The ergodic theory of continued fraction maps

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Euclidean Algorithm and Gauss Map

By Euclidean algorithm, any rational number $a/b > 1$ can be expressed as

$$x = \frac{a}{b} = a_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \, c_{n-1} + \frac{1}{c_n}}}},$$

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where $c_0, \ldots, c_n$ are natural numbers with $c_n > 1$, except for $n = 0$. Note $c_n(x) = c_{n-1}(Tx)$ for $n \geq 1$, where

$$Tx = \begin{cases} \left\{ \frac{1}{x} \right\} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

is the famous Gauss map circa 1800.

Regular Continued fraction Expansions

For arbitrary real x we have the regular continued fraction expansion of a real number

$$x = [c_0; c_1, c_2, \ldots] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 \ddots}}}}.$$

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Again $c_n(x) = c_{n-1}(Tx)$ for $n \geq 1$. The terms $c_0, c_1, \cdots$ are called the partial quotients of the continued fraction expansion and the sequence of rational truncates

$$[c_0; c_1, \cdots, c_n] = \frac{p_n}{q_n}, \qquad (n = 1, 2, \cdots)$$

are called the convergents of the continued fraction expansion.

Continued fraction map on [1, 0)

The *Gauss map* $G : [0,1] \to [0,1]$ is the following map:

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \mod 1 & \text{if } 0 < x \le 1 \end{cases}$$

Here $\{x\}$ denotes the *fractional part* of x. We can write $\{x\} = x - [x]$ where $[x]$ is the integer part. Equivalently, $\{x\} = x \mod 1$.

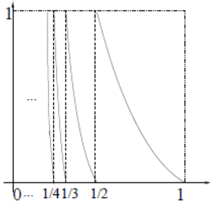
Remark that

$$\left[\frac{1}{x}\right] = n \quad \Leftrightarrow \quad n \le \frac{1}{x} < n+1 \quad \Leftrightarrow \quad \frac{1}{n+1} < x \le \frac{1}{n}.$$

Thus, explicitely, one has the following expression (see the graph in Figure 1.1):

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} - n & \text{if } \frac{1}{n+1} < x \le \frac{1}{n} \end{cases} \quad \text{for } n \in \mathbb{N}.$$

The rescrition of G to an interval of the form $(1/n+1, 1/n]$ is called *branch*. Each *branch* $G : (1/n+1, 1/n] \to [0,1)$ is monotone, surjective (onto $[0,1)$) and invertible (see Figure 1.1).



Dynamical System

By a dynamical system (X, β, μ, T) we mean a set X, together with a σ-algebra β of subsets of X, a probability measure μ on the measurable space (X, β) and a measurable self map T of X that is also measure preserving. i.e. if given an element A of β if we set $T^{-1}A = \{x \in X : Tx \in A\}$ then $\mu(A) = \mu(T^{-1}A)$.

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We say T is weak-mixing, if $(X \times X, \beta \times \beta, \mu \times \mu, T \times T)$ is ergodic

Note weak mixing is strictly stronger than ergodicity.

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Birkhoff's theorem

If (X, β, μ, T) is measure preserving and ergodic and f is integrable we have Birkhoff's pointwise ergodic theorem

$$\overline{f}(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \; = \; \int_X f(x) d\mu \; a.e..$$

If (X, β, μ, T) is not ergodic, we just know this limit is T invariant almost everywhere i.e. $\overline{f}(Tx) = \overline{f}(x)$

Gauss dynamical system and its natural extention

(i) If $X = [0, 1]$, β is the σ-algebra of Borel sets on X, $\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1}$, for $A \in \beta$ and T is the Gauss map then (X, β, μ, T) is weak mixing.

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(ii) If $X = \Omega = ([0,1) \setminus \mathbf{Q}) \times [0,1]$, γ is the σ-algebra of Borel subsets of Ω, ω is the probability measure on the measurable space (Ω, β) defined by $\omega(A) = \frac{1}{(\log 2)}\int_A \frac{dxy}{(1+xy)^2}$, and $\overline{T}(x,y) = (Tx, \frac{1}{[\frac{1}{x}]+y})$. Then the map $\overline{T}$ preserves the measure ω and the dynamical system $(\Omega, \gamma, \omega, \overline{T})$ called the natural extention of (X, β, μ, T) is weak mixing.

Means of convergents

Suppose the function F with domain the non-negative real numbers and range the real numbers is continuous and increasing. For each natural number n and arbitrary non-negative real numbers $a_1, \cdots, a_n$ we define

$$M_{F,n}(a_1, \cdots, a_n) = F^{-1}[\frac{1}{n}\sum_{j=1}^{n} F(a_j)].$$

Then C. Ryll-Nardzewski observed that

$$\lim_{n\to\infty} M_{F,n}(c_1(x), \cdots, c_n(x)) = F^{-1}[\frac{1}{\log 2}\int_{-0}^{1} F(c_1(t))d\frac{dt}{1+t}],$$

almost every where with respect to Lebesgue measure.

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Special cases due to A. Khinchin

(i) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n(x) = \infty a.e.$;
(ii) $\lim_{N\to\infty}(c_1(x)\ldots c_N(x))^{N^{-1}} = \Pi_{k\geq 1}(1 + \frac{1}{k(k+2)})^{\frac{\log k}{\log 2}}$ a.e.

Hurwitz's constants

Recall the inequality $|x - \frac{p_n}{q_n}| \le \frac{1}{q_n^2}$, which is classical and well known. One can check

$$\theta_n(x) = \frac{1}{(T^n x)^{-1} + q_{n-1} q_n^{-1}} = q_n^2 |x - \frac{p_n}{q_n}| \in [0, 1)$$

for each natural number n. Set

$$F(x) = \begin{cases} \frac{z}{\log 2} & x \in [0, \frac{1}{2}); \\ \frac{1}{\log 2}(1 - z + \log 2z) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} |\{1 \le j \le n : \theta_j(x) \le z\}| = F(z),$$

almost everywhere with respect to Lebesgue measure.
W. Bosma, H. Jager andF. Wiedijk 1983. Conjectured H.W. Lenstra Jr.

Good Universality

A sequence of integers $(a_n)_{n=1}^{\infty}$ is called *L^p-good universal* if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^p(X, \mathcal{B}, \mu)$ we have

$$\overline{f}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n} x)$$

existing μ almost everywhere.

Uniform distribution modulo 1

A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed modulo one* if for each interval $I \subseteq [0, 1)$, if $|I|$ denotes its length, we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : \{x_n\} \in I\} = |I|.$$

Subsequence ergodic theory

Lemma (Nair)

If $(\{a_n\gamma\})_{n=1}^{\infty}$ *is uniformly distributed modulo one for each irrational number* γ*, the dynamical system* $(X, \mathcal{B}, \mu, T)$ *is weak-mixing and* $(a_n)_{n\geq 1}$ *is* L^2*-good universal then* $\overline{f}(x)$ *exists and*

$$\overline{f}(x) = \int_X f d\mu$$

μ *almost everywhere.*

Polynomial like sequences

1. *The natural numbers:* The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.

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2. *Polynomial like sequences:* Note if $\phi(x)$ is a polynomial such that $\phi(\mathbb{N}) \subseteq \mathbb{N}$ (Bourgain, Nair) and $p > 1$ then $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$ (Nair) where p_n is n^{th} prime are L^p good universal sequences.

Hartman uniformly distributed sequences

A sequence of integers $(a_n)_{n\geq 1}$ is Hartman uniformly distributed if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(a_n x) = 0$$

for all non-integer x. Equivalenty a sequence is Hartmann uniformly distributed if $(\{a_n\gamma\})_{n\geq 1}$ is uniform distributed modulo 1 for each irrational number γ, and the sequence $(a_n)_{n\geq 1}$ is uniformly distributed in each residue class $\bmod m$ for each natural number $m > 1$.

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Note if $n \in \mathbb{N}$ then $n^2 \not\equiv 3 \mod 4$ so in general the sequences $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$ are not Hartman uniformly distributed. We do however know that if $\beta \in \mathbb{R}\setminus\mathbb{Q}$ then $(\phi(n)\beta)_{n=1}^{\infty}$ and $(\phi(p_n)\beta)_{n=1}^{\infty}$ are uniformly distributed modulo one. Condition H sequences to follow are Hartman uniformly distributed.

Condition H sequences of integers

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$$b_M = \sup_{\{z\} \in [\frac{1}{a(M)}, \frac{1}{2})} \left| \sum_{n : a_n \leq M} e(z a_n) \right| .$$

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$$\frac{b(M) + A_{[a(M)]} + \frac{M}{a(M)}}{A_M} \leq Cc(M).$$

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$$\frac{b(M) + A_{[a(M)]} + \frac{M}{a(M)}}{A_M} \leq Cc(M).$$

Then we say that $\underline{k} = (a_n)_{n=1}^{\infty}$ satisfies condition H. (Nair)

Examples of Hartman uniformly distribution sequences

Sequences satisfying condition H are both Hartman uniformly distributed and L^p-good universal.
Specific sequences of integers that satisfy conditions H include $k_n = [g(n)]$ $(n = 1, 2, \dots)$ where
I. $g(n) = n^{\omega}$ if $\omega > 1$ and $\omega \notin \mathbb{N}$.
II. $g(n) = e^{\log^{\gamma} n}$ for $\gamma \in (1, \frac{3}{2})$.
III. $g(n) = P(n) = b_k n^k + \dots + b_1 n + b_0$ for $b_k, \dots, b_1$ not all rational multiples of the same real number.

4. Suppose $S = (n_k)_{n=1}^{\infty} \subseteq \mathbb{N}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function I_S we may view it as a point in $\Lambda = \{0,1\}^{\mathbb{N}}$ the set of maps from $\mathbb{N}$ to $\{0,1\}$.

Bourgain's random sequences

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We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$ where for each natural number n we have $X_n = \{0,1\}$ and specify the probability π_n on X_n by $\pi_n(\{1\}) = q_n$ with $0 \leq q_n \leq 1$ and $\pi_n(\{0\}) = 1 - q_n$ such that $\lim_{n\to\infty} q_n n = \infty$.

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The desired probability measure on Λ is the corresponding product measure $\pi = \prod_{n=1}^{\infty} \pi_n$. The underlying σ-algebra β is that generated by the "cylinders"

$$\{\lambda = (\lambda_n)_{n=1}^{\infty} \in \Lambda : \lambda_{i_1} = \alpha_{i_1}, \ \dots \ \lambda_{i_r} = \alpha_{i_r}\}$$

for all possible choices of $i_1, \dots, i_r$ and $\alpha_{i_1}, \dots, \alpha_{i_r}$.

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for all possible choices of $i_1, \ldots, i_r$ and $\alpha_{i_1}, \ldots, \alpha_{i_r}$.
Let $(k_n)_{n=1}^{\infty}$ be almost any point in Λ with respect to the measure π.

Means of convergents for subsequences

Suppose the function F with domain the non-negative real numbers and range the real numbers is continuous and increasing. For each natural number n and arbitrary non-negative real numbers $a_1, \cdots,\ a_n$ we define

$$M_{F,n}(a_1, \cdots, a_n) \ = \ F^{-1}[\frac{1}{n}\sum_{j=1}^{n} F(a_j)].$$

Then if $(a_n)_{n\geq 1}$ is L^p good universal and $(\{a_n\gamma\})_{n\geq 1}$ is uniformly distributed modulo one for irrational γ we have

$$\lim_{n\to\infty} M_{F,n}(c_{a_1}(x), \cdots,\ c_{a_n}(x)) \ = \ F^{-1}[\frac{1}{\log 2}\int_{-0}^{1} F(c_1(t))d\frac{dt}{1+t}],$$

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Special cases

(i) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_{a_n}(x) = \infty a.e.$;

(ii) $\lim_{N\to\infty}(c_{a_1}(x)\ldots c_{a_N}(x))^{N^{-1}} = \Pi_{k\geq 1}(1+\frac{1}{k(k+2)})^{\frac{\log k}{\log 2}}$ a.e.

Hurwitz's constants for subsequences

Recall the inequality

$$|x - \frac{p_n}{q_n}| \leq \frac{1}{q_n^2},$$

which is classical and well known. Clearly

$$\theta_n(x) = q_n^2|x - \frac{p_n}{q_n}| \in [0, 1).$$

if for each natural number n. Set

$$F(x) = \begin{cases} \frac{z}{\log 2} & x \in [0, \frac{1}{2}); \\ \frac{1}{\log 2}(1 - z + \log 2z) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

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Then if [$(a_n)_{n\geq 1}$ is L^p good universal and $(\{a_n\gamma\})_{n\geq 1}$ is uniformly distributed modulo one for irrational γ]* we have

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \leq j \leq n : \theta_{a_j}(x) \leq z\}| = F(z),$$

almost everywhere with respect to Lebesgue measure.

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D. Hensley dropped condition * using a different method.

Other sequences attached to the regular continued fraction expansion. I

Suppose z is in $[0,1]$ and for irrational x in $(0,1)$ set

$$Q_n(x) \ = \ \frac{q_{n-1}(x)}{q_n(x)}$$

for each positive integer n. Suppose also that $(a_n)_{n=1}^{\infty}$ satisfies *. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n : Q_{a_j}(x) \ \le \ z\}| \ = \ F_2(z) \ = \ \frac{\log(1 \ + \ z)}{\log 2}$$

almost everywhere with respect to Lebesgue measure.

Other sequences attached to the regular continued fraction expansion II

For irrational x in $(0, 1)$ set

$$r_n(x) = \frac{|x - \frac{p_n}{q_n}|}{|x - \frac{p_{n-1}}{q_{n-1}}|}. \qquad (n = 1, 2, \cdots)$$

Other sequences attached to the regular continued fraction expansion II

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$$r_n(x) = \frac{\left|x - \frac{p_n}{q_n}\right|}{\left|x - \frac{p_{n-1}}{q_{n-1}}\right|}. \qquad (n = 1, 2, \cdots)$$

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Other sequences attached to the regular continued fraction expansion II

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Suppose also that $(a_n)_{n=1}^{\infty}$ satisfes *. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n : r_{a_j}(x) \le z\}| = F_3(z),$$

almost everywhere with respect to Lebesgue measure.

Continued fraction map on [1, 0)

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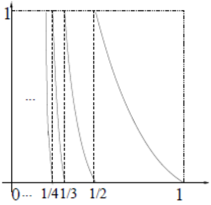

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of $[0,1]$ inductively for k in $\mathbf{N}$ by setting

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Of course some of these sets $P(j_0,\cdots,j_k)$ may be empty. We shall disregard these.

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(v) there exists γ in $(0,1)$ such that

$$|1 \;-\; \frac{T'(x)}{T'(y)}| \;\ll\; |x \;-\; y|^\gamma,$$

for x and y belonging to the same element of $\mathcal{P}_0$.

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The example (a) is known as the β-transformation. Note that in the special case where β is an integer $T_\beta(T_\beta(x)) = \{\beta^2 x\}$. This is not true for non-integer β and this gives the dynamics a quite different character. The example (b) is the famous Gauss map which is associated to the continued fraction expansion of a real number.

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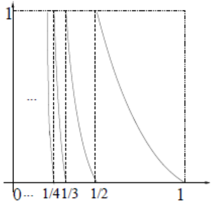

Invariant Measures for Markov Measures

If the map $T : [0,1] \to [0,1]$ is Markov in the sense described above then it preserves a measure η equivalent to Lebesgue measure.
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As a consequence, G. Birkhoff's pointwise ergodic theorem tells us that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_B(T^n(x)) = \eta(B), \tag{1.1}$$

almost everywhere with respect to Lebesgue measure.

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If $\underline{x} = (x_r)_{r=0}^{\infty}$ is a sequence of real numbers such that $0 \le x_r \le 1$ and $f : \mathbf{N}_0 \to \mathbf{R}$ is positive, set

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As a consequence of Birkhoff's theorem and the fact that η is equivalent to Lebesgue measure λ we see that $\lambda(E(\underline{x}, f)) = 0$.

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We say a collection of subsets of M denoted $\mathcal{C}_\delta$ is a δ–cover for E if $E \subseteq \cup_{U \in \mathcal{C}_\delta} U$, and if we set

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Some examples

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(iii) Cantor's middle third set : Let

$$C = \{x \in [0, 1) : x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \text{ s.}t.x_n \in \{0, 2\}\}.$$

C is well known to be uncountanle.
One can show $dim(C) = \frac{\log 2}{\log 3}$.

Abercrombie, Nair

For each sequence $\underline{x} = (x_r)_{r=0}^{\infty}$ of real numbers in $[0, 1]$ and positive function $f : \mathbf{N}_0 \to \mathbf{R}$ such that $f(r) \gg r^2$, the Hausdorff dimension of $E(\underline{x}, f)$ is 1.

A special case

An immediate consequence is the following result.
For $x_0 \in [0,1]$ set

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Take $x_0 = 0$ and T is the Gauss continued fraction map.

Thus the set of $x \in [0,1]$ with bound convergents had dimension 1.

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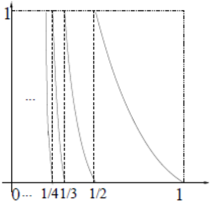

Equivalent characterisations of badly approximability

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(i) and (ii) are equivalent.

Corollary : (V. Jarnik 1929) : The set of badly approximable numbers has Hausdorff dimension 1

The Field of Formal Power series

Let $\mathbb{F}_q$ denote the finite field of q elements, where q is a power of a prime p. If Z is an indeterminate, we denote by $\mathbb{F}_q[Z]$ and $\mathbb{F}_q(Z)$ the ring of polynomials in Z with coefficients in $\mathbb{F}_q$ and the quotient field of $\mathbb{F}_q[Z]$, respectively.

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That is,

$$\mathbb{F}_q((Z^{-1})) = \{a_n Z^n + \cdots + a_0 + a_{-1}Z^{-1} + \cdots : n \in \mathbb{Z},\ a_i \in \mathbb{F}_q\}$$

and we have $|a_n Z^n + a_{n-1}Z^{n-1} + \cdots| = q^n$ $(a_n \neq 0)$ and $|0| = 0$, where q is the number of elements of $\mathbb{F}_q$.

Haar measure on the field of Formal Power Series

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It was shown that the equation $\mu_q(B(\alpha; q^n)) = q^n$ completely characterizes Haar measure here.

Continued fractions on $\mathbb{F}_q((Z^{-1}))$

For each $\alpha \in \mathbb{F}_q((Z^{-1}))$, we can uniquely write

$$\alpha = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \ddots}} = [A_0; A_1, A_2, \dots],$$

where $(A_n)_{n=0}^{\infty}$ is a sequence of polynomials in $\mathbb{F}_q[Z]$ with $|A_n| > 1$ for all $n \geq 1$.

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We define recursively the two sequences of polynomials $(P_n)_{n=0}^{\infty}$ and $(Q_n)_{n=0}^{\infty}$ by

$$P_n = A_n P_{n-1} + P_{n-2} \qquad \text{and} \qquad Q_n = A_n Q_{n-1} + Q_{n-2},$$

with the initial conditions $P_0 = A_0$, $Q_0 = 1$, $P_1 = A_1 A_0 + 1$ and $Q_1 = A_1$.

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Then we have $Q_n P_{n-1} - P_n Q_{n-1} = (-1)^n$, and whence P_n and Q_n are coprime. In addition, we have $P_n/Q_n = [A_0; A_1, \dots, A_n]$.

Continued fractions map on $\mathbb{F}_q((Z^{-1}))$

Define T_q on the unit ball
$B(0;1) = \{a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots : a_i \in \mathbb{F}_q\}$ by

$$T_q\alpha = \left\{\frac{1}{\alpha}\right\} \qquad \text{and} \qquad T0 = 0.$$

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Here $\{a_nZ^n + \cdots + a_0 + a_{-1}Z^{-1} + \cdots\} = a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots$ denotes its fractional part.

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$$\{a_nZ^n + \cdots + a_0 + a_{-1}Z^{-1} + \cdots\} = a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots$$

denotes its fractional part.
We note that if $\alpha = [0; A_1(\alpha), A_2(\alpha), \ldots]$, then we have, for all $m, n \geq 1$,

$$T^n\alpha = [0; A_{n+1}(\alpha), A_{n+2}(\alpha), \ldots] \qquad \text{and} \qquad A_m(T^n\alpha) = A_{n+m}(\alpha).$$

Exactness the CF map on $\mathbb{F}_q((Z^{-1}))$

Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system consisting of a set X with the σ-algebra $\mathcal{B}$ of its subsets, a probability measure μ, and a transformation $T : X \to X$. We say that $(X, \mathcal{B}, \mu, T)$ is *measure-preserving* if, for all $E \in \mathcal{B}$, $\mu(T^{-1}E) = \mu(E)$.

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Let $\mathcal{N} = \{E \in \mathcal{B} \colon \mu(E) = 0 \text{ or } \mu(E) = 1\}$ denote the trivial σ-algebra of subsets of $\mathcal{B}$ of either null or full measure.

We say that the measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is *exact* if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N},$$

where $T^{-n}\mathcal{B} = \{T^{-n}E \colon E \in \mathcal{B}\}$.

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Theorem

The dynamical system $(B(0;1), \mathcal{B}, \mu_q, T_q)$ is exact. (Lertchoosakul, Nair)

Exactness implies mixing, ergodicity

If $(X, \mathcal{B}, \mu, T)$ is exact, then a number of strictly weaker properties arise. Firstly, for any natural number n and any $E_0, E_1, \ldots, E_n \in \mathcal{B}$, we have

$$\lim_{j_1, \ldots, j_n \to \infty} \mu(E_0 \cap T^{-j_1} E_1 \cap \cdots \cap T^{-(j_1 + \cdots + j_n)} E_n) = \mu(E_0)\mu(E_1) \cdots \mu(E_n).$$

This is called *mixing of order n*.

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This is called *mixing of order n*. This implies

$$\lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} |\mu(E_0 \cap T^{-j}E_1) - \mu(E_0)\mu(E_1)| = 0$$

which is called *weak mixing*.

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Weak-mixing property implies the condition that if $E \in \mathcal{B}$ and if $T^{-1}E = E$, then either $\mu(E) = 0$ or $\mu(E) = 1$. This last property is referred to as *ergodicity* in measurable dynamics. All these implications are known to be strict in general.

Good Universality

- A sequence of integers $(a_n)_{n=1}^{\infty}$ is called *L^p-good universal* if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^p(X, \mathcal{B}, \mu)$ we have
$$\overline{f}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n} x)$$
existing μ almost everywhere.
- A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed modulo one* if for each interval $I \subseteq [0, 1)$, if $|I|$ denotes its length, we have
$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : \{x_n\} \in I\} = |I|.$$

Subsequence ergodic theory

Lemma

If $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ, the dynamical system $(X, \mathcal{B}, \mu, T)$ is weak-mixing and $(a_n)_{n\geq 1}$ is L^2-good universal then $\overline{f}(x)$ exists and

$$\overline{f}(x) = \int_X f d\mu$$

μ almost everywhere.(Nair)

V. Berthe, H. Nakada

Specializing for instance to the case where $F(x) = \log_q x$, we recover the positive characteristic analogue of Khinchin's famous result that

$$\lim_{n\to\infty} |A_1(\alpha)\cdots A_n(\alpha)|^{\frac{1}{n}} = q^{\frac{q}{q-1}}$$

almost everywhere with respect to Haar measure.

Let $(a_n)_{n=1}^{\infty}$ be an L^p-good universal sequence with, for any irrational number γ, $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1. Suppose that $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a continuous increasing function with

$$\int_{B(0;1)} |F(|A_1(\alpha)|)|^p \, d\mu < \infty.$$

For each $n \in \mathbb{N}$ and arbitrary non-negative real numbers $d_1, \ldots, d_n$, we define

$$M_{F,n}(d_1, \ldots, d_n) = F^{-1}\left(\frac{F(d_1) + \cdots + F(d_n)}{n}\right).$$

Then we have

$$\lim_{n\to\infty} M_{F,n}(|A_{a_1}(\alpha)|, \ldots, |A_{a_n}(\alpha)|) = F^{-1}\left(\int_{B(0;1)} F(|A_1(\alpha)|) \, d\mu\right)$$

almost everywhere with respect to Haar measure.

Let $(a_n)_{n=1}^{\infty}$ be an L^p-good universal sequence with, for any irrational number γ, $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1. Suppose that $H : \mathbb{N}^m \to \mathbb{R}$ is a function with

$$\int_{B(0;1)} |H(|A_1(\alpha)|, \ldots, |A_m(\alpha)|)|^p \, d\mu < \infty.$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} H(|A_{a_j}(\alpha)|, \ldots, |A_{a_j+m-1}(\alpha)|)$$

$$= \sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}} \right)$$

almost everywhere with respect to Haar measure.

New application 1

Let $(a_n)_{n=1}^{\infty}$ be an L^p-good universal sequence with, for any irrational number γ, $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \deg(A_{a_j}(\alpha)) = \frac{q}{q-1}$$

almost everywhere with respect to Haar measure. (Lertchoosakul, Nair)

Apply with $f(\alpha) = \sum_{n=1}^{\infty} n \cdot \chi_{\{q^n\}}(|A_1(\alpha)|)$.

New application 2

Let $(a_n)_{n=1}^{\infty}$ be an L^p-good universal sequence with, for any irrational number γ, $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1. Then, for any $A \in \mathbb{F}_q[Z]^*$,

$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{1 \leq j \leq n \colon A_{a_j}(\alpha) = A\} = |A|^{-2}$$

almost everywhere with respect to Haar measure. (Lertchoosakul, Nair)
Apply with $f(\alpha) = \chi_{\{A\}}(A_1(\alpha))$.

New application 3

Let $(a_n)_{n=1}^{\infty}$ be an L^p-good universal sequence with, for any irrational number γ, $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1. Then, for any natural numbers $k < l$,

$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{1 \le j \le n\colon \deg(A_{a_j}(\alpha)) = l\} = \frac{q-1}{q^l},$$
$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{1 \le j \le n\colon \deg(A_{a_j}(\alpha)) \ge l\} = \frac{1}{q^{l-1}},$$
$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{1 \le j \le n\colon k \le \deg(A_{a_j}(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right)$$

almost everywhere with respect to Haar measure. (Lertchoosakul, Nair)

Apply with $f_1(\alpha) = \chi_{\{q^l\}}(|A_1(\alpha)|)$, $f_2(\alpha) = \chi_{[q^l,\infty)}(|A_1(\alpha)|)$, and $f_3(\alpha) = \chi_{[q^k,q^l)}(|A_1(\alpha)|)$, respectively.

The Gal-Koksma Theorem

Let S be a measurable set. For any non-negative integers M and N, let $\varphi(M, N; x) \geq 0$ be a function defined on S such that

(i) $\varphi(M, 0; x) = 0$ for all $M \geq 0$;

(ii) $\varphi(M, N; x) \leq \varphi(M, N'; x) + F(M + N', N - N'; x)$ for all $M, N \geq 0$ and $0 \leq N' \leq N$.

Suppose that, for all $M \geq 0$,

$$\int_S \varphi(M, N; x)^p \, dx = O(\phi(N)),$$

where $\phi(N)/N$ is a non-decreasing function. Then, given any $\epsilon > 0$, we have

$$\varphi(0, N; x) = o(\phi(N)^{\frac{1}{p}} (\log N)^{1+\frac{1}{p}+\epsilon})$$

almost everywhere $x \in S$.

New application 4

Suppose that $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a function such that

$$\int_{B(0;1)} |F(|A_1(\alpha)|)|^2 \, d\mu(\alpha) < \infty.$$

Then, given any $\epsilon > 0$, we have

$$\frac{1}{N} \sum_{n=1}^{N} F(|A_1(T^n \alpha)|) = \int_{B(0;1)} F(|A_1(\alpha)|) \, d\mu(\alpha) + o(N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\epsilon})$$

almost everywhere with respect to Haar measure.

New application 5

Suppose that $H : \mathbb{N}^m \to \mathbb{R}$ is a function such that

$$\int_{B(0;1)} |H(|A_1(\alpha)|, |A_2(\alpha)|, \ldots, |A_m(\alpha)|)|^2 \, d\mu(\alpha) < \infty.$$

Then, given any $\epsilon > 0$, we have

$$\frac{1}{N} \sum_{n=1}^{N} H(|A_1(T^n\alpha)|, |A_2(T^n\alpha)|, \ldots, |A_m(T^n\alpha)|)$$
$$= \sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}} \right) + o(N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\epsilon})$$

almost everywhere with respect to Haar measure.

A special case

Specializing for instance to the case $F(x) = \log_q x$, we establish the positive characteristic analogue of the quantitative version of Khinchin's famous result that

$$|A_1(\alpha)\cdots A_N(\alpha)|^{\frac{1}{N}} = q^{\frac{q}{q-1}} + o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}) \qquad (1)$$

almost everywhere with respect to Haar measure. Results for means other than the geometric mean can be obtained by making different choices of F and H.

New application 6

Given any $\epsilon > 0$, we have

$$\frac{1}{N}\sum_{n=1}^{N}\deg(A_n(\alpha)) = \frac{q}{q-1} + o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon})$$

almost everywhere with respect to Haar measure.

New application 7

Given any $A \in \mathbb{F}_q[Z]^*$ and $\epsilon > 0$, we have

$$\frac{1}{N} \cdot \#\{1 \leq n \leq N \colon A_n(\alpha) = A\} = |A|^{-2} + o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon})$$

almost everywhere with respect to Haar measure.

New application 8

Let $k < l$ be two natural numbers. Given any $\epsilon > 0$, we have

$$\frac{1}{N} \cdot \#\{1 \leq n \leq N \colon \deg(A_n(\alpha)) = l\} = \frac{q-1}{q^l} + o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}),$$

$$\frac{1}{N} \cdot \#\{1 \leq n \leq N \colon \deg(A_n(\alpha)) \geq l\} = \frac{1}{q^{l-1}} + o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon}),$$

$$\frac{1}{N} \cdot \#\{1 \leq n \leq N \colon k \leq \deg(A_n(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right) + o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\epsilon})$$

almost everywhere with respect to Haar measure.

Definition

The p-**adic absolute value** of $a \in \mathbb{Q}$ is defined by

$$|a|_p = p^{-\alpha} \quad \text{and} \quad |0|_p = 0.$$

p-adic numbers

Let p be a prime. Any nonzero rational number a can be written in the form $a = p^{\alpha}(r/s)$ where $\alpha \in \mathbb{Z}, r, s \in \mathbb{Z}$ and $p \nmid r, p \nmid s$.

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The p-adic absolute value $|.|_p$ satisfies the following **properties**:

1. $|a|_p = 0$ if and only if $a = 0$,
2. $|ab|_p = |a|_p|b|_p$ for all $a, b \in \mathbb{Q}_p$,
3. $|a + b|_p \leq |a|_p + |b|_p$ for all $a, b \in \mathbb{Q}_p$,
4. $|a + b|_p \leq \max\{|a|_p, |b|_p\}$ for all $a, b \in \mathbb{Q}_p$.

The p-adic absolute value is *non-archimedian*.

The topology of $\mathbb{Q}_p$

Let $a \in \mathbb{Q}_p$ and $r \geq 0$ be a real number. The **open ball** of radius r centered at a is the set

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

The **closed ball** of radius r and center a is the set

$$\overline{B}(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}.$$

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The **ring of p-adic integers** is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Next, we will consider the set
$p\mathbb{Z}_p = \{px : x \in \mathbb{Z}_p\} = \{x \in \mathbb{Q}_p : |x|_p < 1\}$.

p-adic continued fraction expansion

Let p be a prime. We will consider the continued fraction expansion of a p-adic integer $x \in p\mathbb{Z}_p$ in the form

$$x = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \ddots}}} \tag{2}$$

where $b_n \in \{1, 2, \ldots, p-1\}$, $a_n \in \mathbb{N}$ for $n = 1, 2, \ldots$.

p-adic continued fraction map

For $x \in p\mathbb{Z}_p$ define the map $T_p : p\mathbb{Z}_p \to p\mathbb{Z}_p$ to be

$$T_p(x) = \frac{p^{v(x)}}{x} - \left(\frac{p^{v(x)}}{x} \bmod p\right) = \frac{p^{a(x)}}{x} - b(x) \tag{3}$$

where $v(x)$ is the p-adic valuation of x, $a(x) \in \mathbb{N}$ and $b(x) \in \{1, 2, \dots, p-1\}$.

p-adic continued fraction map

For $x \in p\mathbb{Z}_p$ define the map $T_p : p\mathbb{Z}_p \to p\mathbb{Z}_p$ to be

$$T_p(x) = \frac{p^{v(x)}}{x} - \left(\frac{p^{v(x)}}{x} \bmod p\right) = \frac{p^{a(x)}}{x} - b(x) \qquad (4)$$

where $v(x)$ is the p-adic valuation of x, $a(x) \in \mathbb{N}$ and $b(x) \in \{1, 2, \dots, p-1\}$.

We will consider the dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ where $\mathcal{B}$ is σ-algebra on $p\mathbb{Z}_p$ and μ is Haar measure on $p\mathbb{Z}_p$.
For the Haar measure it it is the case that $\mu(pa + p^m\mathbb{Z}_p) = p^{1-m}$.

Properties of the p-adic continued fraction map

The following properties are due to Hirsch and Washington (2011).

- T_p is measure-preserving with respect to μ, i.e. $\mu(T_p^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.

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- T_p is measure-preserving with respect to μ, i.e. $\mu(T_p^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.
- T_p is ergodic, i.e. $\mu(B) = 0$ or 1 for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$.

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- T_p is measure-preserving with respect to μ, i.e. $\mu(T_p^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.
- T_p is ergodic, i.e. $\mu(B) = 0$ or 1 for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$.
- The p-adic analogue of Khinchin's Theorem: For almost all $x \in p\mathbb{Z}_p$ the p-adic continued fraction expansion satisfies

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{p}{p-1}.$$

Other properties of the p-adic continued fraction map

Definition

Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. The transformation T is *exact* if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N}.$$

where $\mathcal{N} = \{B \in \mathcal{B} \mid B = \emptyset \text{ a.e. or } B = X \text{ a.e.}\}$.

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

The p-adic continued fraction map T_p is exact.

Other properties of the p-adic continued fraction map

Because $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is exact, it implies other strictly weaker properties:

- T_p is strong-mixing, i.e. for all $A, B \in \mathcal{B}$ we have

$$\lim_{n\to\infty} \mu(T_p^{-n} A \cap B) = \mu(A)\mu(B)$$

  which impies

- T_p is weak-mixing, i.e. for all $A, B \in \mathcal{B}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T_p^{-j} A \cap B) - \mu(A)\mu(B)| = 0$$

  which implies

- T_p is ergodic, i.e. $\mu(B) = 0$ or 1 for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$.

Good Universality

- A sequence of integers $(a_n)_{n=1}^{\infty}$ is called *L^p-good universal* if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^p(X, \mathcal{B}, \mu)$ we have
$$\overline{f}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n} x)$$
existing μ almost everywhere.
- A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed modulo one* if for each interval $I \subseteq [0, 1)$, if $|I|$ denotes its length, we have
$$\lim_{N \to \infty} \frac{1}{N} \#\{n \leq N : \{x_n\} \in I\} = |I|.$$

Subsequence ergodic theory

Lemma

If $(\{a_n\gamma\})_{n=1}^{\infty}$ *is uniformly distributed modulo one for each irrational number* γ*, the dynamical system* $(X, \mathcal{B}, \mu, T)$ *is weak-mixing and* $(a_n)_{n\geq 1}$ *is* L^2*-good universal then* $\overline{f}(x)$ *exists and*

$$\overline{f}(x) = \int_X f d\mu$$

μ *almost everywhere.*

Results

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

For any L^p-good universal sequence $(k_n)_{n\geq 1}$ where $(\{k_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{k_n} = \frac{p}{p-1},$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b_{k_n} = \frac{p}{2},$$

almost everywhere with respect to Haar measure on $p\mathbb{Z}_p$.

Results

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$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : a_{k_n} = i\} = \frac{p-1}{p^i};$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : a_{k_n} \geq i\} = \frac{1}{p^{i-1}};$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : i \leq a_{k_n} < j\} = \frac{1}{p^{i-1}}\left(1 - \frac{1}{p^j}\right);$$

almost everywhere with respect to Haar measure on $p\mathbb{Z}_p$.

Partitions

Let $(X, \mathcal{A}, m)$ be a probability space where X is a set, $\mathcal{A}$ is a σ-algebra of its subsets and m is a probability measure. A partition of $(X, \mathcal{A}, m)$ is defined as a denumerable collection of elements $\alpha = \{A_1, A_2, \dots\}$ of $\mathcal{A}$ such that

$$A_i \cap A_j = \emptyset$$

for all $i, j \in I, i \neq j$ and

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Given partitions $\alpha = \{A_1, A_2, \dots\}$ and $\beta = \{B_1, B_2, \dots\}$ we define the join of α and β to be the partition

$$\alpha \vee \beta = \{A_i \cap B_j | A_i \in \alpha, B_j \in \beta\}$$

.

Entropy of a Partition

For a finite partition $\alpha = \{A_1, \dots, A_n\}$ we define its entropy $H(\alpha) = -\sum_{i=1}^{n} m(A_i) \log m(A_i)$. Let $\mathcal{A}' \subset \mathcal{A}$ be a sub-σ-algebra.

Then we define the conditional entropy of α given $\mathcal{A}'$ to be $H(\alpha|\mathcal{A}') = -\sum_{i=1}^{n} m(A_i|\mathcal{A}') \log m(A_i|\mathcal{A}')$.

Here of course $m(A|\mathcal{A}')$ means $\mathbb{E}(\chi_A|\mathcal{A}')$ where $\mathbb{E}(.|\mathcal{A}')$ denotes the projection operator $L^1(X, \mathcal{A}, m) \to L^1(X, \mathcal{A}', m)$ and χ_A is the characteristic function of the set A.

Entropy of a transformation

The entropy of a measure-preserving transformation T relative to a partition α is defined to be

$$h_m(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

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$$h_m(T,\alpha) = \lim_{n\to\infty} H\left(\alpha | \bigvee_{i=1}^{n} T^{-i}\alpha\right) = H\left(\alpha | \bigvee_{i=1}^{\infty} T^{-i}\alpha\right). \quad (5)$$

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We define the measure-theoretic entropy of T with respect to the measure m (irrespective of α) to be $h_m(T) = \sup_\alpha h_m(T, \alpha)$ where the supremum is taken over all finite or countable partitions α from $\mathcal{A}$ with $H(\alpha) < \infty$.

Theorem (Jaššová,Nair)

Let $\mathcal{B}$ denote the Haar σ-algebra restricted to $p\mathbb{Z}_p$ and let μ denote Haar measure on $p\mathbb{Z}_p$. Then the measure-preserving transformation $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ has measure-theoretic entropy $\frac{p}{p-1} \log p$.

Isomorphism of measure preserving transformations

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They are said to be isomorphic if there exist sets $M_1 \subseteq X_1$ and $M_2 \subseteq X_2$ with $m_1(M_1) = 1$ and $m_2(M_2) = 1$ such that $T_1(M_1) \subseteq M_1$ and $T_2(M_2) \subseteq M_2$ and such that there exists a map $\phi : M_1 \to M_2$ satisfying $\phi T_1(x) = T_2 \phi(x)$ for all $x \in M_1$ and $m_1(\phi^{-1}(A)) = m_2(A)$ for all $A \in \beta_2$.

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The importance of measure theoretic entropy, is that two dynamical systems with different entropies can not be isomorphic.

Bernoulli Space

Suppose (Y, α, l) is a probability space, and let $(X, \beta, m) = \Pi_{-\infty}^{\infty}(Y, \alpha, l)$ i.e. the bi-infinite product probability space.

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Any measure preserving transformation isomorphic to a Bernoulli process will be refered to as Bernoulli.

Ornstein's theorem

The fundamental fact about Bernoulli processes, famously proved by D. Ornstein in 1970, is that Bernoulli processes with the same entropy are isomorphic.

The natural extention

To any measure-preserving transformation, (X, β, m, T_0) set $X^\infty = \Pi_{n=0}^{\infty} X$ and set

$$X_{T_0} = \{\underline{x} = (x_0, x_1, \dots) \in X^\infty : x_n = T_0(x_{n+1}), x_n \in X, n = 0, 1, 2, \dots\}.$$

Let $T : X_{T_0} \to X_{T_0}$ be defined by

$$T((x_0, x_1, \dots,)) = (T(x_0), x_0, x_1, \dots,).$$

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The map T is bijective on X_{T_0}. If T_0 preserves a measure m, then we can define a measure $\overline{m}$ on X_{T_0}, by defining $\overline{m}$ on the cylinder sets $C(A_0, A_1, \dots, A_k) = \{\underline{x} : x_0 \in A_0, x_1 \in A_1, \dots, x_k \in A_k\}$ by

$$\overline{m}(C(A_0, A_1, \dots, A_k)) = m(T_0^{-k}(A_0) \cap T_0^{-k+1}(A_1) \cap \dots \cap A_k),$$

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for $k \geq 1$. One can check that the invertable transformation $(X_{T_0}, \overline{\beta}, \overline{m}, T_0)$ called the natural extention of (X, β, m, T_0) is measure preserving as a consequence of the measure preservation of the transformation (X, β, m, T_0).

Fundamental Dynamical property of the Schneider Map

Theorem (Jaššová,Nair)

Suppose $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is the Schneider continued fraction map. Then the dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ has a natural extention that is Bernoulli.

This property implies all the mixing properties of the map and via ergodic theorems all the properties of averages of convergents. Also, via Ornstein's theorem, it is isomphorphic as a dynamical system to all Bernoulli shifts with the same entropy and hence is completely classified.

Absolute values on topological fields

Let K denote a topological field. By this we mean that the field K is a locally compact group under the addition , with respect a topology (which in our case is discrete). This ensures that K comes with a translation invariant Haar measure μ on K, that is unique up to scalar multiplication. For an element $a \in K$, we are now able it absolute value, as

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An absolute value is a function $|.| : K \to \mathbb{R}_{\geq 0}$ such that (i) $|a| = 0$ if and only if $a = 0$; (ii) $|ab| = |a||b|$ for all $a, b \in K$ and (iii) $|a + b| \leq |a| + |b|$ for all pairs $a, b \in K$. The absolute value just defined gives rise to a metric defined by $d(a, b) = |a - b|$ with $a, b \in K$, whose topology coincides with original topology on the field K.

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(b) spaces where (iii)* is not true called archimedean spaces.

From now on we shall concern ourselves solely with non-archimedean fields.

Valuations and absolute values

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Let $K^* = K\backslash\{0\}$. A map $v : K^* \to \mathbb{R}$ is a valuation if
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To our initial valuation we associate the valuation described as follows. Pick $0 < \alpha < 1$ and write $|a| = \alpha^{v(a)}$, i.e., let $v(a) = \log_\alpha |a|$. Then $v(a)$ is a valuation, an additive version of $|a|$.

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The set of units in $\mathcal{O}$ are
$\mathcal{O}^\times = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$
and $\mathcal{M} = \{x \in K : v(x) > 0\} = \{x \in K : |x| < 1\}$ is an ideal in $\mathcal{O}$.
$k = \mathcal{O}/\mathcal{M}$ is a field, called the residue field of v or of K.

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Then every $x \in K$ can be written uniquely as $x = u\pi^n$ with $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}_{\geq 0}$. Also every $x \in \mathcal{M}$ can be written uniquely as $x = u\pi^n$ for a unit $u \in \mathcal{O}^\times$ and $n \geq 1$.

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Moreover, every ideal $I \subset \mathcal{O}$ is principal, as $(0) \neq I \subset \mathcal{O}$ implies $I = (\pi^n)$ where $n = \min\{v(x) : x \in I\}$, so $\mathcal{O}$ is a principal ideal domain (PID).

There are two examples

(i) The p-adic numbers $\mathbb{Q}_p$ and their finite extentions. For instance if $K = \mathbb{Q}_p$ then $\mathcal{O} = \mathbb{Z}_p$ $\mathcal{M} = p\mathbb{Z}_p$. Here we can take $\pi = p$.

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(ii) The field of formal power series $K = \mathbb{F}_q((X^{-1}))$ for $q = p^n$ for some prime p, with $\mathcal{O} = \mathbb{F}_q[X]$ and $\mathcal{M} = I(x)\mathbb{F}_q[X]$ for some irreducible polynomial I. Here we can take $\pi = I$.

These two are the only two possibilities. This is the structure theorem for non-archemedian fields.

Schneider's Map on an arbitrary non-archemedean field

We define the map $T_v : \mathcal{M} \to \mathcal{M}$ defined by

$$T_v(x) = \frac{\pi^{v(x)}}{x} - b(x)$$

where $b(x)$ denotes the residue class to which $\frac{\pi^{v(x)}}{x}$ in k.
This gives rise to the continued fraction expansion of $x \in \mathcal{M}$ in the form

$$x = \cfrac{\pi^{a_1}}{b_1 + \cfrac{\pi^{a_2}}{b_2 + \cfrac{\pi^{a_3}}{b_3 + \ddots}}} \tag{7}$$

where $b_n \in k^\times, a_n \in \mathbb{N}$ for $n = 1, 2, \ldots$.

The start of continued fractions on a non-archemedean field

The rational approximants to $x \in \mathcal{M}$ arise in a manner similar to that in the case of the real numbers as follows. We suppose $A_0 = b_0, B_0 = 1, A_1 = b_0 b_1 + \pi^{a_1}, B_1 = b_1$. Then set

$$A_n = \pi^{a_n} A_{n-2} + b_n A_{n-1} \text{ and } B_n = \pi^{a_n} B_{n-2} + b_n B_{n-1} \quad (8)$$

for $n \geq 2$. A simple inductive argument gives for $n = 1, 2, \ldots$.

$$A_{n-1} B_n - A_n B_{n-1} = (-1)^n \pi^{a_1 + \ldots + a_n}. \quad (9)$$

Dynamics of the Schneider's map on a non-archemedean field

The map $T_v : \mathcal{M} \to \mathcal{M}$ preserves Haar measure on $\mathcal{M}$. We also have the following.

Theorem

Let $\mathcal{B}$ denote the Haar σ-algebra restricted to $\mathcal{M}$ and let μ denote Haar measure on $\mathcal{M}$. Then the measure-preserving transformation $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has measure-theoretic entropy $\frac{|k|}{|k^\times|} \log(|k|)$.

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Suppose $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is as in our first theorem. Then the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has a natural extension that is Bernoulli.

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Theorem

Suppose $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is as in our first theorem. Then the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has a natural extension that is Bernoulli.

This tells us the isomorphism class of the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is determined by its residue class field irrespective of the characteristic. This means for different p each Sneider map on the p-adics non-isomorphic.

Results

Theorem (Nair,Jaššová)

For any L^p-good universal sequence $(k_n)_{n\geq 1}$ where $(\{k_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_{k_n} = \frac{|k|}{|k^{\times}|},$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} b_{k_n} = \frac{|k|}{2},$$

almost everywhere with respect to Haar measure on $\mathcal{M}$.

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$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : a_{k_n} = i\} = \frac{|k^{\times}|}{|k|^{i}};$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : a_{k_n} \geq i\} = \frac{1}{|k|^{i-1}};$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : i \leq a_{k_n} < j\} = \frac{1}{|k|^{i-1}}\left(1 - \frac{1}{|k|^{j}}\right);$$

almost everywhere with respect to Haar measure on $\mathcal{M}$.

Thank you for your attention.