not lattice-congruent to each other. Let us prove that these polytopes are lattice-regular for any $n\geq 3$.

Since the vectors of $\{4,3^{n-2}\}_1^L$ generate lattice, it is lattice-regular.

Now we study the case of $\{4,3^{n-2}\}_2^L$. Denote by $\overline{v}_i$ the vector $\overline{e}_i$ for $i=1,\ldots,n-1$ and by $\overline{v}_n$ the vector $\overline{e}_1+\overline{e}_2+\ldots+\overline{e}_{n-1}+2\overline{e}_n$. The group of lattice symmetries of the cube $\{4,3^{n-2}\}_2^L$ is generated by the linear operators A_k transposing the vectors $\overline{v}_i$ and $\overline{v}_{i+1}$ for $i=1,\ldots,n-1$, and the last one: the composition of the symmetry A_n sending $\overline{v}_1$ to $-\overline{v}_1$ and preserving $\overline{v}_i$ for $i=2,\ldots,n$ and the lattice shift on the vector $\overline{v}_1$.

Let us check that all the linear transformations A_k are lattice-linear. We show explicitly the matrices of A_k in the basis $\overline{e}_i$ for $i=1,\ldots,n$. The matrix of A_k for $k=1,\ldots,n-2$ coincides with the the matrix of the transposition of the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$. The matrix of A_{n-1} coincides with the matrix of identity transformation except the last two columns. These columns contain the coefficients of the vectors $\overline{v}_n$ and $\overline{e}_{n-1}+\overline{e}_n-\overline{v}_n$ respectively. The matrix of A_n coincides with the matrix of identity transformation except the first row, which is $(-1,0,\ldots,0,1)$. Since all these matrices are in $SL(n,\mathbb{Z})$, the cube $\{4,3^{n-2}\}_2^L$ is lattice-regular.

Let us consider now the case of $\{4,3^{n-2}\}_3^L$. Put by definition $\overline{v}_1=\overline{e}_1$, and $\overline{v}_i=\overline{e}_1+2\overline{e}_i$ for $i=2,\ldots,n$. The group of lattice symmetries of the cube $\{4,3^{n-2}\}_3^L$ is generated by the linear operators A_k transposing the vectors $\overline{v}_i$ and $\overline{v}_{i+1}$ for $i=1,\ldots,n-1$, and the last one: the composition of the symmetry A_n sending $\overline{v}_1$ to $-\overline{v}_1$ and preserving $\overline{v}_i$ for $i=2,\ldots,n$, and the lattice shift on the vector $\overline{v}_1$.

Let us check that all the linear transformations A_k are lattice-linear. The matrix of A_1 coincides with the matrix of identity transformation except the second row, which is $(2,-1,\ldots,-1)$. The matrix of A_k for $k=2,\ldots,n{-}1$ coincides with the matrix transposing the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$. The matrix of A_n coincides with the matrix of identity transformation except the first row, which is $(-1,1,\ldots,1)$. Since all these matrices are in $SL(n,\mathbb{Z})$, the cube $\{4,3^{n-2}\}^L_3$ is lattice-regular.

Conclusion of the proof of Proposition 5.1. Now we prove, that any elementary lattice-regular n-dimensional lattice cube for $n\ge 3$ is lattice-congruent to $\{4,3^{n-2}\}^L_1$, $\{4,3^{n-2}\}^L_2$, or $\{4,3^{n-2}\}^L_3$. by the induction on n.

The base of induction. Any face of any three-dimensional lattice-regular cube is lattice-regular.

Suppose, that the faces of three-dimensional lattice-regular cube C are lattice-congruent to $\{4\}^L_1$. Then C is lattice-congruent to the cube generated by the origin and the vectors $\overline{e}_1$, $\overline{e}_2$, and $a_1\overline{e}_1+a_2\overline{e}_2+a_3\overline{e}_3$. By Corollary 5.3 we can choose a_3 equals either 1 or 2. If $a_3=1$ then C is lattice-congruent to $\{4,3\}^L_1$. If $a_3=2$, then we can choose a_1 and a_2 being 0, or 1. Direct calculations show, that the only possible case is $a_1=1$, and $a_2=0$, i.e. the case of $\{4,3\}^L_2$.

Suppose now, that the faces of three-dimensional lattice-regular cube C are lattice-congruent to $\{4\}^L_2$. Then C is lattice-congruent to the cube generated by the origin and the vectors $\overline{e}_1$, $\overline{e}_1+2\overline{e}_2$, and $a_1\overline{e}_1+a_2\overline{e}_2+a_3\overline{e}_3$. By Corollary 5.3 we can choose a_3 equals either 1 or 2. Then we can also choose a_1 and a_2 being 0, or 1. Direct calculations show, that the only possible case $a_1=1$, $a_2=0$, and $a_3=2$ corresponds to $\{4,3\}^L_3$.

The step of induction. Suppose that any elementary lattice-regular $(n{-}1)$-dimensional lattice cubes $(n>3)$ are lattice-congruent to $\{4,3^{n-3}\}^L_1$, $\{4,3^{n-3}\}^L_2$, or $\{4,3^{n-3}\}^L_3$. Let us prove that any elementary lattice-regular n-dimensional lattice cubes are lattice-congruent to $\{4,3^{n-2}\}^L_1$, $\{4,3^{n-2}\}^L_2$, or $\{4,3^{n-2}\}^L_3$.

Any face of any lattice-regular cube is lattice-regular. Suppose, that the faces of $(n{-}1)$-dimensional lattice-regular cube C are lattice-congruent to $\{4,3^{n-3}\}^L_1$. Then C is lattice-congruent to the cube C' generated by the origin and the vectors $\overline{v}_i=\overline{e}_i$ for $i=1,\ldots,n-1$ and the vector $\overline{e}_n=a_1\overline{e}_1+\ldots+a_n\overline{e}_n$.

By Corollary 5.3 we can choose a_n equals either 1 or 2. If $a_n=1$, then the lattice volume of C is $n!$ and it is lattice-congruent to $\{4,3^{n-2}\}^L_1$. If $a_n=2$, then we can choose a_i being 0, or 1 for $i=1,\ldots,n{-}1$. Consider a symmetry of C' transposing the vectors $\overline{v}_k$ and $\overline{v}_{k+1}$ for $k=1,\ldots,n{-}2$. This transformation is linear and its matrix coincides with the matrix of the transposition of the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$, except the last column. The n-th column contains the coordinates of the vector

$$\tfrac{a_k-a_{k+1}}{2}(\overline{e}_k-\overline{e}_{k+1})+\overline{e}_n.$$

Since the transformation is lattice,

$$a_k\equiv a_{k+1}(\mathrm{mod}\,2)\text{ for }k=1,\ldots,n{-}2.$$

Since any a_i is either zero or unit, the above imply $a_1=a_2=\ldots=a_{n-1}$. If $a_1=0$, then the vector $\overline{v}_n$ is not of the unit lattice length, but the vector $\overline{v}_1$ is of the unit length, so C' is not lattice-regular. If $a_1=1$ then C' coincides with $\{4,3^{n-2}\}^L_2$.

Suppose, that the faces of $(n{-}1)$-dimensional lattice-regular cube C are lattice-congruent to $\{4,3^{n-3}\}^L_2$. Then C is lattice-congruent to the cube C' generated by the origin and the vectors $\overline{v}_i=\overline{e}_i$, for $i=1,\ldots,n-2$, $\overline{v}_{n-1}=\overline{e}_1+\ldots+\overline{e}_{n-2}+2\overline{e}_{n-1}$, and $\overline{v}_n=a_1\overline{e}_1+\ldots+a_n\overline{e}_n$.

Consider a symmetry of C' transposing the vectors $\overline{v}_{n-1}$ and $\overline{v}_n$. This transformation is linear and its matrix coincides with the matrix of identity transformation except the last two columns. These columns contain the coefficients of the following two vectors respectively:

$$\sum_{j=1}^{n}\tfrac{a_j}{2}\overline{e}_j-\tfrac{1}{2}\sum_{j=1}^{n-2}\overline{e}_j,\quad\text{and}\quad\sum_{j=1}^{n-2}\Big(\tfrac{a_{n-1}-2a_j-a_{n-1}a_j+2}{2a_n}\overline{e}_j\Big)+\tfrac{4-a_{n-1}^2}{2a_n}\overline{e}_{n-1}-\tfrac{a_{n-1}}{2}\overline{e}_n.$$

Since the described transformation is lattice, the integers a_{n-1} and a_n are even, and a_{n-2} is odd.

Consider now a symmetry of C' transposing the vectors $\overline{v}_{n-2}$ and $\overline{v}_{n-1}$. This transformation is linear and its matrix coincides with the matrix of identity transformation except the last three columns. These columns contain the coefficients of the following three vectors respectively:

$$\overline{v}_{n-1}, \quad -\sum_{j=1}^{n-3}\overline{e}_j-\overline{e}_{n-1}, \quad \text{and} \quad \tfrac{a_{n-1}-a_{n-2}}{a_n}\sum_{j=1}^{n-3}\overline{e}_j+2\tfrac{a_{n-1}-a_{n-2}}{a_n}\overline{e}_{n-1}+\overline{e}_n.$$

Since the described transformation is lattice, the integer $a_{n-1}-a_{n-2}$ is even, and thus a_{n-2} is even. We come to the contradiction with the divisibility of a_{n-2} by 2. So C is not lattice-regular.

Suppose, that the faces of $(n-1)$-dimensional lattice-regular cube C are lattice-congruent to $\{4,3^{n-3}\}_3^L$. Then C is lattice-congruent to the cube C' generated by the origin and the vectors $\overline{v}_1=\overline{e}_1$, $\overline{v}_i=\overline{e}_1+2\overline{e}_i$ for $i=2,\ldots,n-1$, and $\overline{v}_n=a_1\overline{e}_1+\ldots+a_n\overline{e}_n$. By Corollary 5.3 we can choose a_n equals either 1 or 2. Then we choose a_i being 0, or 1 for $i=1,\ldots n-1$.

Consider a symmetry of C' transposing the vectors $\overline{v}_{n-1}$ and $\overline{v}_n$. This transformation is linear and its matrix coincides with the matrix of identity transformation except the last two columns. These columns contain the coefficients of the following two vectors respectively:

$$-\tfrac{1}{2}\overline{e}_1+\sum_{j=1}^{n}\tfrac{a_j}{2}\overline{e}_j, \quad \text{and} \quad \tfrac{(1-a_1)(2+a_{n-1})}{2a_n}\overline{e}_1-\sum_{j=2}^{n-2}\left(\tfrac{a_j(a_{n-1}+2)}{2a_n}\overline{e}_j\right)+\tfrac{4-a_{n-1}^2}{2a_n}\overline{e}_{n-1}-\tfrac{a_{n-1}}{2}\overline{e}_n.$$

Since the described transformation is lattice, the integer a_1 is odd, and the integers a_i for $i=1,\ldots n$ are even. Thus

$$a_n=2, \qquad a_1=1, \quad \text{and} \quad a_2=\ldots=a_{n-1}=0.$$

Therefore C' coincides with $\{4,3^{n-2}\}_3^L$.

We have already studied all possible n-dimensional cases. This proves the statement for the dimension n.

All statements of Proposition 5.1 are proven. □

6. Lattice-regular lattice generalized octahedra.

In this section we describe all lattice-regular lattice generalized octahedra for all integer dimensions greater than 2.

Proposition 6.1. *All elementary lattice-regular n-dimensional lattice generalized octahedra for $n\ge 3$ are lattice-congruent to $\{3^{n-2},4\}_1^L$, $\{3^{n-2},4\}_2^L$, or $\{3^{n-2},4\}_3^L$. The generalized octahedra $\{3^{n-2},4\}_1^L$, $\{3^{n-2},4\}_2^L$, and $\{3^{n-2},4\}_3^L$ are not lattice-congruent to each other.*

We show the (three-dimensional) octahedra $\{3,4\}_1^L$, $\{3,4\}_2^L$, and $\{3,4\}_3^L$ on Figure 6.

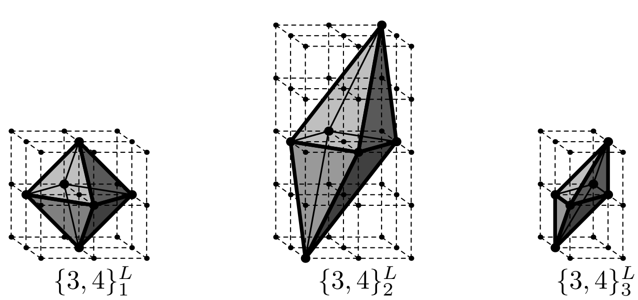

FIGURE 6. Three-dimensional elementary lattice-regular convex lattice tetrahedra.

Proof. Consider an arbitrary elementary n-dimensional lattice-regular generalized octahedron P. Let the vertices $V_1, \dots, V_{2n}$ of P be enumerated in such a way that for any positive integer $i \leq n$ there exist a lattice symmetry exchanging V_i and V_{i+n} and preserving any other vertex. So, $V_iV_jV_{i+n}V_{j+n}$ is a lattice-regular square for $i \neq j$. Therefore, the midpoints of the segments V_iV_{i+n} coincide for $i = 1, 2, \dots, n$. Denote the common midpoint of the segments V_iV_{i+n} by A.

Suppose the point A is lattice. Consider the lattice cube with the vertices $A \pm \overline{AV_1} \pm \overline{AV_2} \pm \dots \pm \overline{AV_n}$, we denote it by $C(P)$. The cube $C(P)$ is also lattice-regular.

Note that the lattice-regular generalized octahedra P' and P'' have lattice-congruent cubes $C(P')$ and $C(P'')$ iff P' and P'' are lattice-congruent.

Since P is elementary, the segment AV_1 is of unit lattice length. Therefore, the cube $C(P)$ is lattice-congruent to the 2-multiple of some $\{4, 3^{n-2}\}_k^L$, for $k = 1, 2, 3$. If $C(P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_1^L$, or to the 2-multiple of $\{4, 3^{n-2}\}_2^L$, then P is lattice-congruent to $\{3^{n-2}, 4\}_1^L$, or to $\{3^{n-2}, 4\}_2^L$, respectively. If $C(P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_3^L$, then P is not elementary.

Suppose now, the common midpoint A of the diagonals is not lattice. If the lattice length of V_1V_{n+1} equals $2k+1$ for some positive k, then the generalized octahedron P is not elementary. Suppose the segment V_1V_{n+1} is of unite lattice length. Consider a 2-multiple to the polygon P and denote in by $2P$. Since the segment V_1V_{n+1} is of unit lattice length, the cube $C(2P)$ is the 2-multiple of some $\{4, 3^{n-2}\}_k^L$, for $k = 1, 2, 3$. If $C(2P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_1^L$, or of $\{4, 3^{n-2}\}_2^L$, then P is not a lattice polytope. If $C(2P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_3^L$, then P is lattice-congruent to $\{3^{n-2}, 4\}_3^L$.

The generalized octahedra $\{3^{n-2}, 4\}_1^L$, $\{3^{n-2}, 4\}_2^L$, and $\{3^{n-2}, 4\}_3^L$ are lattice-regular, since so are the cubes $C(\{3^{n-2}, 4\}_1^L)$, $C(\{3^{n-2}, 4\}_2^L)$, and $C(2\{3^{n-2}, 4\}_3^L)$.

The generalized octahedra $\{3^{n-2}, 4\}_1^L$, $\{3^{n-2}, 4\}_2^L$, and $\{3^{n-2}, 4\}_3^L$ are not lattice-congruent to each-other, since the corresponding elementary cubes $C(\{3^{n-2}, 4\}_1^L)$, $C(\{3^{n-2}, 4\}_2^L)$, and $C(2\{3^{n-2}, 4\}_3^L)$ are not lattice-congruent. □

7. Proof of Theorem 2.2.

In this section we obtain proof of Theorem 2.2 by combining the results of propositions from the previous sections and describing the remaining low-dimensional cases.

Consider any convex lattice-regular lattice polytope. Since it is lattice-regular and convex it is combinatorically regular. Therefore, by Theorem B it is combinatorically isomorphic to one of the Euclidean polytopes of Theorem A. In Section 3 we gave the description of the two-dimensional case. In Sections 4, 5, and 6 we studied the cases of lattice-regular polytopes combinatorically isomorphic to regular symplices ($\{3^{n-1}\}$), regular cubes ($\{4, 3^{n-2}\}$), and regular generalized octahedra ($\{3^{n-2}, 4\}$) respectively.

Now we will study the remaining special cases of three- and four-dimensional regular polytopes.

7.1. **Three-dimensional icosahedra and dodecahedra.** We have already classified all lattice-regular elementary tetrahedra, cubes, and octahedra. There is no lattice-regular dodecahedron, since there is no lattice-regular pentagon. There is no lattice-regular icosahedron, since there is no lattice-affine transformation with a fixed point of order 5.

So, the classification in the three-dimensional case is completed.

7.2. **Four-dimensional 24-sells, 120-sells, and 600-sells.** *The case of 24-sell.* Suppose that P is a lattice-regular 24-sell. It contains 16 vertices such that the subgroup of the group of all lattice-symmetries of P preserving these 16 vertices is isomorphic to the group of the symmetries of the four-dimensional cube. So we can naturally define a combinatorial-regular cube associated with this 16 vertices. Any two-dimensional face of this cube is an Euclidean parallelogram, since such face is the diagonal section containing four lattice points of some lattice-regular octahedron. If all two-dimensional faces are parallelograms, then all three-dimensional faces are parallelepipeds and these 16 vertices are vertices of a four-dimensional parallelepiped. Denote this parallelepiped by C. Since all transformations of C are lattice-affine, the polytope C is a lattice-regular cube. (Note that for any 24-sell there exist exactly three such (distinct) cubes).

Suppose that C is a lattice-regular cube generated by the origin and some vectors $\overline{v}_i$ for $i = 1, 2, 3, 4$. Consider also the coordinates $(*, *, *, *)_v$ corresponding to this basis. Let the point $(a_1, a_2, a_3, a_4)_v$ of P connected by edges with the vertices of C is in the plane with the unit last coordinate. Then, the point $(2-a_1, a_2, a_3, 2-a_4)_v$ is also a vertex. Thus, the point (a_1, a_2, a_3, a_4-1) is also a vertex. Note that the points $(a_1, a_2, a_3, a_4)_v$ and $(a_1, a_2, a_3, a_4-2)_v$ are symmetric about the center of the cube: $(1/2, 1/2, 1/2, 1/2)_v$. So $a_4-1/2 = 1/2 - a_4-2$. Thus, $a_4 = 3/2$. Similar calculations show that $a_1 = a_2 = a_3 = 1/2$. Therefore, the eight points do not contained in C coincide with the following points:

$$O + 1/2(\overline{v}_1+\overline{v}_2+\overline{v}_3+\overline{v}_4) \pm \overline{v}_i, \quad \text{for } i = 1, 2, 3, 4.$$

Consider a lattice-regular primitive 24-sell P and C one of the corresponding cubes. Since the edges of C are the edges of P, the cube C is also primitive. Let us study all three possible cases of lattice-affine types of C.

Suppose C coincides with $\{4, 3, 3\}_1^L$. Then the remaining points

$$O + 1/2(\overline{e}_1+\overline{e}_2+\overline{e}_3+\overline{e}_4) \pm \overline{e}_i, \quad \text{for } i = 1, 2, 3, 4$$

are not lattice. Therefore, the case of $\{4, 3, 3\}_1^L$ is impossible.

In the case of $\{4, 3, 3\}_2^L$ and $\{4, 3, 3\}_3^L$ all the vertices are lattice and in our notation coincide with $\{3, 4, 3\}_1^L$ and $\{3, 4, 3\}_2^L$ respectively. Straightforward calculations shows, that both resulting 24-sells are lattice-regular.

The case of 120-sell. The 120-sell is non-realizable as an lattice-regular lattice polytope, since its two-dimensional faces should be lattice-regular lattice pentagons. By the above, lattice-regular lattice pentagons are not realizable.

The case of 600-sell. Consider an arbitrary polytope with topological structure of 600-sell having one vertex at the origin O. Let OV be some edge of this 600-sell. The group of symmetries of an abstract 600-sell with fixed vertex O is isomorphic to the group of symmetries of an abstract icosahedron. The group of symmetries of an abstract 600-sell with fixed vertices O and V_1 is isomorphic to the group of symmetries of an abstract pentagon. So, there exist a symmetry A of the 600-sell of order 5 preserving the vertices O and V_1. If the polytope P is lattice-regular, then this symmetry is lattice-linear. Since A^5 is the identity transformation and the space is four-dimensional, the characteristic polynomial of A in the variable x is either $x-1$ or $x^4+x^3+x^2+x+1$. Since $A(\overline{OV_1}) = \overline{OV_1}$, the characteristic polynomial of A is divisible by $x-1$. If the characteristic polynomial is $x-1$, then the operator A is the identity operator of order 1 and not of order 5. Therefore, there is no lattice-regular lattice polytope with the combinatorial structure of the 600-sell.

7.3. **Conclusion of the proof of Theorem 2.2.** We have studied all possible combinatorical cases of lattice-regular polytopes. The proof of Theorem 2.2 is completed. □

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On an Invariant Möbius Measure and the Gauss–Kuzmin Face Distribution

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Dedicated to my teacher
Vladimir Igorevich Arnold

Abstract—We study Möbius measures of the manifold of n-dimensional continued fractions in the sense of Klein. By definition any Möbius measure is invariant under the natural action of the group of projective transformations $\mathrm{PGL}(n+1)$ and is an integral of some form of the maximal dimension. It turns out that all Möbius measures are proportional, and the corresponding forms are written explicitly in some special coordinates. The formulae obtained allow one to compare approximately the relative frequencies of the n-dimensional faces of given integer-affine types for n-dimensional continued fractions. In this paper we make numerical calculations of some relative frequencies in the case of $n = 2$.

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INTRODUCTION

Consider an n-dimensional real vector space with a lattice of integer points in it. The boundary of the convex hull of all integer points contained inside one of the n-dimensional invariant cones for a hyperbolic n-dimensional linear operator without multiple eigenvalues is called a *sail* in the sense of Klein. The set of all sails of such an n-dimensional operator is called an $(n-1)$-*dimensional continued fraction* in the sense of Klein (see Section 2 for more detail). Any sail is a polyhedral surface. In this work we study the frequencies of faces of multidimensional continued fractions.

On the manifold of n-dimensional continued fractions in the sense of Klein, there exists a unique, up to multiplication by a constant function, form of the highest dimension that is invariant under the natural action of the group of projective transformations $\mathrm{PGL}(n+1)$. A measure corresponding to the integral of such a form is called a *Möbius measure*. In the present paper we deduce explicit formulae for calculating invariant forms in special coordinates. These formulae can be used to answer some statistical questions of the theory of multidimensional continued fractions. As an example, we present the results of approximate calculations of the frequencies for certain two-dimensional faces of two-dimensional continued fractions.

The problem of generalizing ordinary continued fractions was posed by C. Hermite [39] in 1839. One of the most interesting geometrical generalizations was introduced by F. Klein in 1895 in [20] and [21]. Unfortunately, due to the computational complexity of multidimensional continued fractions, no significant advances in the study of their properties were made 100 years ago. While originally studying A-graded algebras [3], V.I. Arnold faced with the theory of multidimensional continued fractions in the sense of Klein. Since 1989 he has formulated many problems on the geometry and statistics of multidimensional continued fractions, reviving the interest in the study of multidimensional continued fractions (see [4, 7]).

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Multidimensional continued fractions in the sense of Klein are used in various branches of mathematics. J.-O. Moussafir [31] and O.N. German [12] studied the connection between the sails of multidimensional continued fractions and Hilbert bases. In [36] H. Tsuchihashi established a relationship between periodic multidimensional continued fractions and multidimensional cusp singularities. This relationship generalizes the classical relationship between ordinary periodic continued fractions and two-dimensional cusp singularities known before. The combinatorial topological multidimensional generalization of the Lagrange theorem for ordinary continued fractions was obtained by E.I. Korkina in [24], and the corresponding algebraic generalization, by G. Lachaud (see [28]).

A large number of examples of two-dimensional periodic continued fractions were constructed by E.I. Korkina [23, 25, 26], G. Lachaud [28, 29], A.D. Bruno and V.I. Parusnikov [9, 35], as well as by the present author [13, 14]. A part of these two-dimensional continued fractions can be found at the website [8] by K. Briggs. A few examples of three-dimensional continued fractions in the four-dimensional space were constructed by the author in [19]. Algorithms for constructing multidimensional continued fractions are described in the works by R. Okazaki [33], J.-O. Moussafir [32], and the present author [15].

For the first time the statement on the statistics of numbers as elements of ordinary continued fractions was formulated by C.F. Gauss in his letters to P.-S. Laplace (see [11]). This statement (see Section 1) was proven later by R.O. Kuzmin [27], and still later was reproven by P. Lévy [30]. Further investigations in this direction were made by E. Wirsing in [10]. (The basic notions of the theory of ordinary continued fractions are described in the books [38] by A.Ya. Khinchin and [7] by V.I. Arnold.) In 1989 V.I. Arnold generalized statistical problems to the case of one-dimensional and multidimensional continued fractions in the sense of Klein (see [6] (in particular, Problem 1993-11) and [4, 5]).

The one-dimensional case was studied in detail by M.O. Avdeeva and V.A. Bykovskii in [1, 2]. In the two-dimensional and multidimensional cases, V.I. Arnold formulated many problems on the statistics of sail characteristics of multidimensional continued fractions such as the amount of triangular, quadrangular, etc., faces, as well as their integer areas, lengths of edges, etc. The major part of these problems is open nowadays, while some of them have been solved almost completely.

M.L. Kontsevich and Yu.M. Suhov in [22] proved the existence of the above-mentioned statistics. Recently V.A. Bykovskii and M.A. Romanov used the Monte-Carlo method to calculate the frequencies for some types of faces of sails. In the present paper we calculate a natural Möbius measure on the manifold of all n-dimensional continued fractions in the sense of Klein in special coordinates. In particular, this allows us to approximately calculate the relative frequencies of multidimensional faces of multidimensional continued fractions.

Note that the Möbius measure is also used in the theory of energies of knots and graphs (see the works of M.H. Freedman et al. [37], J. O'Hara [34], and the present author [16]). In the case of one-dimensional continued fractions, the Möbius measure is induced by the relativistic measure of the three-dimensional de Sitter world.

This paper is organized as follows. In Section 1 we give necessary notions of the theory of ordinary continued fractions. In particular, we give a definition of the Gauss–Kuzmin statistics. Further, in Section 2 we describe a smooth manifold structure for the set of all n-dimensional continued fractions and define a Möbius measure on it. In Section 3 we study the relative frequencies of faces of one-dimensional continued fractions. These frequencies are proportional to the frequencies of the Gauss–Kuzmin statistics. In Section 4 we study the relative frequencies of faces of multidimensional continued fractions. Finally, in Section 5 we present the results of approximate calculations of relative frequencies for some faces of two-dimensional continued fractions.

1. ONE-DIMENSIONAL CONTINUED FRACTIONS AND GAUSS–KUZMIN STATISTICS

Let α be an arbitrary rational number. Suppose that

$$\alpha = a_0 + 1/\big(a_1 + 1/\big(a_2 + \ldots + 1/(a_{n-1} + 1/a_n)\ldots\big)\big),$$

where a_0 is an integer and the remaining a_i, $i = 1, \ldots, n$, are positive integers. The expression on the right-hand side of this equality is called a *decomposition of* α *into a finite ordinary continued fraction* and denoted by $[a_0, a_1, \ldots, a_n]$. If the total number $n+1$ of elements of the decomposition is even, then the continued fraction is said to be *even*, and if this number is odd, then the continued fraction is said to be *odd*.

Let a_0 be an integer and $a_1, \ldots, a_n, \ldots$ be an infinite sequence of positive integers. Denote by r_n the rational number $[a_0, \ldots, a_{n-1}]$. For such integers a_i, the sequence (r_n) always converges to some real α. The limit

$$\lim_{n\to\infty}[a_0, a_1, \ldots, a_{n-1}]$$

is called the *decomposition of* α *into an infinite ordinary continued fraction* and denoted by $[a_0, a_1, a_2, \ldots]$.

Ordinary continued fractions possess the following basic properties.

Proposition 1.1. (a) *Any rational number has exactly two distinct decompositions into a finite ordinary continued fraction*; *one of them is even, and the other is odd.*

(b) *Any irrational number has a unique decomposition into an infinite ordinary continued fraction.*

(c) *A decomposition into a finite ordinary continued fraction is rational.*

(d) *A decomposition into an infinite ordinary continued fraction is irrational.*

Notice that for any finite continued fraction $[a_0, a_1, \ldots, a_n]$, where $a_n \neq 1$, the following holds:

$$[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1].$$

This equality determines a one-to-one correspondence between the sets of even and odd finite continued fractions.

Let α be some irrational number between zero and unity, and let $[0, a_1, a_2, a_3, \ldots]$ be its ordinary continued fraction. Denote by $z_n(\alpha)$ the real $[0, a_n, a_{n+1}, a_{n+2}, \ldots]$.

Let $m_n(x)$ denote the measure of the set of reals α contained in the interval $[0; 1]$ such that $z_n(\alpha) < x$. In his letters to P.-S. Laplace, C.F. Gauss formulated without proofs the following theorem. Later it was proved by R.O. Kuzmin [27], and still later by P. Lévy [30].

Theorem 1.2 (Gauss–Kuzmin)**.** *For* $0 \le x \le 1$ *the following relation holds*:

$$\lim_{n\to\infty} m_n(x) = \frac{\ln(1+x)}{\ln 2}.$$

For an arbitrary integer $k > 0$, denote by $P_n(k)$ the measure of the set of all reals α of the interval $[0; 1]$ such that each of them has the number k at the nth position. The limit $\lim_{n\to\infty} P_n(k)$ is called the *frequency of* k for ordinary continued fractions and denoted by $P(k)$.

Corollary 1.3. *For any positive integer* k *the following equality holds*:

$$P(k) = \frac{1}{\ln 2}\ln\left(1 + \frac{1}{k(k+2)}\right).$$

Proof. Notice that $P_n(k) = m_n\bigl(\frac{1}{k}\bigr) - m_n\bigl(\frac{1}{k+1}\bigr)$. Now the statement of the corollary follows from the Gauss–Kuzmin theorem. $\square$

The problem of V.I. Arnold on the asymptotic behavior of the frequencies of integers as elements of ordinary continued fractions for rational numbers with bounded numerators and denominators was completely studied by V.A. Bykovskii and M.O. Avdeeva in [1, 2]. It turns out that such frequencies coincide with the frequencies $P(k)$ defined above.

2. MULTIDIMENSIONAL CONTINUED FRACTIONS IN THE SENSE OF KLEIN

2.1. Geometry of ordinary continued fractions. Consider a two-dimensional plane with standard Euclidean coordinates. A point is said to be *integer* if both its coordinates are integer. The *integer length* of a segment AB with integer vertices A and B is the ratio of its Euclidean length to the minimum Euclidean length of integer vectors contained in the segment AB; we denote it by $\mathrm{l}\ell(AB)$. The *integer* (*nonoriented*) *area* of a polygon P is the ratio of its Euclidean area to the minimum Euclidean area of triangles with integer vertices; we denote it by $\mathrm{lS}(P)$. The quantity $\mathrm{lS}(P)$ coincides with the doubled Euclidean area of the polygon P.

For an arbitrary real $\alpha \geq 1$ we consider an angle in the first orthant defined by the rays $\{(x,y)\,|\,y=0,\ x\geq 0\}$ and $\{(x,y)\,|\,y=\alpha x,\ x\geq 0\}$. The boundary of the convex hull of the set of all integer points in the closure of this angle, except the origin O, is a broken line consisting of segments and possibly of a ray or two rays contained in the sides of the angle. The union of all segments of this broken line is called the *sail* of the angle. The sail of the angle is a finite broken line for rational α and an infinite broken line for irrational α. Denote the point with coordinates $(1,0)$ by A_0, and denote all the other vertices of the broken line consecutively by $A_1, A_2, \ldots$. Let $a_i = \mathrm{l}\ell(A_iA_{i+1})$ for $i=0,1,2,\ldots$ and $b_i = \mathrm{lS}(A_{i-1}A_iA_{i+1})$ for $i=1,2,3,\ldots$; then the following equality holds:

$$\alpha = [a_0, b_1, a_1, b_2, a_2, b_3, a_3, \ldots].$$

In Fig. 1 we examine an example with $\alpha = 7/5 = [1,2,2]$.

2.2. Definition of multidimensional continued fractions. Based on the geometrical construction described in the previous subsection, F. Klein introduced the following geometrical generalization of ordinary continued fractions to the multidimensional case (see [20, 21]).

Consider arbitrary $n+1$ hyperplanes in $\mathbb{R}^{n+1}$ such that their intersection consists of a unique point, the origin. The complement of the union of these hyperplanes consists of 2^{n+1} open orthants. Consider one of them. The boundary of the convex hull for the set of all integer points of the closure of the orthant except the origin is called the *sail* of the orthant. The set of all 2^{n+1} sails is called

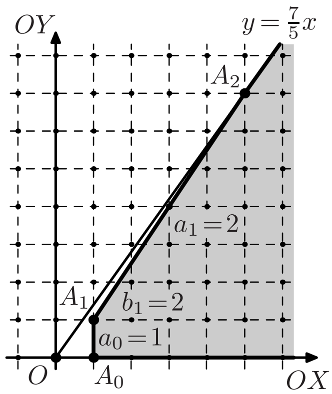


Fig. 1. The sail for the continued fraction of $7/5 = [1,2,2]$.

the *n-dimensional continued fraction* related to the given $n+1$ hyperplanes. The intersection of a hyperplane with the sail is said to be a *k-dimensional face of the sail* if it is contained in some k-dimensional plane and is homeomorphic to a k-dimensional disc. (See also [14].)

Two multidimensional faces of multidimensional continued fractions are said to be *integer-linear (-affine) equivalent* if there exist a linear (affine) transformation preserving the integer lattice and taking one face to the other. A class of all integer-linear (-affine) equivalent faces is called the *integer-linear (-affine) type* of any face of this class.

Let us define one useful integer-linear invariant of a plane. Consider an arbitrary k-dimensional plane π that does not contain the origin and whose integer vectors generate a sublattice of rank k in the lattice of all integer vectors. Let the Euclidean distance from the origin to the plane π equal ℓ. Denote by ℓ_0 the minimum nonzero Euclidean distance to π from the integer points of the plane (of dimension $k+1$) spanned by the given plane π and the origin. The ratio ℓ/ℓ_0 is called the *integer distance* from the origin to the plane π.

Let us now describe one of the original problems of V.I. Arnold on the statistics of faces of multidimensional continued fractions. Note that for any real hyperbolic operator with distinct eigenvalues, there exists a unique multidimensional continued fraction corresponding to it. One should take invariant hyperplanes for the action of the operator as hyperplanes that define the corresponding multidimensional continued fraction. Let us consider only three-dimensional hyperbolic operators that are defined by integer matrices with rational eigenvalues. Denote the set of all such operators by A_3. A continued fraction for any operator in A_3 consists of finitely many faces. Denote by $A_3(m)$ the set of all operators in A_3 such that the sum of absolute values of all coefficients of each of these operators is no greater than m. The number of such operators is finite. Let us calculate the number of triangles, quadrangles, and so on, among the continued fractions constructed for the operators in $A_3(m)$. As m tends to infinity, we have a general distribution of the frequencies for triangles, quadrangles, and so on. Arnold's problem includes the study of the properties of such a distribution (for instance, *what is more frequent, triangles or quadrangles*? *what is the frequency of integer points inside the faces*? etc.). Note that this problem has not yet been completely studied. Surely, the questions formulated above can be easily generalized to the multidimensional case.

V.I. Arnold has also formulated statistical problems for special algebraic periodic multidimensional continued fractions. For more information, see [4, 5].

2.3. Smooth manifold of n-dimensional continued fractions. Denote the set of all continued fractions of dimension n by CF_n. Let us describe a natural structure of a smooth nonsingular nonclosed manifold on the set CF_n.

Consider an arbitrary continued fraction that is defined by an unordered collection of hyperplanes $(\pi_1,\dots,\pi_{n+1})$. The enumeration of planes here is arbitrary, without any ordering. Denote by l_i, $i=1,\dots,n+1$, the intersection of all the above hyperplanes except the hyperplane π_i. Obviously, $l_1,\dots,l_{n+1}$ are *independent* straight lines (i.e., they are not contained in a hyperplane) passing through the origin. These straight lines form an unordered collection of independent straight lines. On the other hand, any unordered collection of $n+1$ independent straight lines uniquely determines some continued fraction.

Denote the sets of all ordered collections of $n+1$ independent and dependent straight lines by FCF_n and Δ_n, respectively. We say that FCF_n is a space of n-dimensional *framed continued fractions.* Also denote by S_{n+1} the permutation group acting on ordered collections of $n+1$ straight lines. In this notation we have

$$\mathrm{FCF}_n=\big(\underbrace{\mathbb{RP}^n\times\mathbb{RP}^n\times\ldots\times\mathbb{RP}^n}_{n+1\text{ times}}\big)\setminus\Delta_n \qquad\text{and}\qquad \mathrm{CF}_n=\mathrm{FCF}_n/S_{n+1}.$$

Therefore, the sets FCF_n and CF_n admit natural structures of smooth manifolds that are induced by the structure of the Cartesian product of $n+1$ projective spaces $\mathbb{RP}^n$. Note also that FCF_n is an

$(n+1)!$-fold covering of CF_n. We call the map of "forgetting" the order in the ordered collections the *natural projection* of the manifold FCF_n to the manifold CF_n and denote it by p, $p\colon \mathrm{FCF}_n \to \mathrm{CF}_n$.

2.4. Möbius measure on the manifolds of multidimensional continued fractions. The group $\mathrm{PGL}(n+1,\mathbb{R})$ of transformations of $\mathbb{R}\mathrm{P}^n$ takes the set of all straight lines passing through the origin in the $(n+1)$-dimensional space into itself. Hence, $\mathrm{PGL}(n+1,\mathbb{R})$ naturally acts on the manifolds CF_n and FCF_n. Furthermore, the action of $\mathrm{PGL}(n+1,\mathbb{R})$ is transitive; i.e., it takes any (framed) continued fraction to any other. Note that for any n-dimensional (framed) continued fraction, the subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$ taking this continued fraction to itself is of dimension n.

Definition 2.1. A form of the manifold CF_n (respectively, FCF_n) is said to be a *Möbius form* if it is invariant under the action of $\mathrm{PGL}(n+1,\mathbb{R})$.

The transitivity of the action of $\mathrm{PGL}(n+1,\mathbb{R})$ implies that all n-dimensional Möbius forms of the manifolds CF_n and FCF_n are proportional, if they exist.

Let ω be some volume form of a manifold M. Denote by μ_ω a measure on the manifold M that is defined by the equality

$$\mu_\omega(S) = \left| \int_S \omega \right|$$

for any open measurable set S contained in the same connected component of M.

Definition 2.2. A measure μ on the manifold CF_n (FCF_n) is said to be a *Möbius measure* if there exists a Möbius form ω of CF_n (FCF_n) such that $\mu = \mu_\omega$.

Note that any two Möbius measures of CF_n (FCF_n) are proportional.

Remark 2.3. The projection p takes the Möbius measures of the manifold FCF_n to the Möbius measures of the manifold CF_n, thus establishing an isomorphism between the spaces of Möbius measures for CF_n and FCF_n. Since the manifold of framed continued fractions possesses a simpler chart system, all formulae of the work are given for the manifold of framed continued fractions. To calculate a measure of some set F of the manifold of unframed continued fractions, one should take $p^{-1}(F)$, calculate the Möbius measure of the obtained set of the manifold of framed continued fractions, and divide the result by $(n+1)!$.

3. ONE-DIMENSIONAL CASE

3.1. Explicit formulae for the Möbius form. Let us write out explicitly the Möbius forms of the manifold FCF_1 of framed one-dimensional continued fractions in special charts.

Consider a vector space $\mathbb{R}^2$ equipped with a standard metric. Let l be an arbitrary straight line in $\mathbb{R}^2$ that does not pass through the origin, and let us choose some Euclidean coordinates O_lX_l on it. Denote by $\mathrm{FCF}_{1,l}$ a chart of the manifold FCF_1 that consists of all ordered pairs of straight lines each of which intersects l. Let us assign to any point of $\mathrm{FCF}_{1,l}$ (i.e., to a collection of two straight lines) coordinates (x_l, y_l), where x_l and y_l are the coordinates on l for the intersections of l with the first and the second straight lines of the collection, respectively. Denote by $|\overline{v}|_l$ the Euclidean length of a vector $\overline{v}$ in the coordinates $O_lX_lY_l$ of the chart $\mathrm{FCF}_{1,l}$. Note that the chart $\mathrm{FCF}_{1,l}$ is the space $\mathbb{R}\times\mathbb{R}$ minus its diagonal.

Consider the following form in the chart $\mathrm{FCF}_{1,l}$:

$$\omega_l(x_l, y_l) = \frac{dx_l \wedge dy_l}{|x_l - y_l|_l^2}.$$

Proposition 3.1. *The measure* μ_{ω_l} *coincides with the restriction of some Möbius measure to* $\mathrm{FCF}_{1,l}$.

Proof. The transformations of the group $\mathrm{PGL}(2,\mathbb{R})$ are in one-to-one correspondence with the set of all projective transformations of the projectivization of the straight line l. Note that the expression

$$\frac{\Delta x_l \Delta y_l}{|x_l - y_l|_l^2}$$

is an infinitesimal cross-ratio of four points with coordinates x_l, y_l, $x_l + \Delta x_l$, and $y_l + \Delta y_l$. Hence, the form $\omega_l(x_l, y_l)$ is invariant under the transformations (of the everywhere dense set) of the chart $\mathrm{FCF}_{1,l}$ that are induced by projective transformations of l. Therefore, the measure μ_{ω_l} coincides with the restriction of some Möbius measure to $\mathrm{FCF}_{1,l}$. □

Corollary 3.2. *The restriction of an arbitrary Möbius measure to the chart* $\mathrm{FCF}_{1,l}$ *is proportional to* μ_{ω_l}.

Proof. The statement follows from the proportionality of any two Möbius measures. □

Consider now the manifold FCF_1 as a set of ordered pairs of distinct points on a circle $\mathbb{R}/\pi\mathbb{Z}$ (this circle is a one-dimensional projective space obtained from the unit circle by identifying its antipodal points). The doubled angular coordinate φ of the circle $\mathbb{R}/\pi\mathbb{Z}$ induced by the coordinate x of the straight line $\mathbb{R}$ naturally defines coordinates (φ_1, φ_2) of the manifold FCF_1.

Proposition 3.3. *The form* $\omega_l(x_l, y_l)$ *is extendable to some form* ω_1 *of* FCF_1. *In the coordinates* (φ_1, φ_2) *the form* ω_1 *can be written as follows*:

$$\omega_1 = \frac{1}{4}\cot^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) d\varphi_1 \wedge d\varphi_2.$$

We leave the proof of Proposition 3.3 as an exercise for the reader.

3.2. Relative frequencies of faces of one-dimensional continued fractions. Without loss of generality, in this subsection we consider only the Möbius form ω_1 of Proposition 3.3. Denote the natural projection of the form μ_{ω_1} to the manifold of one-dimensional continued fractions CF_1 by μ_1.

Consider an arbitrary segment F with endpoints at integer points. Denote by $\mathrm{CF}_1(F)$ the set of continued fractions that contain the segment F as a face.

Definition 3.4. The quantity $\mu_1(\mathrm{CF}_1(F))$ is called the *relative frequency* of the face F.

Note that the relative frequencies of faces of the same integer-linear type coincide. Any face of a one-dimensional continued fraction lies at unit integer distance from the origin. Thus, the integer-linear type of a face is defined by its integer length (the number of inner integer points plus one). Denote the relative frequency of an edge of integer length k by $\mu_1(''k'')$.

Proposition 3.5. *For any positive integer* k *the following equality holds*:

$$\mu_1(''k'') = \ln\left(1 + \frac{1}{k(k+2)}\right).$$

Proof. Consider a particular representative of the integer-linear type of a length-k segment: the segment with endpoints $(0,1)$ and $(k,1)$. A one-dimensional continued fraction contains this segment as a face if and only if one of the straight lines defining the fraction intersects the interval with endpoints $(-1,1)$ and $(0,1)$ while the other straight line intersects the interval with endpoints $(k,1)$ and $(k+1,1)$ (see Fig. 2).

For the straight line l defined by the equation $y = 1$, we calculate the Möbius measure of the Cartesian product of the described pair of intervals. According to the previous subsection, this

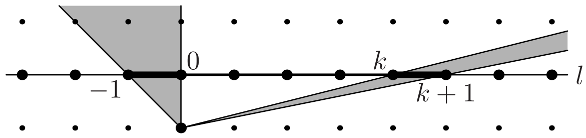


Fig. 2. The rays defining a continued fraction should lie in the domain colored in gray.

quantity coincides with the relative frequency $\mu_1(''k'')$. So,

$$\mu_1(''k'') = \int_{-1}^{0}\int_{k}^{k+1} \frac{dx_l\,dy_l}{(x_l-y_l)^2} = \int_{k}^{k+1}\left(\frac{1}{y_l}-\frac{1}{y_l+1}\right)dy_l = \ln\left(\frac{(k+1)(k+1)}{k(k+2)}\right) = \ln\left(1+\frac{1}{k(k+2)}\right).$$

This proves the proposition. □

Remark 3.6. Note that the argument of the logarithm, $\frac{(k+1)(k+1)}{k(k+2)}$, is the cross-ratio of the points $(-1,1)$, $(0,1)$, $(k,1)$, and $(k+1,1)$.

Corollary 3.7. *The relative frequency $\mu_1(''k'')$ coincides up to the factor*

$$\ln 2 = \int_{-1}^{0}\int_{1}^{+\infty} \frac{dx_l\,dy_l}{(x_l-y_l)^2}$$

with the Gauss–Kuzmin frequency $P(k)$ of occurrence of k in a continued fraction. □

4. MULTIDIMENSIONAL CASE

4.1. Explicit formulae for the Möbius form. Let us now write out explicitly the Möbius forms for the manifold FCF_n of framed n-dimensional continued fractions for arbitrary n.

Consider $\mathbb{R}^{n+1}$ with the standard metric on it. Let π be an arbitrary hyperplane of the space $\mathbb{R}^{n+1}$ with chosen Euclidean coordinates $OX_1\dots X_n$ that does not pass through the origin. By a chart $\mathrm{FCF}_{n,\pi}$ of the manifold FCF_n we mean the set of all collections of $n+1$ ordered straight lines each of which intersects π. Let the intersection of π with the ith line be a point with coordinates $(x_{1,i},\dots,x_{n,i})$ on the plane π. For an arbitrary tetrahedron $A_1\dots A_{n+1}$ in the plane π, we denote by $V_\pi(A_1\dots A_{n+1})$ its oriented Euclidean volume in the coordinates $OX_{1,1}\dots X_{n,1}X_{1,2}\dots X_{n,n+1}$ of the chart $\mathrm{FCF}_{n,\pi}$. Denote by $|\overline{v}|_\pi$ the Euclidean length of the vector $\overline{v}$ in the coordinates $OX_{1,1}\dots X_{n,1}X_{1,2}\dots X_{n,n+1}$ of the chart $\mathrm{FCF}_{n,\pi}$. Note that the chart $\mathrm{FCF}_{n,\pi}$ is everywhere dense in $(\mathbb{R}^n)^{n+1}$.

Consider the following form in the chart $\mathrm{FCF}_{n,\pi}$:

$$\omega_\pi(x_{1,1},\dots,x_{n,n+1}) = \frac{\bigwedge_{i=1}^{n+1}\left(\bigwedge_{j=1}^{n} dx_{j,i}\right)}{\left(V_\pi(A_1\dots A_{n+1})\right)^{n+1}}.$$

Proposition 4.1. *The measure μ_{ω_π} coincides with the restriction of some Möbius measure to* $\mathrm{FCF}_{n,\pi}$.

Proof. The transformations of the group $\mathrm{PGL}(n+1,\mathbb{R})$ are in one-to-one correspondence with the set of all projective transformations of the hyperplane π. Let us show that the form ω_π is invariant under the transformations (of the everywhere dense set) of the chart $\mathrm{FCF}_{n,\pi}$ that are induced by projective transformations of the hyperplane π.

At each point of the tangent space to $\mathrm{FCF}_{n,\pi}$, define a new basis corresponding to the directions of edges of the corresponding tetrahedron in π. Namely, consider an arbitrary point $(x_{1,1},\dots,x_{n,n+1})$

of the chart $\mathrm{FCF}_{n,\pi}$ and the tetrahedron $A_1 \ldots A_{n+1}$ in the hyperplane π corresponding to this point. Let

$$\overline{f}_{ij} = \frac{\overline{A_j A_i}}{|\overline{A_j A_i}|_\pi}, \qquad i,j = 1,\ldots,n+1, \quad i \neq j.$$

The basis constructed above continuously depends on the point of the chart $\mathrm{FCF}_{n,\pi}$. By dv_{ij} we denote the 1-form corresponding to the coordinate along the vector $\overline{f}_{ij}$ of $\mathrm{FCF}_{n,\pi}$.

Denote by $A_i = A_i(x_{1,i},\ldots,x_{n,i})$ the point with coordinates $(x_{1,i},\ldots,x_{n,i})$, $i = 1,\ldots,n+1$, which depends on the coordinates of the plane π. Let us rewrite the form ω_π in the new coordinates:

$$\omega_\pi(x_{1,1},\ldots,x_{n,n+1}) = \prod_{i=1}^{n+1}\left(\frac{V_\pi(A_i A_1 \ldots A_{i-1}A_{i+1}\ldots A_{n+1})}{\prod_{k=1,\,k\neq i}^{n+1}|\overline{A_k A_i}|_\pi}\right)\frac{dv_{21}\wedge dv_{31}\wedge\ldots\wedge dv_{n,n+1}}{\big(V_\pi(A_1\ldots A_{n+1})\big)^{n+1}}$$

$$= (-1)^{\left[\frac{n+3}{4}\right]}\frac{dv_{21}\wedge dv_{12}}{|\overline{A_1A_2}|_\pi^2}\wedge\frac{dv_{32}\wedge dv_{23}}{|\overline{A_2A_3}|_\pi^2}\wedge\ldots\wedge\frac{dv_{n+1,n}\wedge dv_{n,n+1}}{|\overline{A_nA_{n+1}}|_\pi^2},$$

where $[a]$ denotes the maximal integer not exceeding a.

As in the one-dimensional case, the expression

$$\frac{\Delta v_{ij}\Delta v_{ji}}{|\overline{A_iA_j}|^2}$$

for the infinitesimal increments Δv_{ij} and Δv_{ji} is the infinitesimal cross-ratio of the four points A_i, A_j, $A_i + \Delta v_{ji}\overline{f}_{ji}$, and $A_j + \Delta v_{ij}\overline{f}_{ij}$ of the straight line A_iA_j. Therefore, the form ω_π is invariant under the transformations (of the everywhere dense set) of the chart $\mathrm{FCF}_{n,\pi}$ that are induced by projective transformations of the hyperplane π. Hence, the measure μ_{ω_π} coincides with the restriction of some Möbius measure to $\mathrm{FCF}_{n,\pi}$. □

Corollary 4.2. *The restriction of an arbitrary Möbius measure to the chart* $\mathrm{FCF}_{n,\pi}$ *is proportional to* μ_{ω_π}.

Proof. The statement follows from the proportionality of any two Möbius measures. □

Let us fix an origin O_{ij} on the straight line A_iA_j. The integral of the form dv_{ij} (respectively, dv_{ji}) over the segment $O_{ij}P$ defines a *coordinate* v_{ij} (v_{ji}) of the point P on the straight line A_iA_j. As in the one-dimensional case, consider a projectivization of the straight line A_iA_j. Denote the angular coordinates by φ_{ij} and φ_{ji}, respectively. In these coordinates

$$\frac{dv_{ij}\wedge dv_{ji}}{|\overline{A_iA_j}|_\pi^2} = \frac{1}{4}\cot^2\left(\frac{\varphi_{ij}-\varphi_{ji}}{2}\right)d\varphi_{ij}\wedge d\varphi_{ji}.$$

Then, the following is true.

Corollary 4.3. *The form* ω_π *extends to some form* ω_n *of* FCF_n. *In the coordinates* v_{ij} *the form* ω_n *is as follows*:

$$\omega_n = \frac{(-1)^{\left[\frac{n+3}{4}\right]}}{2^{n(n+1)}}\left(\prod_{i=1}^{n+1}\prod_{j=i+1}^{n+1}\cot^2\left(\frac{\varphi_{ij}-\varphi_{ji}}{2}\right)\right)\left(\bigwedge_{i=1}^{n+1}\left(\bigwedge_{j=i+1}^{n+1}d\varphi_{ij}\wedge d\varphi_{ji}\right)\right).$$

4.2. Relative frequencies of faces of multidimensional continued fractions. As in the one-dimensional case, without loss of generality, we consider the form ω_n of Corollary 4.3. Denote by μ_n the projection of the measure μ_{ω_n} to the manifold CF_n of multidimensional continued fractions.

Consider an arbitrary polytope F with vertices at integer points. Denote by $\mathrm{CF}_n(F)$ the set of n-dimensional continued fractions that contain the polytope F as a face.

Definition 4.4. The value $\mu(\mathrm{CF}_n(F))$ is called the *relative frequency* of a face F.

The relative frequencies of faces of the same integer-linear type coincide.

Problem 1. *Find integer-linear types of n-dimensional faces with the highest relative frequencies. Is it true that the number of integer-linear types of faces with relative frequencies greater than some constant is finite? Find its asymptotics as the constant tends to infinity.*

Problem 1 is open for $n \geq 2$.

Conjecture 2. *The relative frequencies of faces are proportional to the frequencies of faces in the sense of Arnold (see Subsection 2.2).*

This conjecture is checked in the present paper for the case of one-dimensional continued fractions. It is still open in the n-dimensional case for $n \geq 2$.

5. EXAMPLES OF CALCULATING RELATIVE FREQUENCIES FOR FACES IN THE TWO-DIMENSIONAL CASE

5.1. A method for calculating relative frequencies. Let us describe in detail a method for calculating relative frequencies in the two-dimensional case.

Consider the space $\mathbb{R}^3$ with the standard metric on it. Let π be an arbitrary plane in $\mathbb{R}^3$ that does not pass through the origin and is endowed with a fixed system of Euclidean coordinates $O_\pi X_\pi Y_\pi$. Let $\mathrm{FCF}_{2,\pi}$ be the corresponding chart of the manifold FCF_2 (see the previous section). For an arbitrary triangle ABC on the plane π, we denote by $S_\pi(ABC)$ its oriented Euclidean area in the coordinates $O_\pi X_1Y_1X_2Y_2X_3Y_3$ of the chart $\mathrm{FCF}_{2,\pi}$. Denote by $|\overline{v}|_\pi$ the Euclidean length of the vector $\overline{v}$ in the coordinates $O_\pi X_1Y_1X_2Y_2X_3Y_3$ of the chart $\mathrm{FCF}_{2,\pi}$. Consider the following form in the chart $\mathrm{FCF}_{2,\pi}$:

$$\omega_\pi(x_1,y_1,x_2,y_2,x_3,y_3) = \frac{dx_1\wedge dy_1\wedge dx_2\wedge dy_2\wedge dx_3\wedge dy_3}{\big(S_\pi((x_1,y_1)(x_2,y_2)(x_3,y_3))\big)^3}.$$

Note that the oriented area S_π of the triangle $(x_1,y_1)(x_2,y_2)(x_3,y_3)$ can be expressed in the coordinates x_i, y_i as follows:

$$S_\pi((x_1,y_1)(x_2,y_2)(x_3,y_3)) = \frac{1}{2}\big(x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1 + x_2y_1 - x_1y_2\big).$$

For the approximate computation of relative frequencies of faces, it is useful to rewrite the form ω_π in the dual coordinates (see Remark 5.2 below). Define a triangle ABC in the plane π by three straight lines l_1, l_2, and l_3, where l_1 passes through B and C, l_2 passes through A and C, and l_3 passes through A and B. Define the straight line l_i $(i = 1,2,3)$ in π by an equation (but first we move the origin to some inner point of the triangle by a parallel translation of π)

$$a_ix + b_iy = 1$$

for the variables x and y. Then, if we know the 6-tuple of numbers $(a_1,b_1,a_2,b_2,a_3,b_3)$, we can restore the triangle in a unique way.

Proposition 5.1. *In the coordinates a_1,b_1,a_2,b_2,a_3,b_3 the form ω_π can be written as follows:*

$$-\frac{8da_1\wedge db_1\wedge da_2\wedge db_2\wedge da_3\wedge db_3}{(a_3b_2 - a_2b_3 + a_1b_3 - a_3b_1 + a_2b_1 - a_1b_2)^3}. \qquad \square$$

So, we reduce the computation of the relative frequency for the face F, i.e., the value of $\mu_2(\mathrm{CF}_2(F))$, to the computation of the measure $\mu_{\omega_2}(p^{-1}(\mathrm{CF}_2(F)))$. Consider a plane π in $\mathbb{R}^3$ that does not pass through the origin. By Corollary 4.3,

$$\mu_{\omega_2}(p^{-1}(\mathrm{CF}_2(F))) = \mu_{\omega_\pi}\big(p^{-1}(\mathrm{CF}_2(F)) \cap \mathrm{FCF}_{2,\pi}\big).$$

Finally, the computation should be carried out for the measure $\mu_{\omega_\pi}(p^{-1}(\mathrm{CF}_2(F)) \cap \mathrm{FCF}_{2,\pi})$ in the dual coordinates a_i, b_i (see Proposition 5.1).

Remark 5.2. In the coordinates a_i, b_i the computation of the relative frequency often reduces to the estimation of the integral over the disjoint union of a finite number of six-dimensional Cartesian products of three triangles in the coordinates a_i, b_i (see Proposition 5.1). The integration over such a simple domain greatly increases the speed of approximate computations. In particular, the integration can be reduced to the integration over some 4-dimensional domain.

5.2. Some results. In conclusion, we give some results of calculating the relative frequencies for some two-dimensional faces of two-dimensional continued fractions.

It is hardly possible to explicitly calculate the relative frequencies for faces. Nevertheless, one can make approximations of the corresponding integrals. Normally, the greater the area of the integer-linear type of a polygon, the smaller its relative frequency. The most complicated approximate calculations are those for the simplest faces, such as an empty triangle.

Figure 3 shows examples of the following faces: triangular $(0,0,1)$, $(0,1,1)$, $(1,0,1)$ and $(0,0,1)$, $(0,2,1)$, $(2,0,1)$ and a quadrangular $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,0,1)$ faces. For each face the figure presents the plane containing this face. The points painted in light gray correspond to the points at which the rays defining the two-dimensional continued fraction may intersect the plane of the chosen face.

For the majority of integer-linear types, the faces of two-dimensional continued fractions lie at unit integer distance from the origin. Only three infinite series and three particular examples of faces lie at a greater integer distance from the origin (see a detailed description in [17, 18]). As the distance to a face increases, the frequency of the face decreases on the average. The average decrease rate of the frequency is unknown to the author.

In the table we show calculated relative frequencies for 12 integer-linear types of faces. In column “No.” we write a special symbol for the integer-affine type of a face. The index denotes the integer distance from the corresponding face to the origin. In column “Face” we give a picture of the integer-affine type of the face. Further, in column “lS” we write out the integer areas of faces, and in column “ld” we write out the integer distances from the planes of faces to the origin. Finally, in column “μ_2” we show the results of the approximate calculations of the relative frequencies for the corresponding integer-linear types of faces.

Note that in these examples the integer-affine type and the integer distance to the origin determine the integer-linear type of the face.

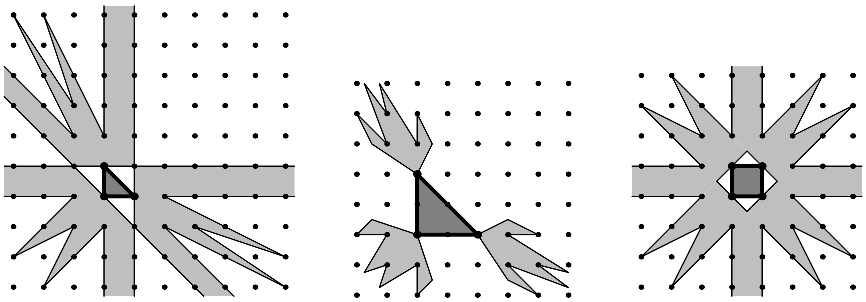
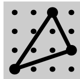
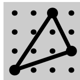
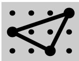
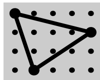
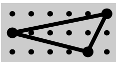
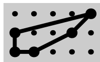
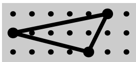


Fig. 3. The points painted in light gray correspond to the points at which the rays defining the two-dimensional continued fraction may intersect the plane of the chosen face.

Some calculated relative frequencies

No.	Face	lS	ld	μ_2	No.	Face	lS	ld	μ_2
$\mathbf{I}_1$		3	1	$1.3990 \cdot 10^{-2}$	$\mathbf{VI}_1$		7	1	$3.1558 \cdot 10^{-4}$
$\mathbf{I}_3$		3	3	$1.0923 \cdot 10^{-3}$	$\mathbf{VI}_2$		7	2	$3.1558 \cdot 10^{-4}$
$\mathbf{II}_1$		5	1	$1.5001 \cdot 10^{-3}$	$\mathbf{VII}_1$		11	1	$3.4440 \cdot 10^{-5}$
$\mathbf{III}_1$		7	1	$3.0782 \cdot 10^{-4}$	$\mathbf{VIII}_1$		7	1	$5.6828 \cdot 10^{-4}$
$\mathbf{IV}_1$		9	1	$9.4173 \cdot 10^{-5}$	$\mathbf{IX}_1$		7	1	$1.1865 \cdot 10^{-3}$
$\mathbf{V}_1$		11	1	$3.6391 \cdot 10^{-5}$	$\mathbf{X}_1$		6	1	$9.9275 \cdot 10^{-4}$

In conclusion of this section we give two simple statements on the relative frequencies of faces.

Statement 5.3. *Faces of the same integer-affine type that lie at an integer distance of* 1 *from the origin and at an integer distance of* 2 *from the origin always have the same relative frequencies* (*see, for example,* $\mathbf{VI}_1$ *and* $\mathbf{VI}_2$ *in the table*).

Denote by A_n the triangle with vertices $(0,0,1)$, $(n,0,1)$, and $(0,n,1)$. Denote by B_n the square with vertices $(0,0,1)$, $(n,0,1)$, $(n,n,1)$, and $(0,n,1)$.

Statement 5.4. *The following equality holds*:

$$\lim_{n\to\infty} \frac{\mu(\mathrm{CF}_n(A_n))}{\mu(\mathrm{CF}_n(B_n))} = 8.$$

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Number Theory/Geometry

Three examples of three-dimensional continued fractions in the sense of Klein

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Abstract

The problem of the investigation of the simplest n-dimensional continued fraction in the sense of Klein for $n \geqslant 2$ was posed by V. Arnold. The answer for the case $n = 2$ can be found in the works of E. Korkina (1995) and G. Lachaud (1995). In present Note we study the case $n = 3$. ***To cite this article: O. Karpenkov, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Trois exemples des fractions continues trois-dimensional en sens de Klein. Le problème de l'étude les plus simple fractions continues n-dimensional en sens de Klein pour $n \geqslant 2$ a été poser de V. Arnold. Le solution pour la case de $n = 2$ a presenté dans les articles de E. Korkina (1995) et G. Lachaud (1995). Dans la Note présente, on étude la case de $n = 3$. ***Pour citer cet article : O. Karpenkov, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Definitions

A point of $\mathbb{R}^{n+1}$ is called *integer* if all its coordinates are integers. A hyperplane is called *integer* if all its integer vectors generate an n-dimensional sublattice of integer lattice. Consider some integer hyperplane and an integer point in the complement to this plane. Let the Euclidean distance from the given point to the given plane equal l. The minimal value of nonzero Euclidean distances from integer points of the space $\mathbb{R}^{n+1}$ to the plane is denoted by l_0. The ratio l/l_0 is said to be the *integer distance* from the given integer point to the given integer hyperplane.

2. Definition of multidimensional continued fraction in the sense of Klein

Consider arbitrary $n+1$ hyperplanes in $\mathbb{R}^{n+1}$ that intersect at the unique point: at the origin. Assume also that all the given planes do not contain any integer point different to the origin. The complement to these hyperplanes consists

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of 2^{n+1} open orthants. Consider one of these orthants. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the *sail* of the orthant. The set of all 2^{n+1} sails is called the *n-dimensional continued fraction* constructed accordingly to the given $n+1$ hyperplanes. Two n-dimensional continued fractions are said to be *equivalent* if there exists a linear lattice preserving transformation of $\mathbb{R}^{n+1}$ taking all sails of one continued fraction to the sails of the other continued faction.

We associate to any hyperbolic irreducible operator A of $\mathrm{SL}(n+1,\mathbb{Z})$ an n-dimensional continued fraction constructed according to the set of all $n+1$ eigen-hyperplanes for A. Any sail of such continued fraction is homeomorphic to $\mathbb{R}^n$. From Dirichlet unity theorem it follows that the group of all $\mathrm{SL}(n+1,\mathbb{Z})$-operators commuting with A and preserving the sails is homeomorphic to $\mathbb{Z}^n$ and its action is free (we denote this group by $\Xi(A)$). A *fundamental domain* of the sail with respect to the action of the group $\Xi(A)$ is a face union that contains exactly one face of the sail from each orbit. (For more information see [1–5].)

3. The examples

Denote by $A_{a,b,c,d}$ the following integer operator

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

Example 1. Consider the operator $A_1 = A_{1,-3,0,4}$. The group $\Xi(A_1)$ is generated by the operators $B_{11} = A_1^{-2}$, $B_{12} = (A_1-E)^2 A_1^{-2}$, and $B_{13} = (A_1-E)^2(A_1+E)A_1^{-2}$. Let us enumerate all three-dimensional faces for one of the fundamental domains of the sail containing the vertex $(0,0,0,1)$. Let $V_{10} = (-3,-2,-1,1)$, $V_{1,4i+2j+k} = B_{11}^i B_{12}^j B_{13}^k (V_{10})$ for $i,j,k \in \{0,1\}$. One of the fundamental domains of the sail contains the following three-dimensional faces: $T_{11} = V_{10}V_{12}V_{14}V_{15}$, $T_{12} = V_{12}V_{14}V_{15}V_{16}$, $T_{13} = V_{12}V_{15}V_{16}V_{17}$, $T_{14} = V_{12}V_{13}V_{15}V_{17}$, $T_{15} = V_{10}V_{12}V_{13}V_{15}$, $T_{16} = V_{10}V_{11}V_{13}V_{15}$, and $T_{17} = V_{10}V_{11}V_{12}V_{13}$ (see Fig. 1 (left)). All listed tetrahedra are taken by some integer affine transformations to the unit basis tetrahedron. The integer distance from the origin to the planes containing the faces $T_{11},\ldots,T_{17}$ equal 4, 3, 2, 4, 3, 2, and 1, respectively.

Statement 1. The continued fraction constructed for any hyperbolic matrix of $\mathrm{SL}(4,\mathbb{Z})$ with irreducible characteristic polynomial over rationals and with the sum of absolute values of the elements smaller than 8 is equivalent to the continued fraction of Example 1.

Statement 2. The symmetry (not commuting with A_1) defined by the matrix

$$\begin{pmatrix} 4 & -16 & 17 & -3 \\ 3 & -11 & 11 & -2 \\ 3 & -8 & 6 & -1 \\ 6 & -8 & -2 & 1 \end{pmatrix}$$

acts on the sail of Example 1. This symmetry permutes the equivalence classes (with respect to the action of $\Xi(A_1)$) of tetrahedra T_{11} and T_{14}, T_{12} and T_{15}, T_{13} and T_{16}, and takes the class of T_{17} to itself.

Example 2. Let us consider the operator $A_2 = A_{1,-4,1,4}$. The group $\Xi(A_2)$ is generated by the operators $B_{21} = A_2^{-2}$, $B_{22} = (A_2-E)^2 A_2^{-2}$, and $B_{23} = (A_2+E)A_2^{-1}$. Let us enumerate all three-dimensional faces for one of the fundamental domains of the sail containing the vertex $(0,0,0,1)$. Let $V_{20} = (-4,-3,-2,0)$, $V_{2,4i+2j+k} = B_{21}^i B_{22}^j B_{23}^k (V_{20})$ for $i,j,k \in \{0,1\}$. One of the fundamental domains of the sail contains the following three-dimensional faces: $T_{21} = V_{20}V_{21}V_{23}V_{24}$, $T_{22} = V_{21}V_{23}V_{24}V_{25}$, $T_{23} = V_{20}V_{22}V_{23}V_{24}$, $T_{24} = V_{22}V_{23}V_{24}V_{26}$, $T_{25} = V_{23}V_{24}V_{25}V_{27}$, and $T_{26} = V_{23}V_{24}V_{26}V_{27}$ (see Fig. 1 (middle)). All listed tetrahedra are taken by some integer affine transformations to the unit basis tetrahedron. The integer distance from the origin to the planes containing the faces $T_{21},\ldots,T_{26}$ equal 1, 2, 2, 4, 8, and 13, respectively.

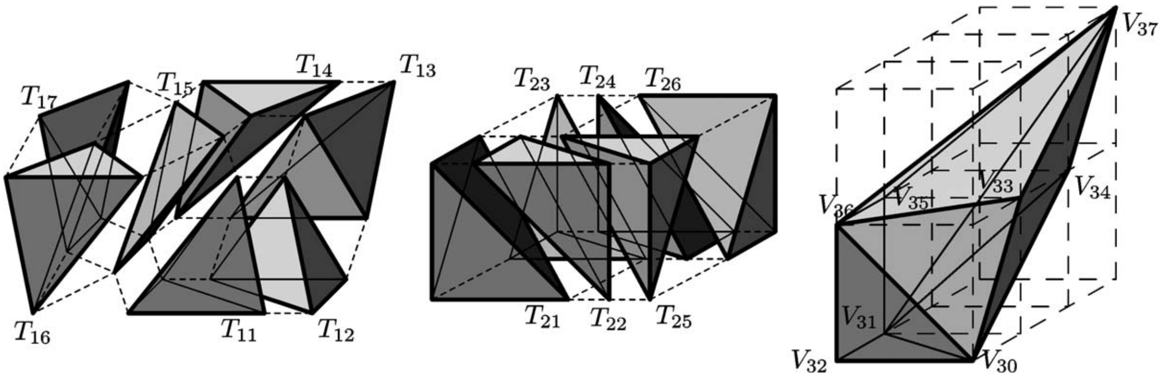

Fig. 1. Gluing faces; see text for details.

Example 3. Consider the operator $A_3 = A_{-1,-3,1,3}$. The group $\Xi(A_3)$ is generated by the operators $B_{31} = A_3^{-2}$, $B_{32} = (A_3 - E)A_3^{-1}$, and $B_{33} = A_3 + E$. Any fundamental domain of the sail with $(0,0,0,1)$ as a vertex contains a unique three-dimensional face. The polyhedron $V_{30}V_{31}V_{32}V_{33}V_{34}V_{35}V_{36}V_{37}$ shown on Fig. 1 (right) is an example of such face, here $V_{30} = (-1,-1,-1,0)$, $V_{31} = B_{33}(V_{30})$, $V_{32} = B_{32}B_{33}(V_{30})$, $V_{33} = B_{31}B_{32}^{-1}(V_{30})$, $V_{34} = B_{32}^{-1}(V_{30})$, $V_{35} = B_{31}B_{33}^2(V_{30})$, $V_{36} = B_{31}B_{33}(V_{30})$, $V_{37} = B_{31}B_{32}^{-1}B_{33}(V_{30})$. The described face is contained in the plane on the unit distance from the origin. The integer volume of the face equals 8.

Example 3 provides the negative answer to the following question for the case of $n = 3$: *is it true, that any n-periodic n-dimensional sail contains an n-dimensional face in some hyperplane on integer distance to the origin greater than one?* The answers for $n = 2, 4, 5, 6, \ldots$ are unknown. The answer to the following question is also unknown to the author: *is it true, that any n-periodic n-dimensional sail contains an n-dimensional face in some hyperplane on unit integer distance to the origin?*

We show with dotted lines (Fig. 1) how to glue the faces to obtain the combinatorial scheme of the described fundamental domains.

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