Lattice structures of multidimensional continued fractions

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Part I

I. Introduction.

Multidimensional continued fractions

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- Algorithmic generalizations (Jacobi-Perron Algorithm, etc.)

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- Dynamical generalizations (Farey tessellation and triangle sequences, etc.)
- Combinatorial description (tangles and rational knots)

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Why to study?

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- Applications to algebraic geometry (toric singularities)

How to study lattices?

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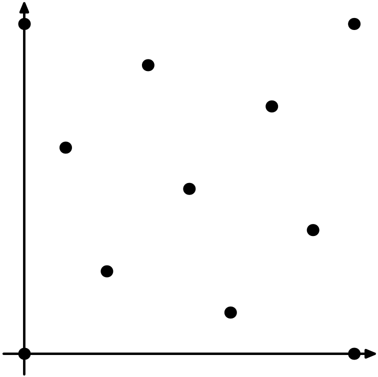

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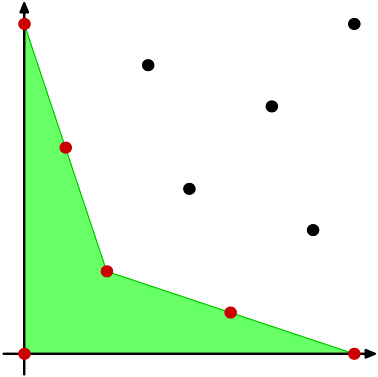

Klein polyhedron.

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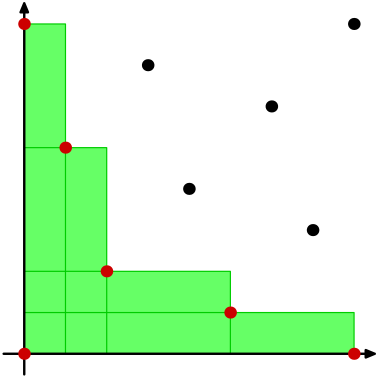

Minkowski-Voronoi complex.

Part II

II. Klein polyhedron.

Continued fractions for 7/5

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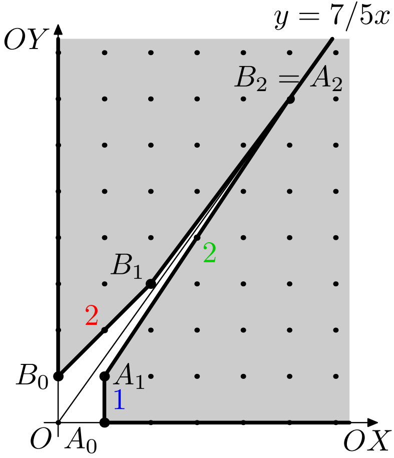
Proposition

Any rational number has a unique odd and even ordinary continued fractions.

Geometry of continued fractions



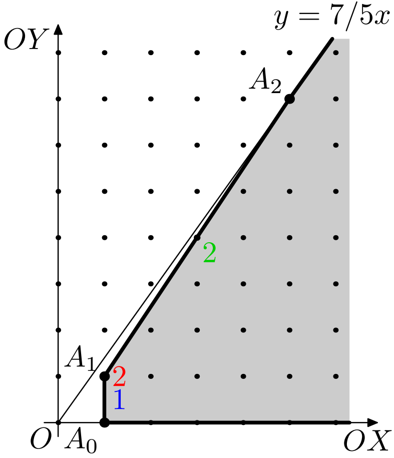
$a_0 = \mathrm{l}\ell(A_0A_1) = 1;$

$a_1 = \mathrm{l}\ell(B_0B_1) = 2;$

$a_2 = \mathrm{l}\ell(A_1A_2) = 2.$

$$7/5 = [1; 2 : 2].$$

$\mathrm{l}\ell(AB)$ — the number of primitive vectors in AB.



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$$\mathrm{lsin}(ABC) = \frac{S(ABC)}{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)}$$ (integer sin-formula).

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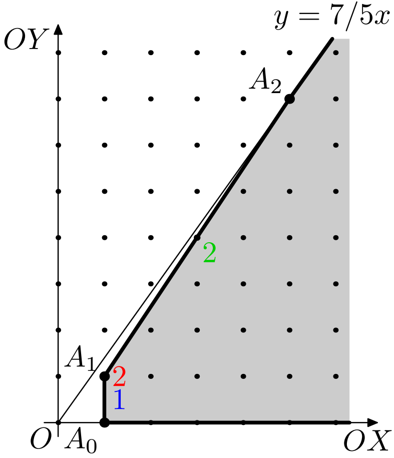

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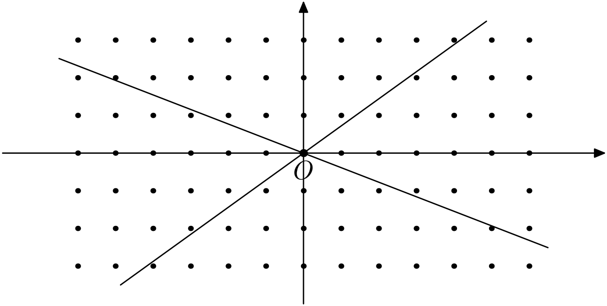
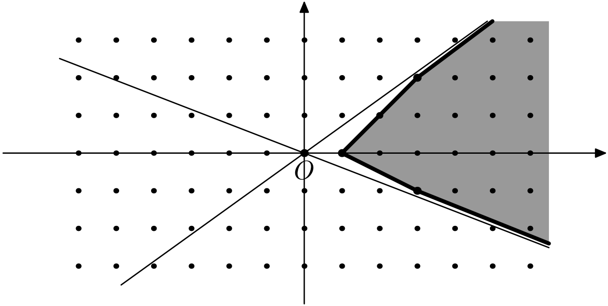
$(a_0, \ldots, a_{2n})$ — lattice length-sine sequence (LLS-sequence).

Multidimensional continued fractions



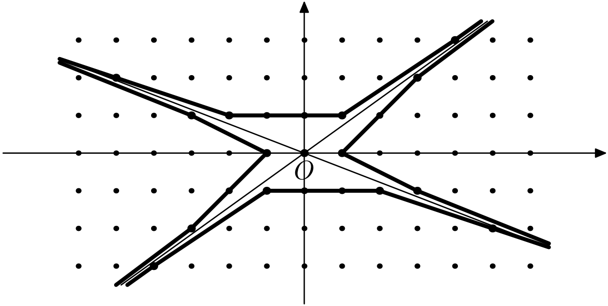
Consider n hyperplanes passing through O.

Multidimensional continued fractions



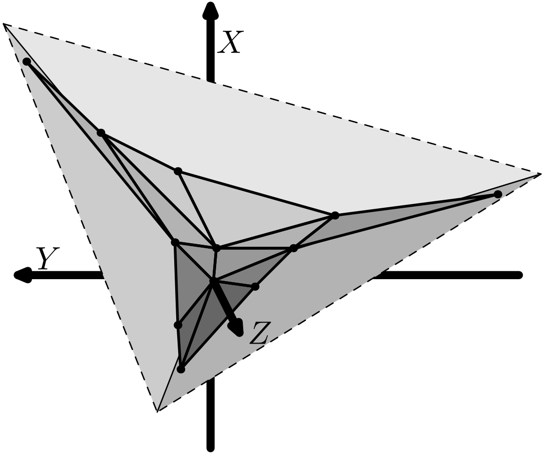
The *sail* for one of the cones, i.e. the boundary of the convex hull of all integer inner points.

Multidimensional continued fractions



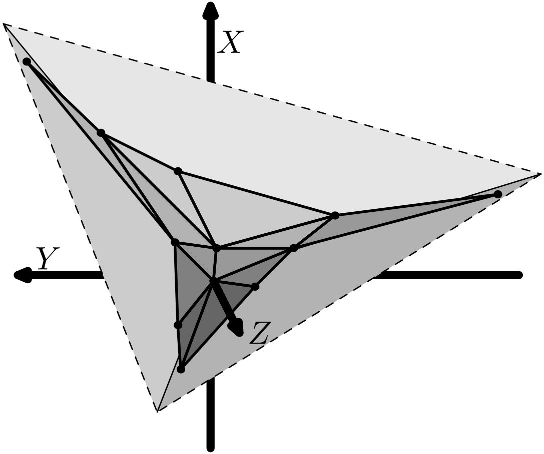
The set of all sails is called *geometric continued fraction* **(Klein, 1895)**.

Multidimensional continued fractions



A sail in 3D.

Multidimensional continued fractions



First question: Which two-dimensional faces can a sail have?

Faces of MCF

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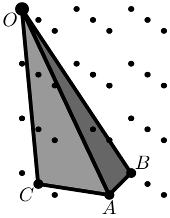

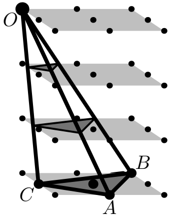

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Faces of MCF

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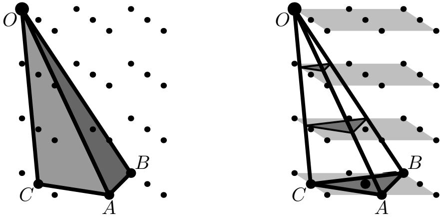

Two different cases

- **The face is at distance 1.**
- **The face is at distance greater than 1.**

Empty simplices

Definition

A simplex is *empty* if it does not contain lattice points distinct to vertices.

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Proposition

All lattice empty triangles are congruent.

Empty tetrahedra

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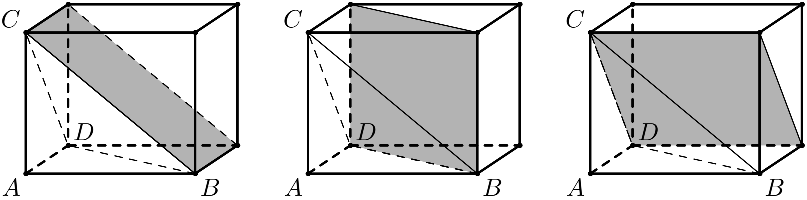
Theorem

(Equivalent to G. K. White, 1964) *If ABCD is empty then the lattice points of the corresponding parallelepiped (except for the vertices) are on one of the planes:*



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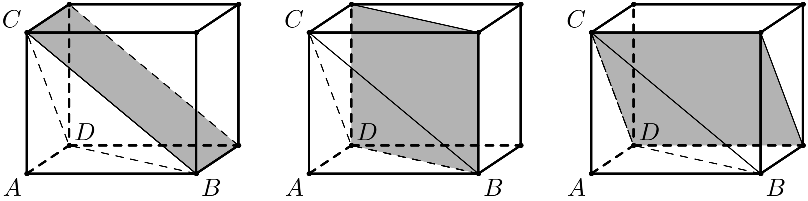


Corollary

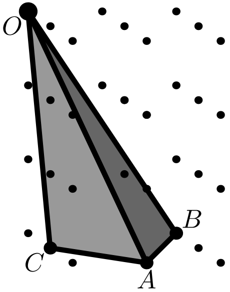
Complete list of empty simplices:

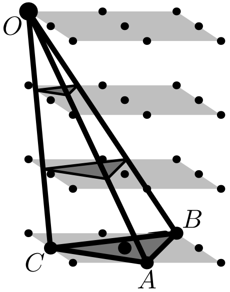
— $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,0,0)$;

— $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(\xi, r-\xi, r)$ *for* $r \geq 2$, $0 < \xi < r$, $\gcd(r,\xi) = 1$.

Next step: empty marked pyramids

A marked pyramid is *empty* if all lattice points distinct to the vertex are in the base.





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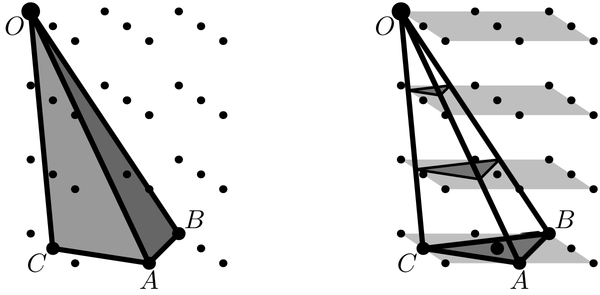

Lattice distance equals 1 – any base.

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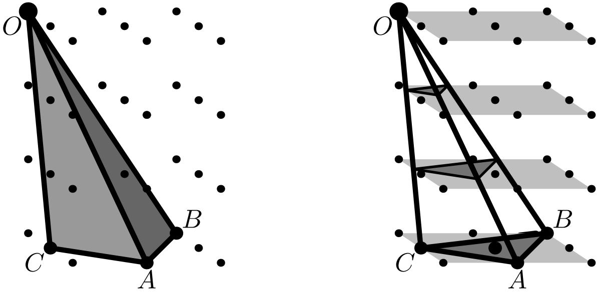

Lattice distance equals 1 – any base.

Lattice distance is greater than 1 – ???

Awful slide

Theorem

(Karpenkov, 2008) *A complete list of 3D empty marked multistory pyramids.*
— *the quadrangular marked pyramids* $M_{a,b}$, *with* $b \geq a \geq 1$;
— *triangular* $T^{\xi}_{a,r}$, *where* $a \geq 1$, *and* $\gcd(\xi, r) = 1$, $r \geq 2$, *and* $0 < \xi \leq r/2$;
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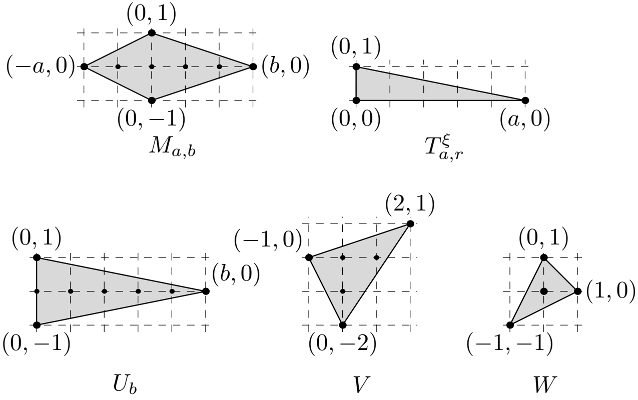
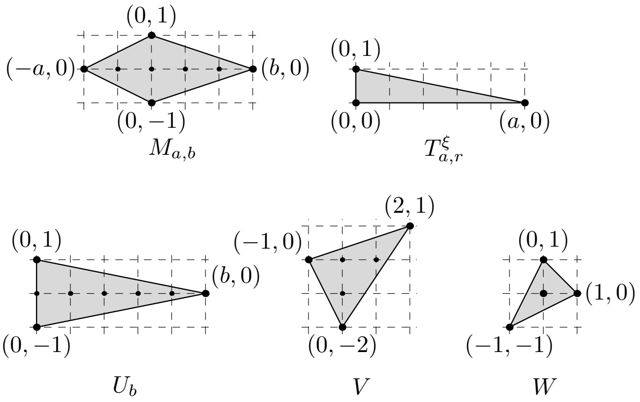
Vertex at the origin. Bases
$M_{a,b}$: $(2,-1,0)$, $(2,-a-1,1)$, $(2,-1,2)$, $(2,b-1,1)$
$T^{\xi}_{a,r}$: $(\xi, r-1, -r)$, $(a+\xi, r-1, -r)$, $(\xi, r, -r)$
U_b: $(2,1,b-1)$, $(2,2,-1)$, $(2,0,-1)$
V: $(2,-2,1)$, $(2,-1,-1)$, $(2,1,2)$
W: $(3,0,2)$, $(3,1,1)$, $(3,2,3)$

Bases empty marked pyramids





Corollary

Any face of MCF at distance > 1 *from O* **is from the list above**.

This corollary is used in for the algorithm to construct MCF.

Empty 4D simplices

Problem

(unsolved, 1964) *What happens in 4D with empty simplices?*

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Problem

What about 3D faces?

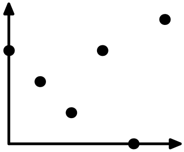
Part III

III. Minkovskii-Voronoi continued fractions.

Coaxial sets in general position.

Definition

A subset $S \subset \mathbb{R}^n_{\geq 0}$ is **axial** if S contains points on each of the coordinate axes.



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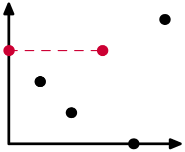
- No two points on other plane parallel to a coordinate plane.



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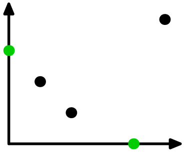


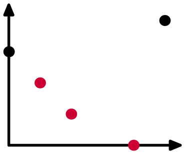
Minkovskii-Voronoi minima and minimal sets

Set

$$\max(A, i) = \max\{x_i \mid (x_1, \ldots, x_n) \in A\}$$

and define the parallelepiped

$$\Pi(A) = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq \max(A, i), i = 1, \ldots, n\}.$$



$\Pi(\text{Red dots})$.

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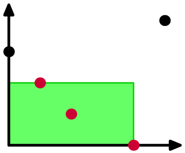
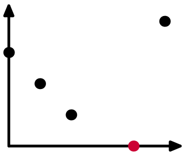


Π(Red dots).

Minkovskii-Voronoi minima and minimal sets

Definition

Let S be an arbitrary subset of $\mathbb{R}^n_{\geq 0}$ (csgp). An element $\gamma \in S$ is called a **Voronoi relative minimum** if the parallelepiped $\Pi(\{\gamma\})$ contains no points of $S \setminus \{\gamma\}$.



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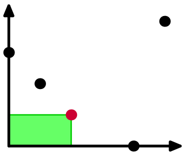

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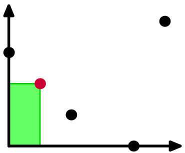

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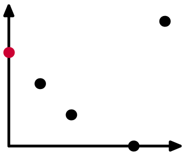

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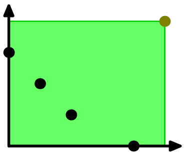

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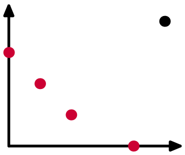

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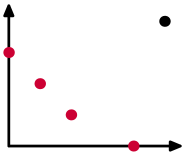

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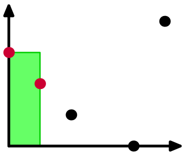

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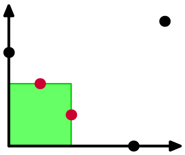

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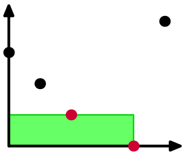

Minkovskii-Voronoi complex

Definition

MV-complex is an $(n-1)$-dimensional complex such that

- the k-dimensional faces are enumerated by the minimal $(n-k)$-element subsets
- a face with minimal subset F_1 is adjacent to a face with a minimal subset $F_2 \neq F_1$ if and only if $F_1 \subset F_2$.

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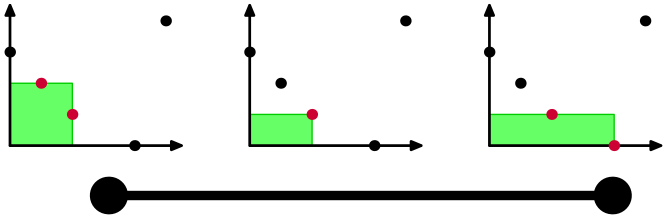

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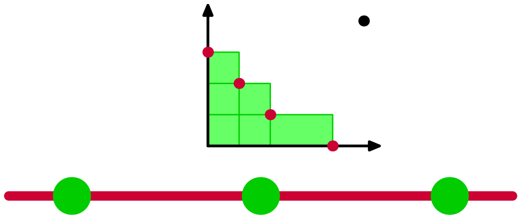

Example of the MV-complex

Consider

$$S_0 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\},$$

where

$$\begin{array}{lll} \gamma_1 = (3,0,0), & \gamma_2 = (0,3,0), & \gamma_3 = (0,0,3), \\ \gamma_4 = (2,1,2), & \gamma_5 = (1,2,1), & \gamma_6 = (2,3,4). \end{array}$$

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Relative minima: $\gamma_1, \ldots, \gamma_5$.

Example of the MV-complex

MV-complex contains 5 vertices, 6 edges, and 5 faces.

Vertices:

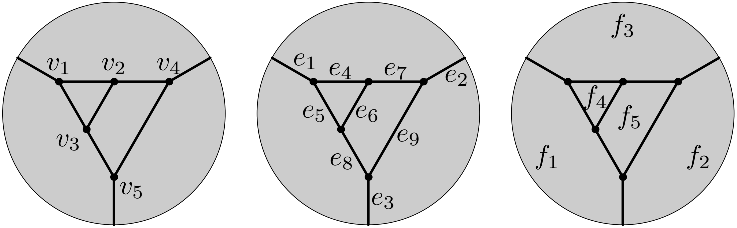
$$v_1 = \{\gamma_1, \gamma_3, \gamma_4\}, \quad v_2 = \{\gamma_3, \gamma_4, \gamma_5\}, \quad v_3 = \{\gamma_1, \gamma_4, \gamma_5\},$$
$$v_4 = \{\gamma_2, \gamma_3, \gamma_5\}, \quad v_5 = \{\gamma_1, \gamma_2, \gamma_5\}.$$

Edges:

$$\begin{array}{lll} e_1 = \{\gamma_1, \gamma_3\}, & e_2 = \{\gamma_3, \gamma_2\}, & e_3 = \{\gamma_1, \gamma_2\}, \\ e_4 = \{\gamma_3, \gamma_4\}, & e_5 = \{\gamma_1, \gamma_4\}, & e_6 = \{\gamma_4, \gamma_5\}, \\ e_7 = \{\gamma_3, \gamma_5\}, & e_8 = \{\gamma_1, \gamma_5\}, & e_9 = \{\gamma_2, \gamma_5\}. \end{array}$$

Faces:

$$f_1 = \{\gamma_1\}, \quad f_2 = \{\gamma_2\}, \quad f_3 = \{\gamma_3\}, \quad f_4 = \{\gamma_4\}, \quad f_5 = \{\gamma_5\}.$$



$MV(S)$ as a tessellation of an open two-dimensional disk.

Tessellations of the plane

Question: **How to describe MV-complexes in 3D?**

Tessellations of the plane

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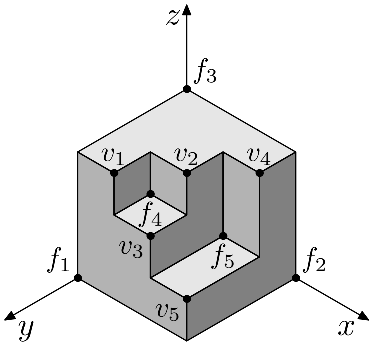
Useful tools:

Minkowski polyhedron for an arbitrary S;

Tessellations of the plane.

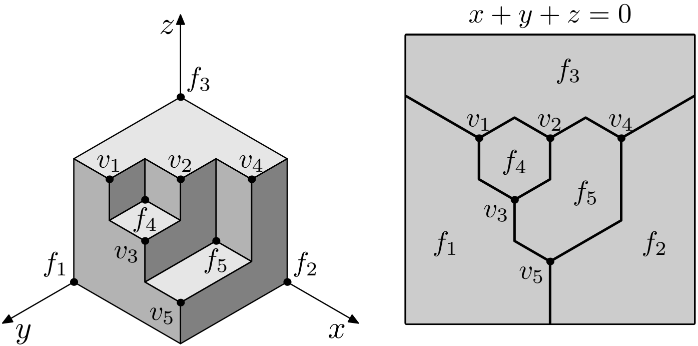
Tessellations of the plane



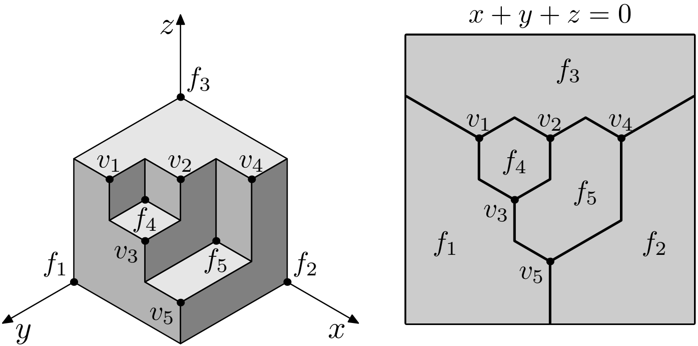
Minkowski polyhedron for a set S (some sort of convex hull):

$$S \oplus \mathbb{R}^3_{\geq 0} = \{s + r \mid s \in S, r \in \mathbb{R}^3_{\geq 0}\}.$$

Tessellations of the plane



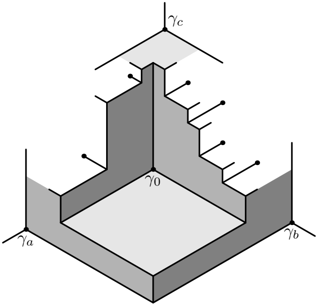
- The Minkowski polyhedron (left)
- Minkowski–Voronoi tessellation (right).

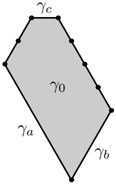


Definition

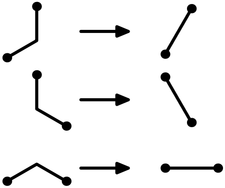
Step 1. Project the Minkowski polyhedron to $x + y + z = 0$.
Step 2. Remove relative minima (i.e., minima of $x + y + z$).
Remove also all edges adjacent to them.
Step 3. Rays to vertices of valence 1.

Linearisation of faces





Linearisation laws for edges



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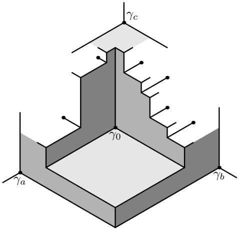
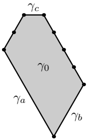
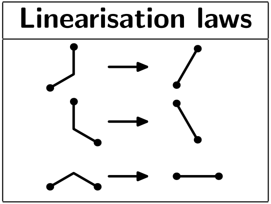
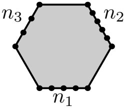






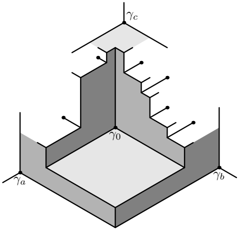
Theorem

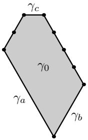
Every linearized finite face is as follows (up to size rescaling):

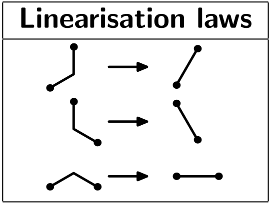


where $n_1, n_2, n_3 \geq 0$.

Linearisation of faces







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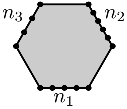

where $n_1, n_2, n_3 \geq 0$.

In our example: $n_1 = 0$, $n_2 = 4$, and $n_3 = 2$.

Diagrams of the tessellation

Definition

A diagram of a tessellation is **canonical** if all its faces are linearized.

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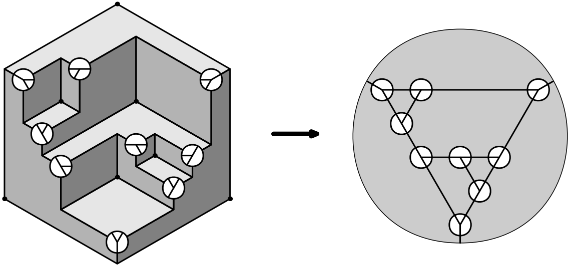

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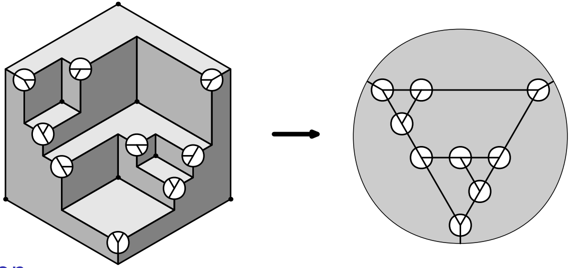

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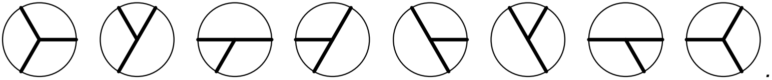
Every vertex of the MV-complex that is one of one of



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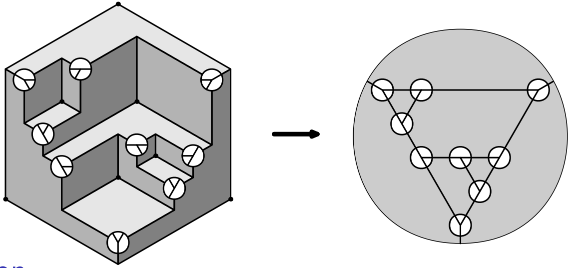

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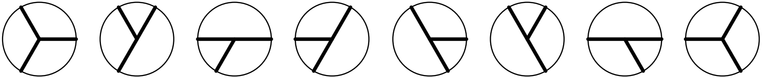
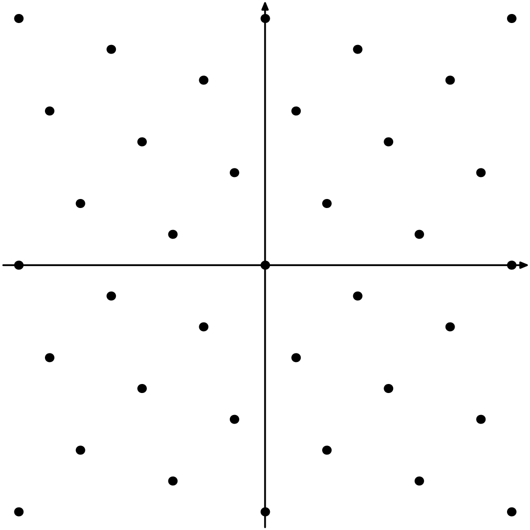


.

Proposition

Every finite tessellation of the plane admits a canonical diagram.

MV for lattices



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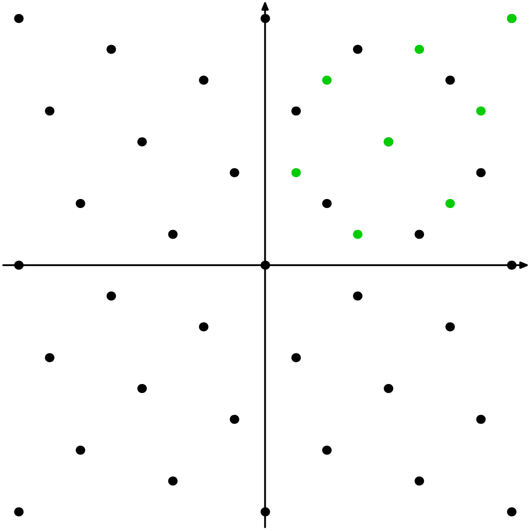

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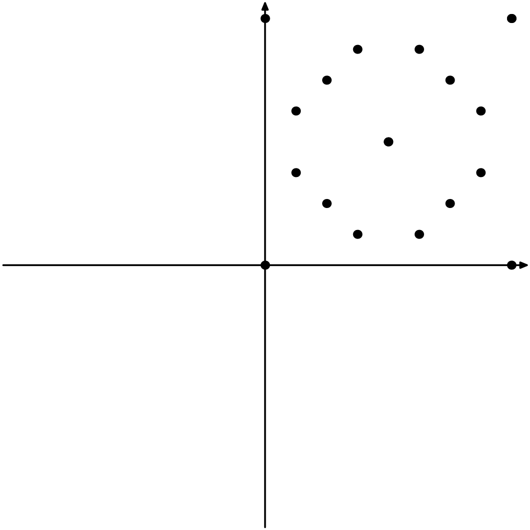

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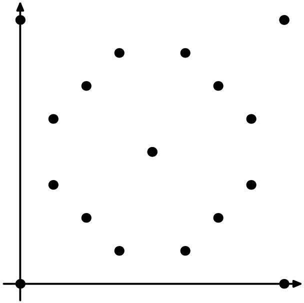

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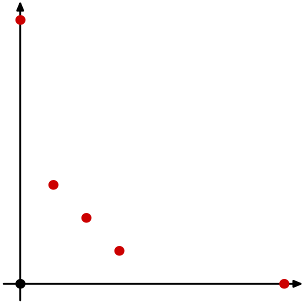

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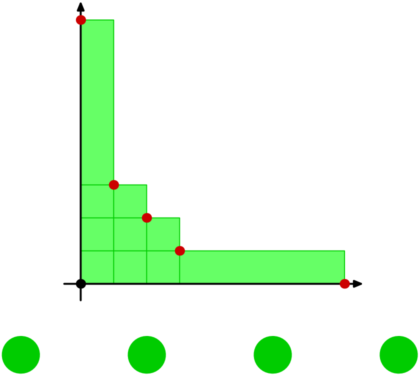

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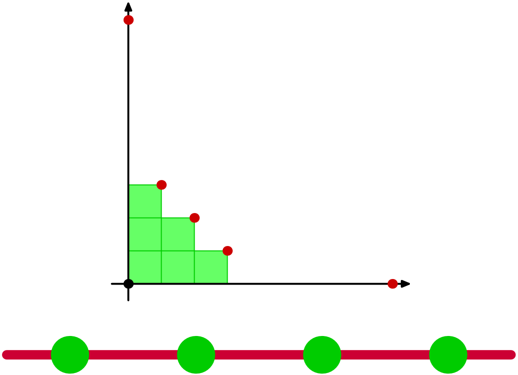

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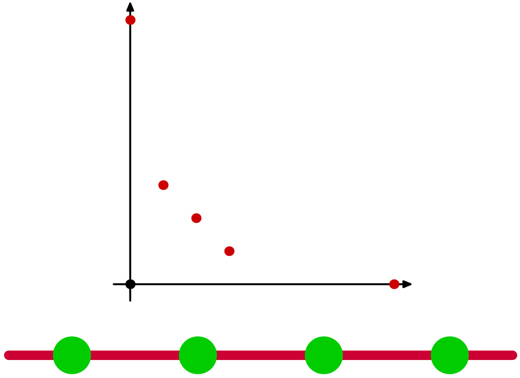
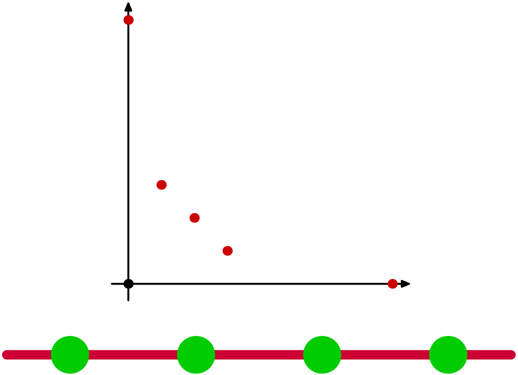


The lattice is generated by $(8,0)$ and $(5,1)$. Here

$$\frac{8}{5} = [1:1;1;1;1].$$



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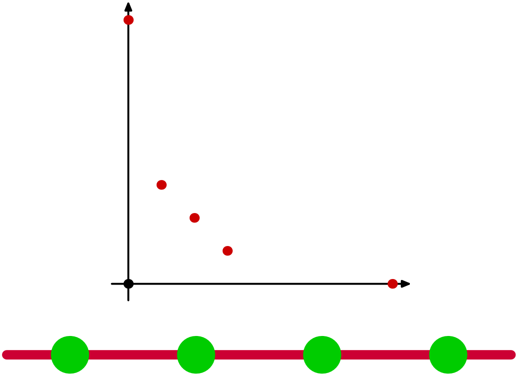
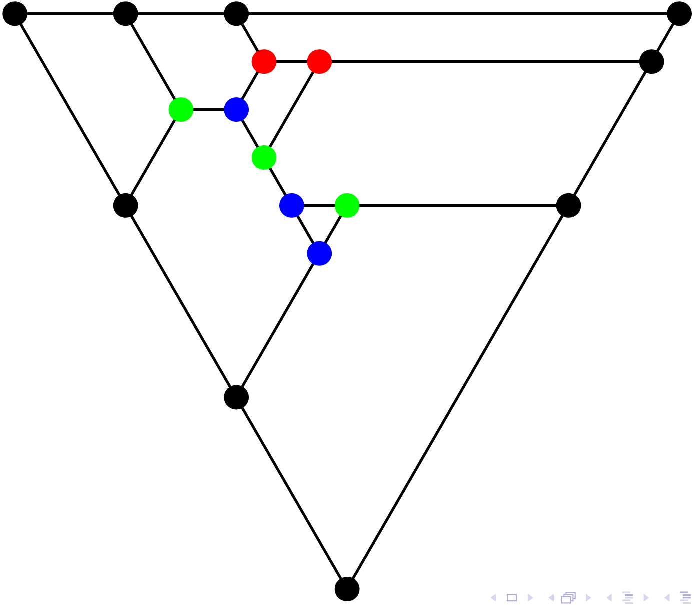
Remark. Here the continued fraction has 5 elements.

MV for lattices



Theorem on combinatorics of continued fractions.
The number of relative minima for a general lattice generated by $(N, 0)$ and $(a, 1)$ coincides with the number of elements for the longest continued fractions of $\frac{a}{N}$.

Lattice examples in 3D



Notation: $L(a, b, N)$

Definition

Let $a, b, N \in \mathbb{Z}_+$. The lattice

$$\Gamma(a, b, N) := \Big\langle (1, a, b), (0, N, 0), (0, 0, N) \Big\rangle$$

is said to be the 1-*rank lattice*.

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All local minima are in
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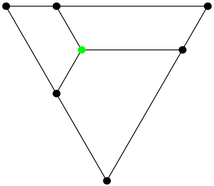
Proposition

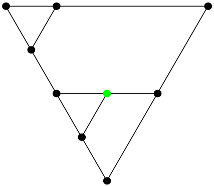
Let $\gcd(a, N) = \gcd(b, N) = 1$.
Then the set of all local minima for $|\Gamma(a, b, N)|$ *is a finite axial set in general position.*

Series of examples

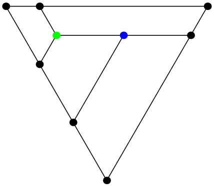
Observation of regularities (A.Ustinov, O.K. '13):



$[t = 1, u = 1]$



$[t = 1, u \geq 2]$



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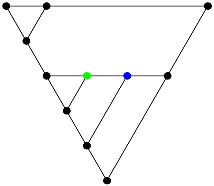

$[t \geq 2, u \geq 2]$

$$L(2, b, N): \quad b = 2t + 1, \quad N = b(2u + 0) + 1.$$

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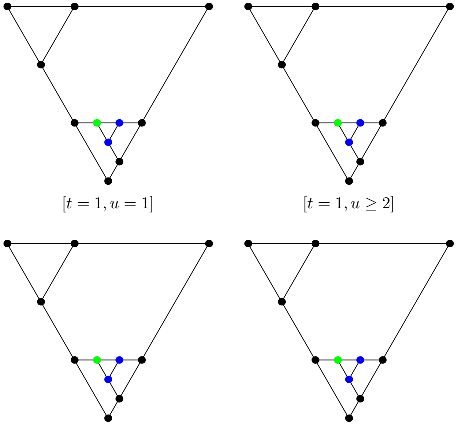

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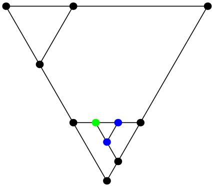

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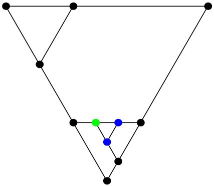

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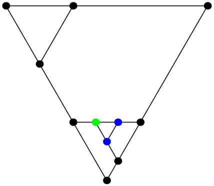

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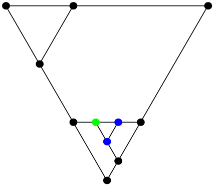

$[t \geq 2, u \geq 2]$

$$L(2, b, N): \quad b = 4t + 3, \quad N = b(2u + 1) + 2.$$

Series of examples

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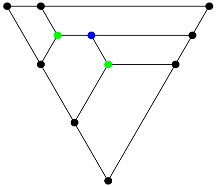

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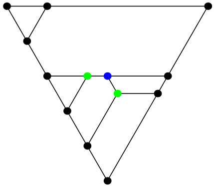

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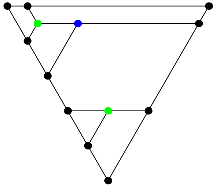

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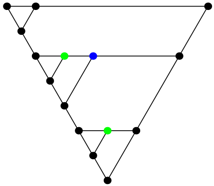

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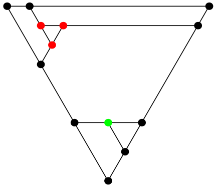

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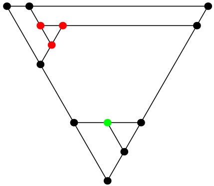

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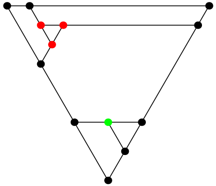

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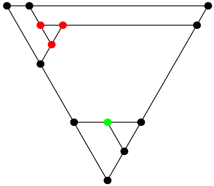

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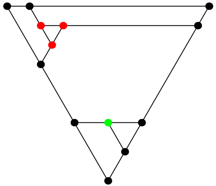
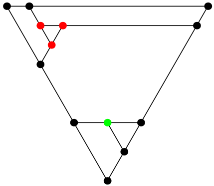
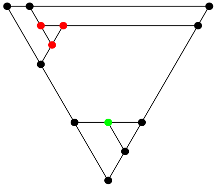


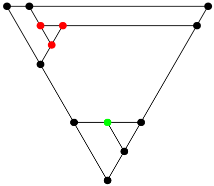
$[t = 1, u = 1]$



$[t = 1, u \geq 2]$



$[t \geq 2, u = 1]$



$[t \geq 2, u \geq 2]$

$$L(2, b, N): \quad b = 6t + 2, \quad N = b(2u + 1) + 3.$$

Series of examples

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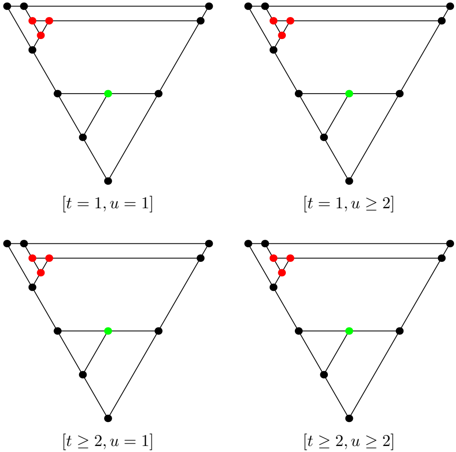

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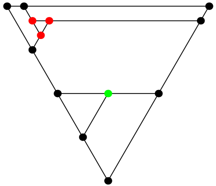

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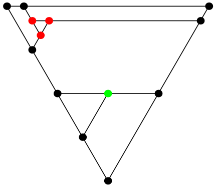
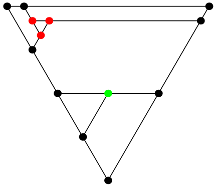


$[t = 1, u \geq 2]$



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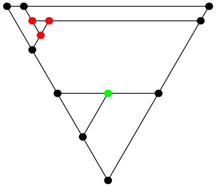

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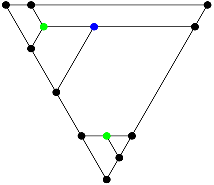

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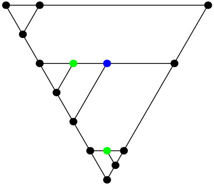

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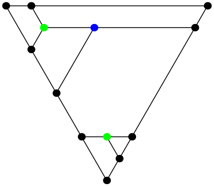

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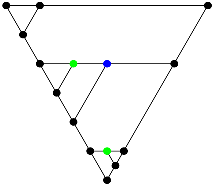

$[t \geq 2, u \geq 2]$

$$L(2, b, N): \quad b = 6t + 5, \quad N = b(2u + 0) + 3.$$

Series of examples

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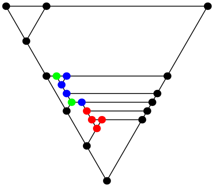

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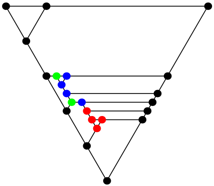

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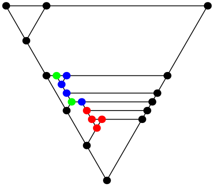

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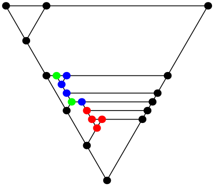

$[t\geq 2, u\geq 2]$

$$L(2,b,N): \quad b = 2\cdot 30t+17, \quad N = b(2u+1)+30.$$

Series of examples

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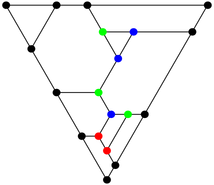

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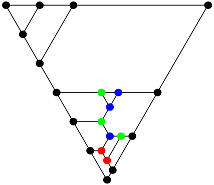

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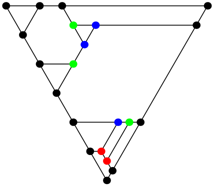

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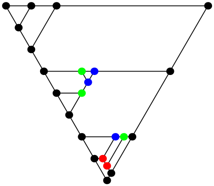

$[t \geq 2, u \geq 2]$

$$L(3, b, N): \quad b = 3 \cdot 5t + 7, \quad N = b(3u + 0) + 5.$$

Theorem on periodicity

MV-complex stabilization theorem (A.Ustinov, O.K.'14). Let

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$$\begin{aligned} b(t) &= \alpha a t + \beta; \\ N(t,u) &= b(t)(au+\gamma) + \alpha = (\alpha a t + \beta)(au+\gamma) + \alpha, \end{aligned}$$

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NOTICE:
$\gcd(a, N) = 1$ and $\gcd(\alpha, \beta) = 1$

$$\Longleftrightarrow$$

$\mathrm{Vrm}(|\Gamma(a, b, N)|)$ is a finite axial set in general position.

Theorem on periodicity

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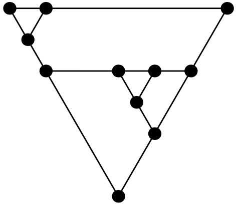
  where t and u are positive integer parameters.
- Suppose $\gcd(a, N) = 1$.

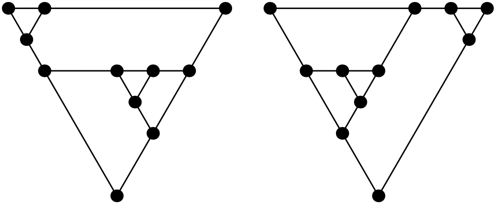
Then the following holds (for $L(a, b, N)$):

- ***t*-stabilization.**
- ***u*-stabilization.**
- **(t, u)-stabilization.**



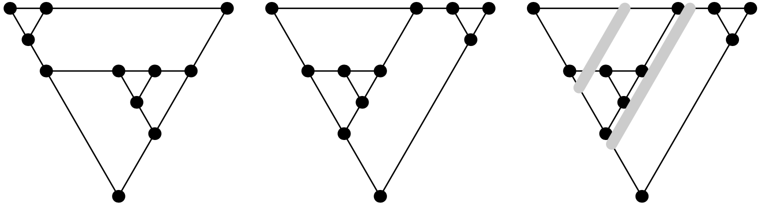
1). Consider a canonical diagram for some S.

Alphabets for diagrams (O.K., & A. Ustinov)



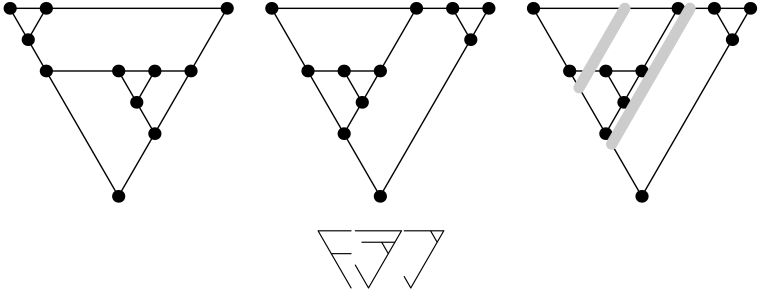
2). Rotate it by $\frac{\pi}{3}$ clockwise.

Alphabets for diagrams (O.K., & A. Ustinov)



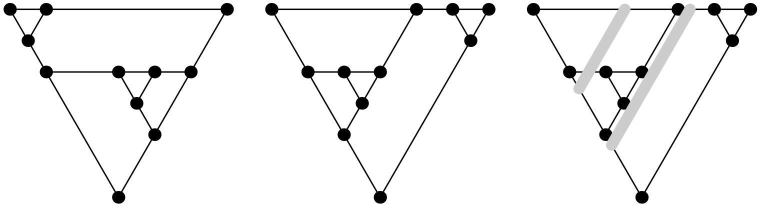
3). Cut it in several parts by parallel cuts.

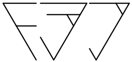
Alphabets for diagrams (O.K., & A. Ustinov)



4). Redraw it in the symbolic form.

Alphabets for diagrams (O.K., & A. Ustinov)

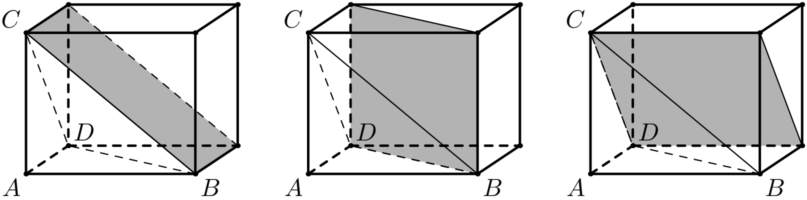




This is the word

Theorem

(Equivalent to G. K. White, 1964) *If ABCD is empty then the lattice points of the corresponding parallelepiped (except for the vertices) are on one of the planes:*



Special case I: White's lattices

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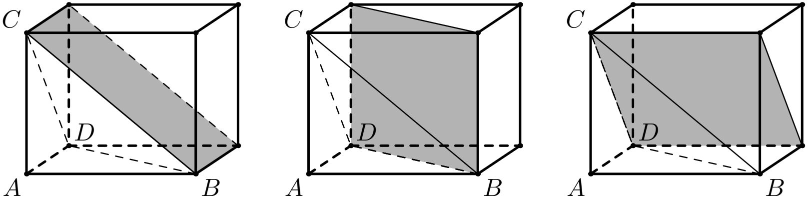

The vectors AB, AC, and AD in this case generate the lattice $L(1, b, N)$.

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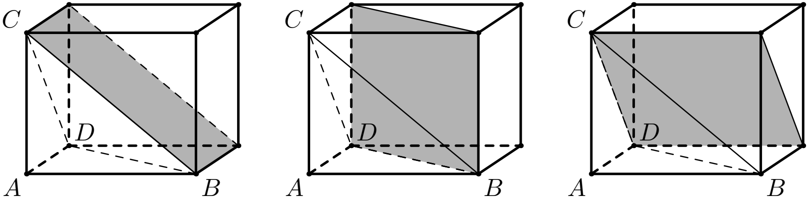

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So $L(1, b, N)$ are **White's lattices**.

Special case I: White's lattices

Theorem

Let $\gcd(b, N) = 1$ *and* $b \leq \frac{N}{2}$.
Then the canonical diagram of $|L(1, b, N)|$ *is*

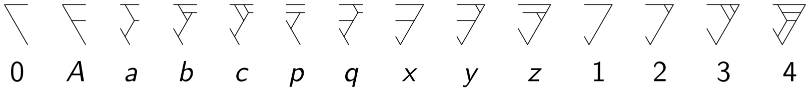
$$\nabla\!\!\!/\,\nabla\!\!\!/\,\nabla\!\!\!/\ldots\nabla\!\!\!/,$$

where
$\#\Big(\text{“}\nabla\!\!\!/\text{”}\Big) = \#\Big($ *elements in the shortest regular c.f. of* $\frac{N}{b}\Big)$.

Special case II: L(2,b,N)

Conjecture

Let $\gcd(b, N) = 1$ and $b \leq \frac{N}{2}$.
Then the canonical diagram of $L(2, b, N)$ is written in the alphabet



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Conjecture

Let $\gcd(b, N) = 1$ and $b \leq \frac{N}{2}$.
Then the canonical diagram of $L(2, b, N)$ is written in the alphabet

0 A a b c p q x y z 1 2 3 4

Remark. Letters 0 and A always take the first position. The rest is separated into blocks.

A *simple block*: 0, 1, 2, 3, or 4.

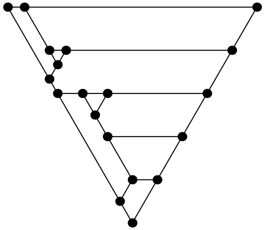
A *nonsimple block*

- starts with A, a, b, or c
- have none or several letters p and q in the middle
- ends with x, y, or z.

We separate such blocks with spaces.

Special case II: L(2,b,N)

Example: $\Gamma(2, 26, 121)$:



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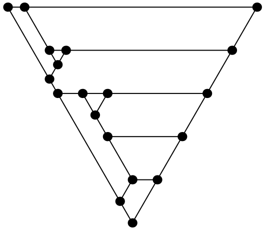

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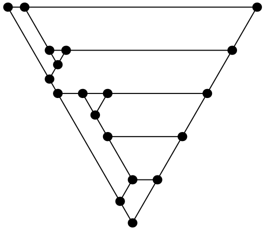

Symbolically: 0 *apz* *bx*.

Special case II: L(2,b,N)

$\alpha = 1$	$\beta = 1,\ \gamma = 0$	$u \geq 2,\ v \geq 2$	0 3 2
$\alpha = 2$	$\beta = 1, 3;\ \gamma = 1$	$u \geq 1,\ v \geq 1$	Az 2
$\alpha = 3$	$\beta = 1;\ \gamma = 0$	$u \geq 2,\ v \geq 2$	0 2 3 2
	$\beta = 2;\ \gamma = 0, 1$	$u \geq 1,\ v \geq 1$	Ax bx
	$\beta = 4;\ \gamma = 0, 1$	$u \geq 1,\ v \geq 1$	0 2 bx
	$\beta = 5;\ \gamma = 0$	$u \geq 2,\ v \geq 1$	Ax 3 2
$\alpha = 4$	$\beta = 1, 5;\ \gamma = 1$	$u \geq 1,\ v \geq 1$	0 bz 2
	$\beta = 3, 7;\ \gamma = 1$	$u \geq 1,\ v \geq 1$	0 apz 2
$\alpha = 5$	$\beta = 1;\ \gamma = 0$	$u \geq 2,\ v \geq 2$	0 3 3 2
	$\beta = 2;\ \gamma = 0, 1$	$u \geq 1,\ v \geq 1$	Az bx
	$\beta = 3;\ \gamma = 0$	$u \geq 2,\ v \geq 2$	Apy 3 2
	$\beta = 4;\ \gamma = 0, 1$	$u \geq 1,\ v \geq 1$	0 4 bx
	$\beta = 6;\ \gamma = 0, 1$	$u \geq 1,\ v \geq 1$	0 3 bx
	$\beta = 7;\ \gamma = 0$	$u \geq 2,\ v \geq 1$	Az 3 2
	$\beta = 8;\ \gamma = 0, 1$	$u \geq 1,\ v \geq 1$	Apy bx
	$\beta = 9;\ \gamma = 0$	$u \geq 2,\ v \geq 1$	0 4 3 2

Some open questions remained

Problem

(General) *Which tessellations are realizable for 1-rank* $L(a, b, N)$ *lattices?*

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Problem

What are the explicit bounds for the asymptotic theorem (is it always 2)?