# CORRECTIONS TO THE BOOK GEOMETRY OF CONTINUED FRACTIONS

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CHAPTER 2.

- The last sentence of Definition 2.7 on page 24 should be:

For the points $A$, $B$, and $C$ in one line we say that $\mathrm{lS}(\triangle ABC) = 0$.

- Several typoes in Definition 2.8 on page 25:

**Definition 2.8.** The *index* of a rational angle $\angle BAC$, denoted by $\mathrm{l}\alpha(\angle BAC)$, is the index of the sublattice generated by all integer vectors of the lines $AB$ and $AC$ in the integer lattice.
In addition, if the points $A$, $B$, and $C$ are collinear, we say that $\mathrm{l}\alpha(\angle BAC) = 0$.

- Excluded formula of Exercise 2.6 on pag 33:

$$\mathrm{lS}(\triangle ABC) = \mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)\,\mathrm{l}\alpha(\angle ABC).$$

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Chapter 3.

- In Figure 3.1 on page 34, the expression for $a_1$ on the left should be as follows:
$$a_1 = \mathrm{l}\ell(B_0B_1) = 2;$$

- The first excluded formula on page 34:
$$A_1 = (1,1), \quad B_1 = (2,3), \quad B_2 = A_2 = (5,7)$$

- The coordinates of type $(p_*, q_*)$ should be replaced by $(q_*, p_*)$ everywhere on pages (34-38). Alternatively, the convergents in Theorem 3.1, Theorem 3.4, and Corollary 3.5 should be reversed.

- In the proof of Theorem 3.1 on page 35 (line 3 of the proof) $A_1 = (1, \lfloor\alpha\rfloor)$

- In the proof of Theorem 3.1 on page 35 (line 5 of the proof)
$$B_1 = (0,1) + a_1(1, \lfloor\alpha\rfloor).$$

- Excluded formula of the inductive step of the algorithm on page 37 should be
$$A_{k+1} = A_k + a_{2k}OB_k \quad \text{and} \quad B_{k+1} = B_k + a_{2k+1}OA_{k+1}.$$

- Proof of Theorem 3.6 on page 38. Some typos.

Let us calculate the index of an integer angle at some vertex of the principal part of one of the two sails. By Theorem 3.1 it is equivalent to calculate the index of the angle between a pair of vectors $(q_{i-1}, p_{i-1})$ and $(q_{i+1}, p_{i+1})$:
$$\mathrm{l}\alpha\left(\angle(q_{i-1}, p_{i-1})(0,0)(q_{i+1}, p_{i+1})\right) = |p_{i-1}q_{i+1} - p_{i+1}q_{i-1}| =$$
$$|p_{i-1}(a_{i+1}q_i + q_{i-1}) - (a_{i+1}p_i + p_{i-1})q_{i-1}| = a_{i+1}|p_{i-1}q_i - p_iq_{i-1}| = a_{i+1}.$$
The second equality follows from Proposition 1.13 while the last follows from Proposition 1.15. □

- Important line added to Edge–angle duality on page 38.

**Edge–angle duality.** From Corollary 3.5 and Theorem 3.6, we get that
$$\mathrm{l}\alpha(\angle A_iA_{i+1}A_{i+2}) = \mathrm{l}\ell(B_iB_{i+1}) \quad \text{and} \quad \mathrm{l}\alpha(\angle B_iB_{i+1}B_{i+2}) = \mathrm{l}\ell(A_{i+1}A_{i+2}).$$
The only exception is the very last angle (for rational slopes) where $\mathrm{l}\alpha$ is smaller than the corresponding $\mathrm{l}\ell$ by one.

- The last excluded formula (page 38).
$$\mathrm{l}\alpha(\angle A_0A_1A_2) = \mathrm{l}\ell(B_0B_1) = a_1 = 2;$$
$$\mathrm{l}\alpha(\angle B_0B_1B_2) = \mathrm{l}\ell(A_1A_2) - 1 = a_2 - 1 = 1.$$

- The label at point $(2,3)$ on Figure 3.4 schould be $2-1$ and not 1 (page 39).

<u>CHAPTER 4</u>.

- Excluded formula of Definition 4.1 (page 42):

$$\frac{\mathrm{lS}(\triangle ABC)}{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)},$$

- Remark 4.2 (page 42):

*Remark 4.2* Notice that the integer sine is well defined, i.e., it does not depend on the choice of points $A$ and $C$. We leave the reader to check this as an exercise.

- The second part of the proof of Proposition 4.3 (page 42):

Hence,

$$\mathrm{lsin}(\angle A'BC') = \mathrm{lS}(\triangle A'BC').$$

The integer area of $\triangle A'BC'$ is the index of the sublattice generated by $BA'$ and $BC'$ in $\mathbb{Z}^2$. Since the vectors $BA'$ and $BC'$ generate all integer points of the lines $AB$ and $BC$, the integer area of $\triangle A'BC'$ is equivalent to the index of the angle. Therefore, we get

$$\mathrm{lsin}(\angle ABC) = \mathrm{l}\alpha(\triangle ABC).$$

If $A$, $B$, and $C$ are on one line, then

$$\mathrm{lsin}(\angle ABC) = \mathrm{l}\alpha(\triangle ABC) = 0.$$

This concludes the proof.

<u>CHAPTER 5</u>.

- In the proof of Theorem 5.7 the following four corrections should be implemented:

*Proof.* Consider an integer angle $\alpha$. Let $\operatorname{ltan}\alpha = p/q$, where $\gcd(p,q) = 1$. By Proposition 5.4(ii), we have $\alpha \cong \operatorname{larctan}(p/q)$. The case $p/q = 1$ is trivial. Consider the case $p/q > 1$.

Let $A = (1,0)$, $B = (q,p)$, and $O = (0,0)$. Suppose that an integer point $C = (q',p')$ of the sail of the angle $\operatorname{larctan}(p/q)$ is the closest integer point to the endpoint $B$. Both coordinates of $C$ are positive integers, since $p/q > 1$. Since the triangle $\triangle BOC$ is empty and the orientation of the pair of vectors $(OB, OC)$ does not coincide with the orientation of the pair of vectors $(OA, OB)$, we have

$$\det \begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = -1.$$

Consider a linear transformation $\xi$ of the two-dimensional plane,

$$\xi = \begin{pmatrix} p-p' & q'-q \\ p & -q \end{pmatrix}.$$

Since $\det(\xi) = -1$, the transformation $\xi$ is integer-linear and changes the orientation. Direct calculations show that the transformation $\xi$ takes the angle $\operatorname{larctan}^t(p/q)$ to the angle $\operatorname{larctan}(p/(p-p'))$. (See the example in Fig.5.1.)

Since $\gcd(p,p') = 1$ and $p > p-p'$, the following holds:

$$\begin{cases} \operatorname{lsin}(\alpha^t) &= p, \\ \operatorname{lcos}(\alpha^t) &= p-p'. \end{cases}$$

Since $qp'-pq' = -1$, we have $qp' \equiv -1 (\operatorname{mod} p)$. Therefore,

$$\operatorname{lcos}\alpha \operatorname{lcos}(\alpha^t) = q(p-p') \equiv 1 (\operatorname{mod} p).$$

From that we have

$$\begin{cases} \operatorname{lsin}(\alpha^t) &= \operatorname{lsin}\alpha, \\ \operatorname{lcos}(\alpha^t) &\equiv \big(\operatorname{lcos}\alpha (\operatorname{mod} \operatorname{lsin}\alpha)\big)^{-1}. \end{cases}$$

This concludes the proof of Theorem 5.7. □

- In the proof of Theorem 5.9 the following two corrections should be implemented:

1) The matrix in the last excluded formula on page 52 should be

$$\det \begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = 1.$$

2) Line 7 of page 53 should start with
"Since $qp'-pq' = 1$, we have $qp' \equiv 1 (\operatorname{mod} p)$."

Chapter 6.

- In the proof of Theorem 6.1 twice swap:

*Proof.* We have

$$\mathrm{lS}(\triangle ABC) = \mathrm{l}\ell(AB)\,\mathrm{l}\ell(AC)\,\mathrm{lsin}(\angle CAB) = \mathrm{l}\ell(BA)\,\mathrm{l}\ell(BC)\,\mathrm{lsin}(\angle CBA)$$
$$= \mathrm{l}\ell(CB)\,\mathrm{l}\ell(CA)\,\mathrm{lsin}(\angle ACB),$$

After inverting the expressions and multiplying by all three integer lengths, we get the statement of the proposition. □

- In the proof of Theorem 6.13 (page 63), line 6 of the proof:
  "The area of the triangle is exactly $\lambda_1\lambda_2 p$. Hence $\lambda_2 p \le d$."

CHAPTER 7.

- In Definition 7.12 (page 72), line 3 of the definition:
  "... is *reduced* if $d > b > a \geq 0$. "

- Algorithm to Construct Reduced Matrices and the following remark, some typos are corrected: (page 77–78).

**Input data.** We are given an $\mathrm{SL}(2,\mathbb{Z})$ matrix $M = [[p, r][q, s]]$. In addition, we suppose that the characteristic polynomial of the matrix is irreducible over $\mathbb{Q}$ (or, equivalently, that it does not have $\pm 1$ as roots) and has two positive real roots.

**Goal of the algorithm.** To construct one of the periods of the LLS sequence for $M$.

**Step 1.** If $q < 0$, then we multiply the matrix $[[p, r][q, s]]$ by $-\mathrm{Id}$. Go to Step 2.

**Step 2.** We have $q \geq 0$. After conjugation of the matrix $[[p, r][q, s]]$ by the matrix $[[1, -\lfloor p/q \rfloor][0, 1]]$ we get the matrix $[[p', r'][q', s']]$, where $0 \leq p' \leq q'$. Go to Step 3.1.

**Step 3.1.** Suppose that $q' = 1$. Then $p' = 0$, $r' = -1$. In addition we have $|s'| > 2$, otherwise, the matrix has either complex roots or rational roots. The algorithm stops, and the output of the algorithm is the matrix $[[0, -1][1, s']]$.

**Step 3.2.1.** Suppose that $q' > 1$ and $s' > q'$. Then the algorithm stops, and the output of the algorithm is the matrix $[[p', r'][q', s']]$.

**Step 3.2.2.** Suppose that $q' > 1$ and $s' < -q'$. Conjugate the matrix $[[p', r'][q', s']]$ by the matrix $[[-1, 1][0, 1]]$ and multiply by $-\mathrm{Id}$. As a result we have the matrix $[[p'', r''][q'', s'']]$ with $q'' = q'$, $p'' = q' - p'$, and $s'' = -q' - s' > 0$. Go to Step 3.2.1 or to Step 3.2.3 depending on $s''$ and $q''$.

**Step 3.2.3.** Suppose that $q' > 1$ and $|s'| \leq |q'|$. Notice that the absolute values of $q'$ and $s'$ do not coincide, since the matrix has unit determinant. Hence $|s| < |q|$. Then we have

$$|r'| = \left| \frac{p's' - 1}{q'} \right| \leq \frac{(q' - 1)^2 + 1}{q'} \leq q' - 1.$$

Go to Step 1 with the matrix $[[s', q'][r', p']]$ with $|r'| < |q'|$. This is the matrix obtained from $[[p', r'][q', s']]$ by conjugation with $[[0, -1][-1, 0]]$.

**Output.** The reduced integer matrix that is conjugate to $\pm M$. The sign is defined by the sign of the trace of the original matrix (i.e., of $p + s$).

noindent *Remark 7.22* During the algorithm we only use conjugations with $\mathrm{GL}(2,\mathbb{Z})$ matrices and multiply by $-\mathrm{Id}$. So in the output one should expect a reduced integer matrix that is conjugate either to $M$ or to $-M$. Nevertheless, since $M$, as well as any reduced matrix, has all positive eigenvalues, the resulting matrix is conjugate exactly to $\pm M$.

- Supremum sign is missing in the excluded formula of Proposition 7.28(ii) (page 81).

$$\sqrt{\Delta(f)}/m(f) = \sup_{i\in\mathbb{Z}}\big(a_i + [0; a_{i+1} : a_{i+2} : \cdots] + [0; a_{i-1} : a_{i-2} : \cdots]\big).$$

- Right after Proposition 7.28 (pages 81-82): insert the following.

The first statement of Proposition 7.28 is a straightforward corollary of the convexity of sails. For further details regarding the second statement of Proposition 7.28 we refer to the original papers of A. Markov [135] and [136].

- Remark 7.289 (page 81). should be as follows.

If the LLS sequence is periodic, the expression of Proposition 7.28($ii$) is

$$\sqrt{\Delta(f)}/m(f) = \sqrt{\Big(a_0 + \frac{1}{\operatorname{ltan}\alpha} + \frac{1}{\operatorname{ltan}\alpha^t}\Big)^2 - \frac{4}{\operatorname{lsin}^2\alpha}} \quad ,$$

where the angle $\alpha$ is integer congruent to $\operatorname{larctan}([a_1; \cdots : a_{2n-1}])$, and where $(a_0, a_1, \ldots, a_{2n-1})$ is some minimal even period of the LLS sequence.

- The first excluded formula of Subsection 7.4.2 (on page 82).

The first elements in the spectrum in the increasing order are as follows:

$$\begin{aligned}
\sqrt{5} &= 1 + [0 : (1)] + [0 : (1)];\\
\sqrt{8} &= 1 + [0 : (2)] + [0 : (2)];\\
\frac{\sqrt{221}}{5} &= 2 + [0; (2 : 1 : 1 : 2)] + [0; (1 : 1 : 2 : 2)],\\
\frac{\sqrt{1517}}{13} &= 2 + [0; (2 : 1 : 1 : 1 : 1 : 2)] + [0; (1 : 1 : 1 : 1 : 2 : 2)],\\
\frac{\sqrt{7565}}{29} &= 2 + [0; (2 : 2 : 2 : 1 : 1 : 2)] + [0; (1 : 1 : 2 : 2 : 2 : 2)],\\
\frac{\sqrt{2600}}{17} &= 2 + [0; (2 : 1 : 1 : 1 : 1 : 1 : 1 : 2)] + [0; (1 : 1 : 1 : 1 : 1 : 1 : 2 : 2)].
\end{aligned}$$

- It would be good to reference Freiman's manuscript regarding the last excluded formula on page 82.

Freiman, G. A., *Diophantine Approximations and the Geometry of Numbers,* 1975, Kalinin Gos. Univ., Kalinin.

- The last line on page 82.

The first gaps, including the segment $(\sqrt{12}, \sqrt{13})$, were calculated by O. Perron in [166] and [167].

- Add the following right before formulation of Theorem 7.31(ii) on page 83.

The value 3 of the spectrum is interesting on its own. For example, as stated in [?] for any real number $x$ and positive integer $n$ we have that

$$2 + [0; (1:1)^n : x] + [0:2:(1:1)^{n-1}:x] = 3.$$

Hence, 3 is in fact represented by infinitely many forms.

Finally for the discrete Markov Spectrum we have the following remarkable theorem.

- Add the following statement at the end of Theorem 7.31(ii) on page 83.

The form representing this value is unique up to the $\mathrm{GL}(2,\mathbb{Z})$ group action and the multiplication by a non-zero constant.

<u>Chapter 8</u>.

- Last excluded formula on page 87:

$$\Xi(A) = \{B \in \mathrm{GL}(2, \mathbb{Z}) \mid \mathrm{AB} = \mathrm{BA}\}.$$

- Line 6 on page 92, should be:

Therefore, there exists an integer $c$ such that the equation

$$q^2 d - p^2 = c$$

has infinitely many integer solutions. ...

Chapter 9.

- Proof of Lemma 9.13 on page 106:

*Proof.* Since the map $T$ is surjective, we also have
$$T^n\big(S \cap I_{(a_1,\ldots,a_n)}\big) = S.$$
We have ...

- Brackets are missing in the first excluded formula of page 107:

$$\frac{\hat{\mu}\big((X \setminus S) \cap B(y, \hat{\mu}(I_{(a_1,\ldots,a_n)}))\big)}{\hat{\mu}(B(y, \hat{\mu}(I_{(a_1,\ldots,a_n)})))} \le 1 - \frac{\ln 2}{2} \frac{\hat{\mu}(S)\hat{\mu}(I_{(a_1,\ldots,a_n)})}{\hat{\mu}(B(y, \hat{\mu}(I_{(a_1,\ldots,a_n)})))} = 1 - \frac{\ln 2}{4}\hat{\mu}(S).$$

- First line of the proof of Theorem 9.14 on page 107:

*Proof of Theorem 9.14.* Consider a subset $S \subset I$. Let ...

- The second sentence of Chapter 9.5 on page 108 should be:

Let $m_n(x)$ denote the Lebesque measure of the set of real numbers $\alpha$ contained in the segment $[0, 1]$ such that $T^n(\alpha) < x$ (here $T$ is the Gauss map).

Chapter 13.

- The excluded formula of Corollary 13.11 on page 179 should be:

$$]c_1, m_1, c_2, m_2, \ldots, m_{n-1}, c_n[= 0.$$

Chapter 15.

- It would be good to make the following example after Definition 15.11:

**Example.** Let

$$A = (0, 0, 0),$$
$$B = (1, 2, 0),$$
$$C = (2, 1, 0),$$
$$D = (0, 0, 1).$$

In order to find the coordinates of $A$, $B$, $C$ and $D$ in IDC-system, we should compute the following integer distances:

$$\text{ld}(B, ACD) = 3,$$
$$\text{ld}(C, ABD) = 3,$$
$$\text{ld}(D, ABC) = 1.$$

Hence the points in the coorinates of the IDC-system will be as follows:

$$A = (0, 0, 0),$$
$$B = (3, 0, 0),$$
$$C = (0, 3, 0),$$
$$D = (0, 0, 1).$$

Let us now consider the point

$$E = (1, 0, 0)$$

in the old system of coordinates. Note that

$$E = A + \frac{1}{3}\big(2(C - A) - (B - A)\big)$$

Hence the new (IDC-system) coordinates of $E$ are

$$\frac{1}{3}\big(2 \cdot (0, 3, 0) - 1 \cdot (3, 0, 0)\big) = (-1, 2, 0).$$

Chapter 21.

- The fist line of page 332; and the same correction in the caption of Figure 21.5:

**Hessenberg Perfect NRS-Matrices** $H^{(0,0,-1)}_{\langle 0,1|1,0,2\rangle}(m,n)$**.** ...

<u>Chapter 23</u>.

- Page 363 line 7:

In a paper [100] . . .

<u>Bibliography</u>.

Renewed references to preprints that has been published after the book appeared:

[99] O. Karpenkov, *Multidimensional Gauss Reduction Theory for conjugacy classes of* $SL(n, \mathbb{Z})$, J. Théor. Nombres Bordeaux, vol. 25, no. 1, 99–109, 2013.

[100] O. Karpenkov, A. Ustinov, *Geometry and combinatorics of Minkowski-Voronoi 3-dimensional continued fractions*, J. Number Theor., vol. 176, 375–419, 2017.

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Please, feel free to write me (see e-mail below) if you have spotted some further typoes/mistakes.

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