Extremal problems for convex lattice polytopes

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A sample problem

Jarník proved in 1926 that

if $\gamma \subset \mathbf{R}^2$ is a (closed) strictly convex curve of length ℓ, then

$$|\gamma \cap \mathbf{Z}^2| \leq \frac{3}{\sqrt[3]{2\pi}} \ell^{2/3} + O(\ell^{1/3}).$$

Here both the exponent $\frac{2}{3}$ and the constant $\frac{3}{\sqrt[3]{2\pi}}$ are best possible. Equivalently,

Theorem (Jarník 1926)

$$\lim_{\ell \to \infty} \max\{\ell^{-2/3}|\gamma \cap \mathbf{Z}^2| : \gamma \textit{ is a convex..}\} = \frac{3}{\sqrt[3]{2\pi}}$$

convex lattice polygons appear instantly:

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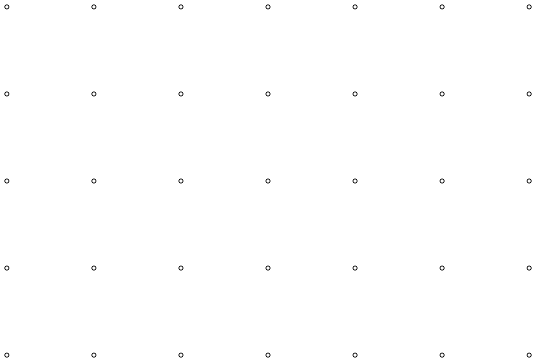
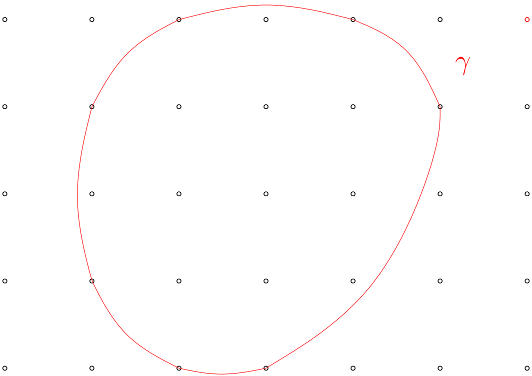
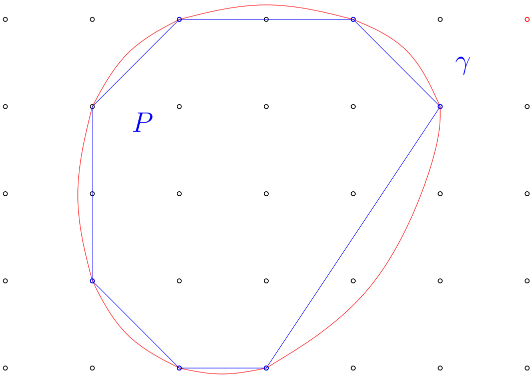


The lattice $\mathbf{Z}^2$ (or $\mathbf{Z}^d$)



The strictly convex curve γ



The convex lattice polygon P whose vertex set is $\gamma \cap \mathbf{Z}^2$

In fact, $P = \operatorname{conv}(\gamma \cap \mathbf{Z}^2)$. Jarník's result says that if P has $n = |\gamma \cap \mathbf{Z}^2|$ vertices, then

$$\ell > \operatorname{per} P \geq \frac{\sqrt{6\pi}}{9} n^{3/2} + O(n^{3/4})$$

with best exponent $3/2$ and best constant $\frac{\sqrt{6\pi}}{9}$.

Theorem

With the min *taken over all convex lattice polygons with n vertices*

$$\lim_{n\to\infty} n^{-3/2} \min \operatorname{per} P = \frac{\sqrt{6\pi}}{9}.$$

This is equivalent to Jarník's theorem. Next comes a quick proof.

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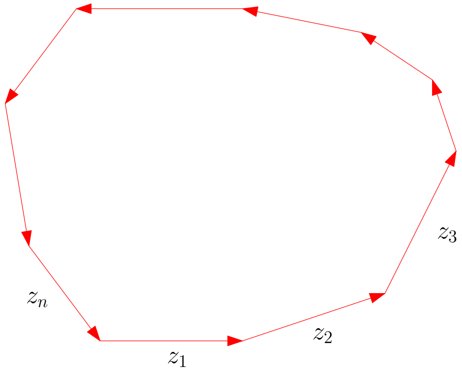
P convex lattice n-gon with minimal perimeter, edges $z_1, z_2, \ldots, z_n \in \mathbf{Z}^2$.



- each $z_i \in \mathbf{Z}^2$ is a primitive vector (primitive: the gcd of the coordinates is 1)
- No z_i, z_j are parallel and same direction
- $\sum_1^n z_i = 0$

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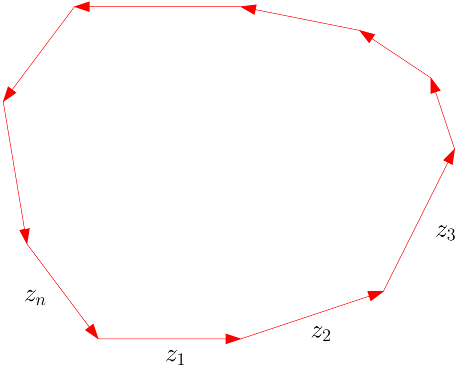

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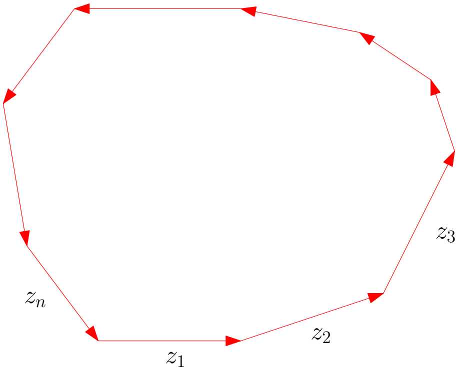

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FACT: $z_1, \ldots, z_n \in \mathbf{P}$ are distinct primitive vectors

Notation: $\mathbf{P} = \mathbf{P}^d \subset \mathbf{Z}^d$ set of primitive vectors

their density in $\mathbf{Z}^2$ is $6/\pi^2$

Let $U = \{u_1, \ldots, u_n\}$ be the set of the n shortest primitive vectors.

$$\text{per}\, P = \sum_1^n ||z_i|| \geq \sum_1^n ||u_i||$$

$\sum_1^n ||u_i||$ can be determined. With $r = \max ||u_i||$

$$U \approx rB^2 \cap \mathbf{P} \text{ and } \frac{6}{\pi^2} r^2 \pi \approx n \text{ so } r \approx \sqrt{\frac{\pi n}{6}}.$$

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Similarly,

$$\begin{aligned}
\text{per } P &\geq \sum_{1}^{n} ||u_i|| \approx \sum_{u \in rB^2 \cap \mathbf{P}} ||u|| \\
&\approx \frac{6}{\pi^2} \int_{rB^2} ||x|| dx \\
&\approx \frac{\sqrt{6\pi}}{9} n^{3/2}.
\end{aligned}$$

Lower bound (for even n): choose the n shortest primitive vectors in pairs $-u, u$, so their sum is zero.

Order the vectors by increasing slope.

This gives the order of edges of a convex lattice polygon P and $\text{per}\, P \approx \frac{\sqrt{6\pi}}{9} n^{3/2}$.

For odd n...

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REMARK. There is a limit shape of the minimizers (after scaling)

MORAL: edge set of P is more important than P
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And for non-symmetric norms?

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And for non-symmetric norms?

$D \in \mathcal{K}^2$ with $0 \in D$ is the unit ball of a (non-symmetric) norm.
Let $\mathcal{P}$ denote the family of all convex lattice polygons.
Each $P \in \mathcal{P}$ has a D-perimeter $\mathrm{per}_D P$. Define

$$L_n(D) = \min\{\mathrm{per}_D P : P \in \mathcal{P},\ P \text{ has } n \text{ vertices}\}$$

Theorem (B.-Enriquez '10)

There is a convex set $P \subset \mathbf{R}^2$ such that the following holds. Let $P_n \in \mathcal{P}$ with n vertices be an arbitrary sequence of minimizers, of $L_n(D)$, translated so that their center of gravity is at the origin. Then the sequence $n^{-3/2}P_n$ tends to P.

P is unique

Proof: convex geometry, number theory, plus calculus of variation

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$\mathcal{K} = \mathcal{K}^d$ the set of convex bodies in $\mathbf{R}^d$ (convex compact sets with non-empty interior)

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THE PROBLEMS

1. Minimal volume. Determine or estimate

$$V_d(n) = \min\{\mathrm{Vol}\, P : P \in \mathcal{P}^d \text{ and } f_0(P) = n\}$$

2. Minimal surface area. Determine or estimate

$$S_d(n) = \min\{S(P) : P \in \mathcal{P}^d \text{ and } f_0(P) = n\}$$

just solved it for $d = 2$.

3. Minimal lattice width. Determine or estimate

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Definition

$K \in \mathcal{K}^d$, $z \in \mathbf{Z}^d$ and $z \neq 0$, then

$$w(K,z) = \max\{z \cdot (x-y) : x,y \in K\}.$$

The lattice width of K is

$$w(K) = \min\{w(K,z) : z \in \mathbf{Z}^d, z \neq 0\}.$$

How many parallel lattice hyperplanes meet K?

FACT. For $P \in \mathcal{P}^d$, $w(P) + 1 =$ minimal number of parallel lattice lines meeting P.

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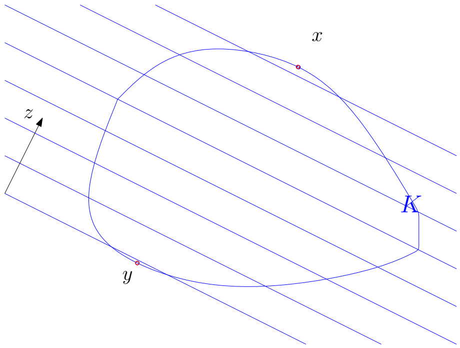
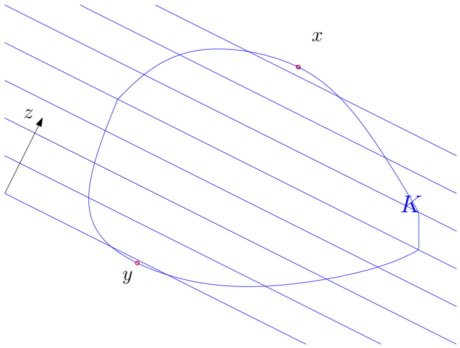


$w(K)$ is invariant under lattice preserving affine transformations



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$P, Q \in \mathcal{P}^d$ are equivalent if P can be carried to Q by a lattice preserving affine transformation.

Equivalent polytopes have the same volume.

Arnold's question. (1980) How many equivalence classes are there in $\mathbf{R}^d$, of volume $\leq V$?

not an extremal question yet ..

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5. Maximal polytopes. Assume $K \in \mathcal{K}^d$ is “large”. Determine

$$\max\{f_0(P) : P \in \mathcal{P}^d,\ P \subset K\}.$$

equivalently, determine or estimate the maximal number of points in $K \cap \mathbf{Z}^d$ that are in convex position,

i.e., none of them is in the convex hull of the others

answers: order of magnitude, asymptotic, precise..

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1. Minimal volume $V_d(n)$

Theorem (Andrews '63)

If $P \in \mathcal{P}^d$ *and* $\mathrm{Vol}\, P > 0$*, then*

$$f_0(P)^{\frac{d+1}{d-1}} \leq c_d \mathrm{Vol}\, P.$$

or with better notation:

$$f_0(P)^{\frac{d+1}{d-1}} \ll \mathrm{Vol}\, P.$$

Corollary

$$n^{\frac{d+1}{d-1}} \ll V_d(n).$$

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If $P \in \mathcal{P}^d$ and $\mathrm{Vol}\, P > 0$, then

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Definition

A tower of $P \in \mathcal{P}^d$ is $F_0 \subset F_1 \subset .. \subset F_{d-1}$ where F_i is an i-dim face of P. $T(P) =$ number of towers of P.

Theorem

If $P \in \mathcal{P}^d$ and $\mathrm{Vol}\, P > 0$, then

$$T(P)^{\frac{d+1}{d-1}} \ll \mathrm{Vol}\, P.$$

implies the same bound for $f_i(P)$.

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FACT. $n^{(d+1)/(d-1)}$ is best possible estimate

Example 1. (Arnold 80') G is the graph of the parabola $y = x^2, \ |x| \leq t$, and

$P = P_t = \text{conv}\,(G \cap \mathbf{Z}^2)$.
Then $f_0(P) = 2t + 1$ and $\text{Area}\, P \approx \frac{2}{3}t^3$.

in d-dim, $G = G_t$ is given by $x_d = x_1^2 + \cdots + x_{d-1}^2 \leq t$,

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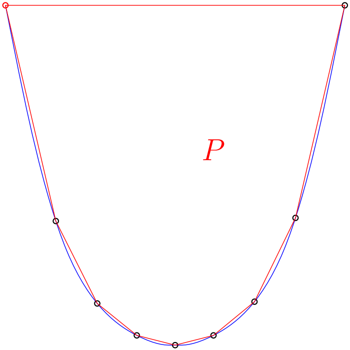
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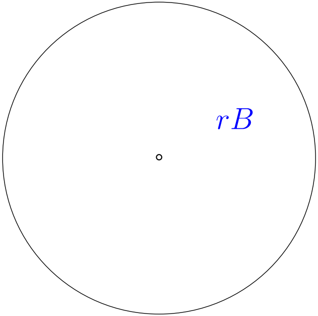
rB

Example 2. (B.-Balog '92 (d=2), B.-Larman '98, all d)
$P_r = \text{conv}\,(rB^d \cap \mathbf{Z}^d)$

the integer convex hull of rB^d

$\text{Vol}\, P_r \approx r^d$ implies via Andrews's theorem

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$\mathrm{Vol}\,(rB^d \setminus P_r) \ll r^{d(d-1)/(d+1)}$.

The proof uses the Flatness Theorem,

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If $P \subset B^d$ is a polytope with $f_0(P) \leq n$, then

$$n^{-2/(d-1)} \ll \mathrm{Vol}\,(B^d \setminus P).$$

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REMARK. Works for all $K \in \mathcal{K}^d$ (instead of B^d) with smooth enough boundary.

OPEN PROBLEM. Does $\lim n^{-\frac{d+1}{d-1}} V_d(n)$ exist????

will come back when $d = 2$.

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Isoperimetric inequality: For all $K \in \mathcal{K}^d$

$$\frac{S(K)^d}{(\mathrm{Vol}\,K)^{d-1}} \geq \frac{S(B^d)^d}{(\mathrm{Vol}\,B^d)^{d-1}}$$

implies $S(P) \gg (\mathrm{Vol}\,P)^{(d-1)/d} \gg f_0(P)^{(d+1)/d}$

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$n^{(d+1)/d} \ll S_d(n)$

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3. Minimal lattice width $w_d(n)$

first $d = 2$. $w(P) + 1$ is the minimal number of consecutive lattice lines intersecting P.

each such line contains at most two vertices of $P \Longrightarrow f_0(P) \leq 2(w(P) + 1)$

FACT. $w_2(n) = \lceil \frac{n}{2} \rceil - 1$

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4. Arnold's question

$P, Q \in \mathcal{P}^d$ are equivalent if a lattice preserving affine transformation maps P to Q.

FACT. $P \sim Q \implies f_0(P) = f_0(Q)$, $w(P) = w(Q)$, $\mathrm{Vol}\, P = \mathrm{Vol}\, Q$.

$N_d(V) =$ number of equivalent classes of $P \in \mathcal{P}^d$ with $\mathrm{Vol}\, P \le V$

$N_2(A)$ for $d = 2$

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Theorem (Arnold 1980)

$A^{1/3} \ll \log N_2(A) \ll A^{1/3} \log A.$

lower bound: let P be the polytope from Example 1 or 2.

Its vertex set $W \Longrightarrow |W| \approx A^{1/3}$.

For each subset $U \subset W$, $\mathrm{conv}\, U \in \mathcal{P}^2$.

there are $2^{|W|} \approx 2^{A^{1/3}}$ such subpolygons. Most of them distinct.

for the upper bound we need:

Lemma (Square lemma)

For every $P \in \mathbf{P}^2$ there is $Q \sim P$ which is contained in the square $[0, 36A]^2$.

So each equivalence class is represented in this square.

Proof follows from Andrews theorem + Square lemma

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Proof follows from Andrews theorem + Square lemma

Theorem (Arnold 1980)

$A^{1/3} \ll \log N_2(A) \ll A^{1/3} \log A.$

lower bound: let P be the polytope from Example 1 or 2.

Its vertex set $W \Longrightarrow |W| \approx A^{1/3}$.

For each subset $U \subset W$, $\mathrm{conv}\, U \in \mathcal{P}^2$.

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Define the Box with parameter $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbf{Z}_+^d$ by

$$\mathrm{Box}(\gamma) = \{x \in \mathbf{R}^d : 0 \le x_i \le \gamma_i,\ i = 1, \dots, d\}.$$

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Lemma (Key lemma)

The number of lattice polytopes contained in $\mathrm{Box}(\gamma)$ *is*

$$\leq \exp\left(c_d(\mathrm{Vol\ Box}(\gamma))^{\frac{d-1}{d+1}}\right)$$

ingredients:

- Minkowski's theorem: outer normals to the facets, of lengths equal to the surface area, determine P uniquely (up to translation),
- for a lattice polytope this outer normal vector is in $\frac{1}{(d-1)!}\mathbf{Z}^d$,
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FACT. The key lemma implies Andrews's theorem

Proof. $P \in \mathcal{P}^d$, $V = \mathrm{Vol}\, P$, we assume $P \subset \mathrm{Box}(\gamma)$ with $\mathrm{Vol}\,\mathrm{Box}(\gamma) \ll V$. Let $f_0(P) = n$.

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there are at least $2^n - 1$ distinct convex lattice polytopes in $\mathrm{Box}(\gamma)$, the subpolytopes of P

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5. Maximal polytopes

better setting: $\mathbf{Z}_t = \frac{1}{t}\mathbf{Z}^d$ where t is large

$K \in \mathcal{K}^d$ is fixed with $\mathrm{Vol}\, K = 1$, say

$\mathcal{P}(K, t)$ family of all convex $\mathbf{Z}_t$-lattice polytopes contained in K

$M(K, t) = \max\{f_0(P) : P \in \mathcal{P}(K, t)\}$,

same as maximal number of points in $\mathbf{Z}_t \cap K$ in convex position

Theorem

Suppose $K \in \mathcal{K}^d$ and $\mathrm{Vol}\, K = 1$. Then

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Yes, when $d = 2$:

Theorem (B.-Prodromou '06)

When $K \subset \mathcal{K}^2$,

$$\lim t^{-2/3} M(K,t) = \frac{3}{(2\pi)^{2/3}} A^*(K)$$

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Minimal area $A(n)$

$A(n) = \min\{\text{Area } P : P \in \mathcal{P}^d,\ f_0(P) = n\} = V_2(n)$ previous notation

Theorem (B.-Tokushige, '04)

$\lim n^{-3}A(n)$ *exists and equals* $0.0185067\ldots$ *most likely.*

FACT. $C \subset \mathbf{R}^2$ is an 0-symmetric convex body, and $|C \cap \mathbf{P}| = n$

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there is a unique (up to translation) convex lattice n-gon, $P(C)$, with edge set $C \cap \mathbf{P}$.

Proof: order the vectors in $C \cap \mathbf{P}$ by increasing slope...

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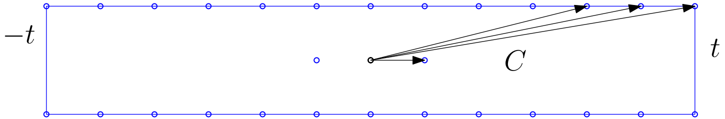
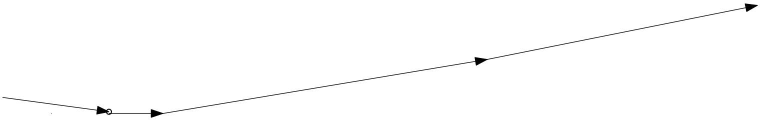
$C = [-t, t] \times [-1, 1]$ with t chosen so that $|C \cap \mathbf{P}| = n \Longrightarrow$

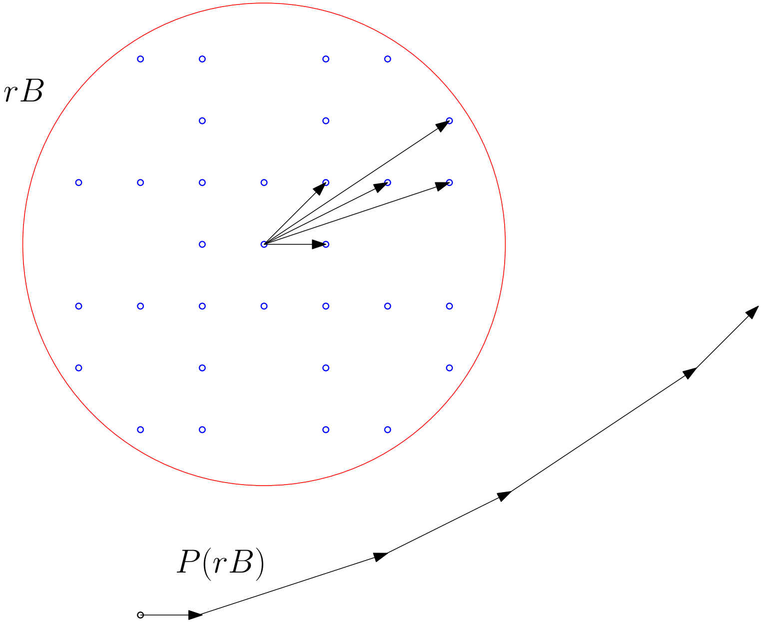
$$\begin{aligned} \text{Area } P(C) &= \left(\frac{1}{48} + o(1)\right) n^3 \\ &= (0.0204085\cdots + o(1))n^3 \end{aligned}$$


$-t$
t
C



$C = rB^2$ with r chosen so that $|C \cap \mathbf{P}| = n \Longrightarrow$

$$\begin{aligned} \text{Area } P(rB^2) &= \left(\frac{1}{54} + o(1)\right) n^3 \\ &= (0.0185185185\cdots + o(1))n^3 \end{aligned}$$



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Lemma (Reduction Lemma)

For even n, $A(n) = M(n)$.

Let C_n be a minimizer for $M(n)$, and w_n be its lattice width.

Theorem

There is a positive constant D such that

$$M(n) \geq \left(\frac{1}{54} - D\frac{\log w_n}{w_n}\right) n^3.$$

$D \approx 5000$

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Determining $M(n)$ with side condition $w(C) = b$ leads to an extremal problem $E(b)$ with b variables

which can be solved by a computer for fixed (not too large) b

$b = 8, 9, 14, 15, \ldots$ gives $M(n)/n^3 < 1/54$

Enough to solve $E(b)$ for $b \leq 10^{10}$

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Arnold's problem in the plane:
$N(A) =$ number of equivalence classes in $\mathcal{P}^2$ of area $\leq A$.

OPEN PROBLEM. Does $\lim A^{-1/3} \log N(A)$ exist????