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1 Definitions

Recall the definition of $\tilde{\Sigma}$ (Equation (6) on page 50 of the paper)

$$\tilde{\Sigma}\big((\alpha, a), (\beta, b), (\gamma, c)\big) = \breve{K}(\alpha\beta).$$

By the definitions of $\otimes$ on page 50 and of L_σ and R_σ on page 36 we have

$$\begin{aligned}
L_\otimes\big((\alpha, a), (\beta, b), (\gamma, c)\big) &= \Big((\alpha, a), \otimes\big((\alpha, a), (\beta, b), (\gamma, c)\big), (\beta, b)\Big) \\
&= \Big((\alpha, a), \Big(\alpha\beta, \tilde{\Sigma}\big((\alpha, a), (\beta, b), (\gamma, c)\big)\Big), (\beta, b)\Big) \\
&= \Big((\alpha, a), \big(\alpha\beta, \breve{K}(\alpha\beta)\big), (\beta, b)\Big), \\
R_\otimes\big((\alpha, a), (\beta, b), (\gamma, c)\big) &= \Big((\alpha, a), \otimes\big((\beta, b), (\gamma, c), (\alpha, a)\big), (\beta, b)\Big) \\
&= \Big((\beta, b), \Big(\beta\gamma, \tilde{\Sigma}\big((\beta, b), (\gamma, c), (\alpha, a)\big)\Big), (\gamma, c)\Big) \\
&= \Big((\beta, b), \big(\beta\gamma, \breve{K}(\beta\gamma)\big), (\gamma, c)\Big).
\end{aligned}$$

2 Corollary 7.19

Corollary 7.19 says

$$\tilde{\Sigma}\Big((\alpha, a), \big(\alpha\beta, \breve{K}(\alpha\beta)\big), (\beta, b)\Big) = \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha^2)}\breve{K}(\alpha\beta) - \breve{K}(\beta).$$

This misleading since this formula holds only for a triple formed by concatenation

$$\Big((\alpha, a), \big(\alpha\beta, \breve{K}(\alpha\beta)\big), (\beta, b)\Big).$$

This formula does not work for an arbitrary triple

$$\big((\alpha, a), (\beta, b), (\gamma, c)\big).$$

It should read like this

Corollary (Corollary 7.19)**.** Let $a = \breve{K}(a)$, $b = \breve{K}(\alpha\beta)$, and $c = \breve{K}(\beta)$. Then

$$\begin{aligned}\tilde{\Sigma}\big((\alpha, a), (\alpha\beta, b), (\beta, c)\big) &= \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)}\breve{K}(\alpha\beta) - \breve{K}(\beta)\\ &= \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)}b - c,\end{aligned}$$

and

$$\begin{aligned}\tilde{\Sigma}\big((\alpha\beta, b), (\beta, c), (\alpha, a)\big) &= \frac{\breve{K}(\beta^2)}{K(\beta)}\breve{K}(\alpha\beta) - \breve{K}(\alpha)\\ &= \frac{\breve{K}(\beta^2)}{K(\beta)}b - a.\end{aligned}$$

From this, the last line in Remark 6.19 should read

$$\begin{aligned}\tilde{\Sigma}\Big((\alpha, a), (\alpha\beta, b), (\beta, c)\Big) &= \Sigma(a, b, c) = 3ab - c,\\ \tilde{\Sigma}\Big((\alpha\beta, b), (\beta, c), (\alpha, a)\Big) &= \Sigma(b, c, a) = 3cb - a.\end{aligned}$$

3 Examples

With this corrected corollary in place we give an example. We use the following notation for a triple (α, β, γ)

$$P(\alpha, \beta, \gamma) = \big(\breve{K}(\alpha), \breve{K}(\beta), \breve{K}(\gamma)\big).$$

Example 1. Let $\alpha = (1, 1, 1, 1)$ and $\beta = (2, 2)$. For the triple $(\alpha, \alpha\beta, \beta)$ we have

$$P(\alpha, \alpha\beta, \beta) = (3, 13, 2).$$

Then

$$\begin{aligned}P\big(L(\alpha, \alpha\beta, \beta)\big) &= P(\alpha, \alpha\alpha\beta, \alpha\beta) = (3, 89, 13),\\ P\big(R(\alpha, \alpha\beta, \beta)\big) &= P(\alpha\beta, \alpha\beta\beta, \beta) = (13, 75, 2).\end{aligned}$$

Further,

$$\begin{aligned}P\big(L^2(\alpha, \alpha\beta, \beta)\big) &= P(\alpha, \alpha\alpha\alpha\beta, \alpha\alpha\beta) = (3, 610, 89),\\ P\big(RL(\alpha, \alpha\beta, \beta)\big) &= P(\alpha\alpha\beta, \alpha\alpha\beta\alpha\beta, \alpha\beta) = (89, 3468, 13),\\ P\big(LR(\alpha, \alpha\beta, \beta)\big) &= P(\alpha\beta, \alpha\beta\alpha\beta\beta, \alpha\beta\beta) = (13, 2923, 75),\\ P\big(R^2(\alpha, \alpha\beta, \beta)\big) &= P(\alpha\beta\beta, \alpha\beta\beta\beta, \beta) = (75, 437, 2).\end{aligned}$$

Now we use $\tilde{\Sigma}$. First note that

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} = 7, \qquad \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)} = 6.$$

$$
\begin{aligned}
L_\otimes\big((\alpha,3),(\alpha\beta,13),(\beta,2)\big) &= \Big((\alpha,3),\Big(\alpha\alpha\beta,\tilde{\Sigma}\big((\alpha,3),(\alpha\beta,13),(\beta,2)\big)\Big),(\alpha\beta,13)\Big)\\
&= \left((\alpha,3),\left(\alpha\alpha\beta,\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)}13-2\right),(\alpha\beta,13)\right)\\
&= \Big((\alpha,3),(\alpha\alpha\beta,7\cdot 13-2=89),(\alpha\beta,13)\Big),\\
R_\otimes\big((\alpha,3),(\alpha\beta,13),(\beta,2)\big) &= \Big((\alpha\beta,13),\Big(\alpha\beta\beta,\tilde{\Sigma}\big((\alpha\beta,13),(\beta,2),(\alpha,3)\big)\Big),(\beta,2)\Big)\\
&= \left((\alpha\beta,13),\left(\alpha\beta\beta,\frac{\breve{K}(\beta^2)}{\breve{K}(\beta)}13-3\right),(\beta,2)\right)\\
&= \Big((\alpha\beta,13),(\alpha\beta\beta,6\cdot 13-3=75),(\beta,2)\Big),
\end{aligned}
$$

For the next triples we just calculate $\tilde{\Sigma}$. We have

$$
\frac{\breve{K}(\alpha\beta\alpha\beta)}{\breve{K}(\alpha\beta)}=39.
$$

Then

$$
\begin{aligned}
\Big(\alpha\alpha\alpha\beta,\tilde{\Sigma}\big((\alpha,3),(\alpha\alpha\beta,89),(\alpha\beta,13)\big)\Big) &= \left(\alpha\alpha\alpha\beta,\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)}\breve{K}(\alpha\alpha\beta)-\breve{K}(\alpha\beta)\right)\\
&= (\alpha\alpha\alpha\beta,7\cdot 89-13=610),
\end{aligned}
$$

$$
\begin{aligned}
\Big(\alpha\alpha\beta\alpha\beta,\tilde{\Sigma}\big((\alpha\alpha\beta,89),(\alpha\beta,13),(\alpha,3)\big)\Big) &= \left(\alpha\alpha\beta\alpha\beta,\frac{\breve{K}(\alpha\beta\alpha\beta)}{\breve{K}(\alpha\beta)}\breve{K}(\alpha\alpha\beta)-\breve{K}(\alpha)\right)\\
&= (\alpha\alpha\beta\alpha\beta,39\cdot 89-3=3468),
\end{aligned}
$$

$$
\begin{aligned}
\Big(\alpha\beta\alpha\beta\beta,\tilde{\Sigma}\big((\alpha\beta,13),(\alpha\beta\beta,75),(\beta,2)\big)\Big) &= \left(\alpha\beta\alpha\beta\beta,\frac{\breve{K}(\alpha\beta\alpha\beta)}{\breve{K}(\alpha\beta)}\breve{K}(\alpha\beta\beta)-\breve{K}(\beta)\right)\\
&= (\alpha\alpha\alpha\beta,39\cdot 75-2=2923),
\end{aligned}
$$

$$
\begin{aligned}
\Big(\alpha\beta\beta\beta,\tilde{\Sigma}\big((\alpha\beta\beta,75),(\beta,2),(\alpha\beta,13)\big)\Big) &= \left(\alpha\beta\beta\beta,\frac{\breve{K}(\beta^2)}{\breve{K}(\beta)}\breve{K}(\alpha\beta\beta)-\breve{K}(\alpha\beta)\right)\\
&= (\alpha\alpha\alpha\beta,6\cdot 75-13=437),
\end{aligned}
$$

4 Theorem 7.15

We change this theorem to the following.

Theorem (Theorem 7.15)**.** *Let n be a positive even integer. Let m and r be non-negative integers such that $m+r>0$. Let α, λ, and ρ be the following sequences of positive integers*

$$
\begin{aligned}
\alpha &= (a_1,\dots,a_n),\\
\lambda &= (b_1,\dots,b_m),\\
\rho &= (c_1,\dots,c_r).
\end{aligned}
$$

Then we have that

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} = \frac{\breve{K}(\lambda\alpha^2\rho) + \breve{K}(\lambda\rho)}{\breve{K}(\lambda\alpha\rho)}. \tag{1}$$

Proof. **The proof when m and r are both positive is the same as in the paper.** Let $\rho = ()$. Equation (11) becomes

$$K(\alpha)\breve{K}(\lambda\alpha) + K_2^{n-1}(\alpha)\breve{K}(\lambda\alpha) - \breve{K}(\lambda\alpha^2) - \breve{K}(\lambda) = 0.$$

Substituting

$$\breve{K}(\lambda\alpha^2) = K(\lambda\alpha)\breve{K}(\alpha) + \breve{K}(\lambda\alpha)K_2^{n-1}(\alpha)$$

into this equation we get

$$K(\alpha)\breve{K}(\lambda\alpha) + K_2^{n-1}(\alpha)\breve{K}(\lambda\alpha) - K(\lambda\alpha)\breve{K}(\alpha) - \breve{K}(\lambda\alpha)K_2^{n-1}(\alpha) - \breve{K}(\lambda) = 0$$

which, after cancelling terms, becomes

$$K(\alpha)\breve{K}(\lambda\alpha) - K(\lambda\alpha)\breve{K}(\alpha) - \breve{K}(\lambda) = 0.$$

Into this equation we substitute the equalities

$$\begin{aligned}\breve{K}(\lambda\alpha) &= K(\lambda)\breve{K}(\alpha) + \breve{K}(\lambda)K_2^{n-1}(\alpha),\\ K(\lambda\alpha) &= K(\lambda)K(\alpha) + \breve{K}(\lambda)K_2^{n}(\alpha),\end{aligned}$$

from which we get

$$\begin{aligned}&K(\alpha)K(\lambda)\breve{K}(\alpha) + K(\alpha)\breve{K}(\lambda)K_2^{n-1}(\alpha)\\ &-\breve{K}(\alpha)K(\lambda)K(\alpha) - \breve{K}(\alpha)\breve{K}(\lambda)K_2^{n}(\alpha) - \breve{K}(\lambda) = 0.\end{aligned}$$

Cancelling terms this becomes

$$K(\alpha)\breve{K}(\lambda)K_2^{n-1}(\alpha) - \breve{K}(\alpha)\breve{K}(\lambda)K_2^{n}(\alpha) - \breve{K}(\lambda) = 0$$

This equation holds since $K(\alpha)K_2^{n-1}(\alpha) - \breve{K}(\alpha)K_2^{n}(\alpha) = 1$, as n is even.

The proof when $\lambda = ()$ is similar, so we don't give as much detail here. Equation (11) is now

$$K(\alpha)\breve{K}(\alpha\rho) + K_2^{n-1}(\alpha)\breve{K}(\alpha\rho) - \breve{K}(\alpha^2\rho) - \breve{K}(\rho) = 0.$$

Splitting the continuant $\breve{K}(\alpha^2\rho)$ and cancelling terms leads us to the equation

$$K_2^{n-1}(\alpha)\breve{K}(\alpha\rho) - \breve{K}(\alpha)K_2^{n+r-1}(\alpha\rho) - \breve{K}(\rho) = 0.$$

Once more splitting the continuants $\breve{K}(\alpha\rho)$ and $K_2^{n+r-1}(\alpha\rho)$ and cancelling terms leads us to the equation

$$K_2^{n-1}(\alpha)K(\alpha)\breve{K}(\rho) - \breve{K}(\alpha)K_2^{n}(\alpha)\breve{K}(\rho) - \breve{K}(\rho) = 0,$$

which holds since $K(\alpha)K_2^{n-1}(\alpha) - \breve{K}(\alpha)K_2^{n}(\alpha) = 1$, as n is even. ■

5 Example 7.22

A triple $\big(\breve{K}(\mu), \breve{K}(\mu\nu), \breve{K}(\nu)\big)$ in a graph is followed by the triples

$$\begin{aligned}\big(\breve{K}(\mu), \breve{K}(\mu^2\nu), \breve{K}(\mu\nu)\big) &= \Big(\breve{K}(\mu), \frac{\breve{K}(\mu^2)}{\breve{K}(\mu)}\breve{K}(\mu\nu) - \breve{K}(\nu), \breve{K}(\mu\nu)\Big),\\ \big(\breve{K}(\mu\nu), \breve{K}(\mu\nu^2), \breve{K}(\nu)\big) &= \Big(\breve{K}(\mu\nu), \frac{\breve{K}(\nu^2)}{\breve{K}(\nu)}\breve{K}(\mu\nu) - \breve{K}(\mu), \breve{K}(\nu)\Big).\end{aligned}$$

Let us call the values

$$\frac{\breve{K}(\mu^2)}{\breve{K}(\mu)} \qquad \text{and} \qquad \frac{\breve{K}(\nu^2)}{\breve{K}(\nu)}$$

the *Markov values of* μ *and* ν respectively.

Example 2. Let $\alpha = (1,1)^n$ and $\beta = (2,2)^m$ for some positive integers n and m. Consider the graph $T_{\alpha,\beta}$. The starting triple in this graph is

$$\big(\breve{K}(\alpha), \breve{K}(\alpha\beta), \breve{K}(\beta)\big),$$

followed by the triples

$$\big(\breve{K}(\alpha), \breve{K}(\alpha^2\beta), \breve{K}(\alpha\beta)\big), \quad \big(\breve{K}(\alpha\beta), \breve{K}(\alpha\beta^2), \breve{K}(\beta)\big).$$

Let $\big(\breve{K}(\gamma), \breve{K}(\gamma\rho), \breve{K}(\rho)\big)$ be any triple in the graph other than the triples

$$\begin{aligned}\big(\breve{K}(\alpha), \breve{K}(\alpha^i\beta), \breve{K}(\alpha^{i-1}\beta)\big) &= \left(\breve{K}(\alpha), \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)}\breve{K}(\alpha^{i-1}\beta) - \breve{K}(\alpha^{i-2}\beta), \breve{K}(\alpha^{i-1}\beta)\right)\\ \big(\breve{K}(\alpha\beta^{i-1}), \breve{K}(\alpha\beta^i), \breve{K}(\beta)\big) &= \left(\breve{K}(\alpha\beta^{i-1}), \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)}\breve{K}(\alpha\beta^{i-1}) - \breve{K}(\alpha\beta^{i-2}), \breve{K}(\beta)\right)\end{aligned} \tag{2}$$

for $i \geq 1$, which we deal with separately. Then both sequences γ and ρ are of the form

$$(1, 1, \ldots, 2, 2).$$

As such they satisfy the conditions of Proposition 7.21, from which we have that

$$\frac{\breve{K}(\gamma^2)}{\breve{K}(\gamma)} = 3\breve{K}(\gamma) \quad \text{and} \quad \frac{\breve{K}(\rho^2)}{\breve{K}(\rho)} = 3\breve{K}(\rho).$$

From this we can build the entire tree $T_{\alpha,\beta}$ if we know the starting triple

$$\big(\breve{K}(\alpha), \breve{K}(\alpha\beta), \breve{K}(\beta)\big)$$

and the two Markov values

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} \qquad \text{and} \qquad \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)}.$$

There are two paths in the tree, given in Equation (2), that depend on the Markov values

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)}, \quad \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)}.$$

These have the values

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} = 3\breve{K}(\alpha), \quad \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)} = 3\breve{K}(\beta),$$

if $n = 1$ and $m = 1$. This is the case of regular Markov numbers. However this is not true for $n > 1$ and $m > 1$, as we show now in Proposition 1.

Remark. We use the following notation for continued fractions. In $[4; (4,4)^i : 7]$ the sequence $(4,4)$ is repeated i-times. In $[2; \langle 4 \rangle]$ the number 4 is repeated infinitely.

Proposition 1. *Let $\alpha = (1,1)^n$ and $\beta = (2,2)^m$ for some integers $n > 1$ and $m > 1$. Prove that*

$$\frac{\breve{K}(\beta^2)}{\breve{K}(\beta)^2} = [2; 1 : (4,1)^{n-2} : 5],$$

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)^2} = \begin{cases} [2; (4,4)^{i-1} : 3] & i = \frac{m+1}{3}, \ i \in \mathbb{Z}, \\ [2; (4,4)^{i-1} : 4] & i = \frac{m}{3}, \ i \in \mathbb{Z}, \\ [2; (4,4)^{i-1} : 4 : 5] & i = \frac{m-1}{3}, \ i \in \mathbb{Z}, \end{cases}$$

Example 3. For $n = 1, \ldots, 5$ and $m = 1, \ldots, 5$ we have

$$\frac{\breve{K}\big((1,1)^2\big)}{\breve{K}\big((1,1)\big)^2} = [3] \qquad \frac{\breve{K}\big((2,2)^2\big)}{\breve{K}\big((2,2)\big)^2} = [3]$$

$$\frac{\breve{K}\big((1,1)^4\big)}{\breve{K}\big((1,1)^2\big)^2} = [2; 3] \qquad \frac{\breve{K}\big((2,2)^4\big)}{\breve{K}\big((2,2)^2\big)^2} = [2; 1 : 5]$$

$$\frac{\breve{K}\big((1,1)^6\big)}{\breve{K}\big((1,1)^3\big)^2} = [2; 4] \qquad \frac{\breve{K}\big((2,2)^6\big)}{\breve{K}\big((2,2)^3\big)^2} = [2; 1 : 4 : 1 : 5]$$

$$\frac{\breve{K}\big((1,1)^8\big)}{\breve{K}\big((1,1)^4\big)^2} = [2; 4 : 5] \qquad \frac{\breve{K}\big((2,2)^8\big)}{\breve{K}\big((2,2)^4\big)^2} = [2; 1 : 4 : 1 : 4 : 1 : 5]$$

$$\frac{\breve{K}\big((1,1)^{10}\big)}{\breve{K}\big((1,1)^5\big)^2} = [2; 4 : 4 : 3] \qquad \frac{\breve{K}\big((2,2)^{10}\big)}{\breve{K}\big((2,2)^5\big)^2} = [2; 1 : 4 : 1 : 4 : 1 : 4 : 1 : 5]$$

Remark. Note that

$$\lim_{m \to \infty} \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)^2} = [2; \langle 4 \rangle] = \sqrt{5},$$

$$\lim_{n \to \infty} \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)^2} = [2; \langle 1, 4 \rangle] = 2\sqrt{2}.$$

The statement in Example 7.22 is therefore not correct. However, if we know the starting sequences $\alpha = (1,1)^n$ and $\beta = (2,2)^m$, then we have can derive the graph $T_{\alpha,\beta}$ using $\tilde{\Sigma}$, the formula in Question 1, and the triple

$$\big(\breve{K}(\alpha), \breve{K}(\alpha\beta), \breve{K}(\beta)\big).$$

6 Other corrections

At the start of Subsection 5.3 we have the formula for Σ. This should be

$$\Sigma(a,b,c) = 3ab - c.$$

Typo in Theorem 6.20: The second bullet point should read "The maps P, Q, and T are as in Fig. 6".

In Figure 8, the box labelled "Generalised Markov triples in $\mathcal{G}_{\otimes}(\mu,\nu)$ (or in $T_{\mu,\nu}$)" is not clear. We add the following remark replacing Remark 6.21.

Remark. Triples in the graph $\mathcal{G}_{\otimes}(\mu,\nu)$ are of the form

$$\big((\alpha,a),(\alpha\beta,c),(\beta,b)\big).$$

There is a trivial map Q from these triples to triples of sequences

$$\big((\alpha,a),(\alpha\beta,c),(\beta,b)\big) \mapsto (\alpha,\alpha\beta,\beta).$$

Maps S and Y are derived from this trivial map.

It is not known to the authors whether there exists a map Q from the triples in $T_{\mu,\nu}$

$$(a,c,b) \mapsto (\alpha,\alpha\beta,\beta).$$

Similarly, maps S and Y are not known to exist from $T_{\mu,\nu}$. For this reason they are marked with dashed lines.