Extremal problems for convex lattice polytopes

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A sample problem

Jarník proved in 1926 that

if $\gamma \subset \mathbb{R}^2$ is a (closed) strictly convex curve of length $\ell$, then

$$|\gamma \cap \mathbb{Z}^2| \leq \frac{3}{\sqrt[3]{2\pi}} \ell^{2/3} + O(\ell^{1/3}).$$

Here both the exponent $\frac{2}{3}$ and the constant $\frac{3}{\sqrt[3]{2\pi}}$ are best possible. Equivalently,

Theorem (Jarník 1926)

$$\lim_{\ell \to \infty} \max_{\ell \to \infty} \{\ell^{-2/3} |\gamma \cap \mathbb{Z}^2| : \gamma \text{ is a convex polygon} \} = \frac{3}{\sqrt[3]{2\pi}}$$

convex lattice polygons appear instantly:
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convex lattice polygons appear instantly:
The lattice $\mathbb{Z}^2$ (or $\mathbb{Z}^d$)
The strictly convex curve $\gamma$
The convex lattice polygon $P$ whose vertex set is $\gamma \cap \mathbb{Z}^2$
In fact, $P = \text{conv} \left( \gamma \cap \mathbb{Z}^2 \right)$. Jarník’s result says that if $P$ has $n = |\gamma \cap \mathbb{Z}^2|$ vertices, then

$$\ell > \text{per } P \geq \frac{\sqrt{6\pi}}{9} n^{3/2} + O(n^{3/4})$$

with best exponent $3/2$ and best constant $\frac{\sqrt{6\pi}}{9}$.

**Theorem**

*With the min taken over all convex lattice polygons with $n$ vertices

$$\lim_{n \to \infty} n^{-3/2} \min \text{ per } P = \frac{\sqrt{6\pi}}{9}.$$

This is equivalent to Jarník’s theorem. Next comes a quick proof.*
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This is equivalent to Jarník’s theorem. Next comes a quick proof.
$P$ convex lattice $n$-gon with minimal perimeter, edges $z_1, z_2, \ldots, z_n \in \mathbb{Z}^2$.

- Each $z_i \in \mathbb{Z}^2$ is a primitive vector (primitive: the gcd of the coordinates is 1)
- No $z_i, z_j$ are parallel and same direction
- $\sum_{1}^{n} z_i = 0$
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**FACT:** $z_1, \ldots, z_n \in \mathbf{P}$ are distinct primitive vectors

Notation: $\mathbf{P} = \mathbf{P}^d \subset \mathbb{Z}^d$ set of primitive vectors

their density in $\mathbb{Z}^2$ is $6/\pi^2$

Let $U = \{u_1, \ldots, u_n\}$ be the set of the $n$ shortest primitive vectors.

$$\text{per } \mathbf{P} = \sum_{1}^{n} \|z_i\| \geq \sum_{1}^{n} \|u_i\|$$

$\sum_{1}^{n} \|u_i\|$ can be determined. With $r = \max \|u_i\|$

$U \approx rB^2 \cap \mathbf{P}$ and $\frac{6}{\pi^2} r^2 \pi \approx n$ so $r \approx \sqrt{\frac{\pi n}{6}}.$
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$$U \approx rB^2 \cap \mathbf{P} \text{ and } \frac{6}{\pi^2} r^2 \pi \approx n \text{ so } r \approx \sqrt{\frac{\pi n}{6}}.$$
Similarly,

$$\text{per } P \geq \sum_{1}^{n} \|u_i\| \approx \sum_{u \in rB^2 \cap P} \|u\|$$

$$\approx \frac{6}{\pi^2} \int_{rB^2} \|x\| \, dx$$

$$\approx \frac{\sqrt{6\pi}}{9} n^{3/2}.$$
Lower bound (for even $n$): choose the $n$ shortest primitive vectors in pairs $-u, u$, so their sum is zero.

Order the vectors by increasing slope.

This gives the order of edges of a convex lattice polygon $P$ and $\text{per } P \approx \frac{\sqrt{6\pi}}{9} n^{3/2}$.

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REMARK. Same method works for every symmetric norm in $\mathbb{R}^2$.

REMARK. There is a limit shape of the minimizers (after scaling)

MORAL: edge set of $P$ is more important than $P$ (and contains the same information)

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And for non-symmetric norms?
$D \in \mathcal{K}^2$ with $0 \in D$ is the unit ball of a (non-symmetric) norm. Let $\mathcal{P}$ denote the family of all convex lattice polygons. Each $P \in \mathcal{P}$ has a $D$-perimeter $\text{per}_D P$. Define

$$L_n(D) = \min\{\text{per}_D P : P \in \mathcal{P}, \ P \text{ has } n \text{ vertices}\}$$

Theorem (B.-Enriquez ’10)

There is a convex set $P \subset \mathbb{R}^2$ such that the following holds. Let $P_n \in \mathcal{P}$ with $n$ vertices be an arbitrary sequence of minimizers, of $L_n(D)$, translated so that their center of gravity is at the origin. Then the sequence $n^{-3/2}P_n$ tends to $P$.

$P$ is unique

Proof: convex geometry, number theory, plus calculus of variation
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Notations:

\( P = P^d \) the set of primitive vectors in \( \mathbb{Z}^d \)

\( K = K^d \) the set of convex bodies in \( \mathbb{R}^d \) (convex compact sets with non-empty interior)

\( P = P^d \) set of convex lattice polytopes,

for \( P \in P \), \( f_0(P) = \) number of vertices of \( P \), \( f_s(P) = \) number of \( s \)-dim faces of \( P \)
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THE PROBLEMS

1. Minimal volume. Determine or estimate

\[ V_d(n) = \min \{ \text{Vol } P : P \in \mathcal{P}^d \text{ and } f_0(P) = n \} \]

2. Minimal surface area. Determine or estimate

\[ S_d(n) = \min \{ S(P) : P \in \mathcal{P}^d \text{ and } f_0(P) = n \} \]

just solved it for \( d = 2 \).

3. Minimal lattice width. Determine or estimate

\[ w_d(n) = \min \{ w(P) : P \in \mathcal{P}^d \text{ and } f_0(P) = n \} \]

where \( w(P) \) is the lattice width of \( P \in \mathcal{P}^d \)
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where \( w(P) \) is the \textbf{lattice width of } P \in \mathcal{P}^d
Definition

If \( K \in \mathcal{K}^d \), \( z \in \mathbb{Z}^d \) and \( z \neq 0 \), then

\[
w(K, z) = \max\{z \cdot (x - y) : x, y \in K\}.
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The lattice width of \( K \) is

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w(K) = \min\{w(K, z) : z \in \mathbb{Z}^d, z \neq 0\}.
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How many parallel lattice hyperplanes meet \( K \)?

**FACT.** For \( P \in \mathcal{P}^d \), \( w(P) + 1 = \text{minimal number of parallel lattice lines meeting } P \).
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$w(K)$ is invariant under lattice preserving affine transformations.
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4. Arnold’s question. How many convex lattice polytopes are there?

$P, Q \in \mathcal{P}^d$ are equivalent if $P$ can be carried to $Q$ by a lattice preserving affine transformation.

Equivalent polytopes have the same volume.

Arnold’s question. (1980) How many equivalence classes are there in $\mathbb{R}^d$, of volume $\leq V$?

not an extremal question yet ..
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$$\max \{ f_0(P) : P \in \mathcal{P}^d, P \subset K \}.$$ 

equivalently, determine or estimate the maximal number of points in $K \cap \mathbb{Z}^d$ that are in convex position,
i.e., none of them is in the convex hull of the others

answers: order of magnitude, asymptotic, precise..
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**Theorem (Andrews ’63)**

If $P \in \mathcal{P}^d$ and $\text{Vol } P > 0$, then

$$f_0(P) \frac{d+1}{d-1} \leq c_d \text{Vol } P.$$ 

or with better notation:

$$f_0(P) \frac{d+1}{d-1} \ll \text{Vol } P.$$ 

**Corollary**

$$n \frac{d+1}{d-1} \ll V_d(n).$$ 

Several proofs, none easy: Andrews ’63, Arnold ’80 ($d = 2$), Konyagin, Sevastyanov ’84 , ($d \geq 2$), W. Schmidt ’86, B.-Vershik ’92, B.-Larman ’98, Reisner-Schütt-Werner ’01, and more
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Definition

A tower of $P \in \mathcal{P}^d$ is $F_0 \subset F_1 \subset \ldots \subset F_{d-1}$ where $F_i$ is an $i$-dim face of $P$. $T(P)$ = number of towers of $P$.

Theorem

If $P \in \mathcal{P}^d$ and $\operatorname{Vol} P > 0$, then

$$T(P)^{\frac{d+1}{d-1}} \ll \operatorname{Vol} P.$$ 

implies the same bound for $f_i(P)$.

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$$T(P) \ll f_0(P) + f_1(P) + \ldots + f_{d-1}(P)$$
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**FACT.** \( n^{(d+1)/(d-1)} \) is best possible estimate

Example 1. (Arnold 80’) \( G \) is the graph of the parabola \( y = x^2, \ |x| \leq t \), and

\[ P = P_t = \text{conv} \ (G \cap \mathbb{Z}^2). \]

Then \( f_0(P) = 2t + 1 \) and \( \text{Area} \ P \approx \frac{2}{3} t^3 \).

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Example 2. (B.-Balog ’92 (d=2), B.-Larman ’98, all d)  
$P_r = \text{conv} \left( rB^d \cap \mathbb{Z}^d \right)$  
the integer convex hull of $rB^d$  
Vol $P_r \approx r^d$ implies via Andrews’s theorem  
\[ f_0(P_r) \ll (\text{Vol } P_r)^{(d-1)/(d+1)} \approx r^{d(d-1)/(d+1)}. \]  
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The proof uses the Flatness Theorem, combined with a statement from approximation theory:

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If \( P \subset B^d \) is a polytope with \( f_0(P) \leq n \), then

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Isoperimetric inequality: For all $K \in \mathcal{K}^d$

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implies $S(P) \gg (\text{Vol } P)^{(d-1)/d} \gg f_0(P)^{(d+1)/d}$

**Corollary**

$$n^{(d+1)/d} \ll S_d(n)$$

Example 2 shows that this is best possible.

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First $d = 2$. $w(P) + 1$ is the minimal number of consecutive lattice lines intersecting $P$.

Each such line contains at most two vertices of $P \implies f_0(P) \leq 2(w(P) + 1)$

**FACT.** $w_2(n) = \lceil \frac{n}{2} \rceil - 1$

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$P, Q \in \mathcal{P}^d$ are equivalent if a lattice preserving affine transformation maps $P$ to $Q$.

**FACT.** $P \sim Q \implies f_0(P) = f_0(Q), w(P) = w(Q), \text{Vol } P = \text{Vol } Q$.

$N_d(V) =$ number of equivalent classes of $P \in \mathcal{P}^d$ with $\text{Vol } P \leq V$

$N_2(A)$ for $d = 2$

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Theorem (Arnold 1980)

\[ A^{1/3} \ll \log N_2(A) \ll A^{1/3} \log A. \]

lower bound: let \( P \) be the polytope from Example 1 or 2.

Its vertex set \( W \) \( \implies \) \( |W| \approx A^{1/3} \).

For each subset \( U \subset W \), \( \text{conv } U \in \mathcal{P}^2 \).

there are \( 2^{|W|} \approx 2^{A^{1/3}} \) such subpolygons. Most of them distinct.

for the upper bound we need:

Lemma (Square lemma)

For every \( P \in \mathcal{P}^2 \) there is \( Q \sim P \) which is contained in the square \([0, 36A]^2\).

So each equivalence class is represented in this square.

Proof follows from Andrews theorem + Square lemma.
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For each subset \( U \subset W \), \( \text{conv} \ U \in \mathcal{P}^2 \).
there are \( 2^{|W|} \approx 2^{A^{1/3}} \) such subpolygons. Most of them distinct.

for the upper bound we need:

Lemma (Square lemma)

For every \( P \in \mathcal{P}^2 \) there is \( Q \sim P \) which is contained in the square \([0, 36A]^2\).

So each equivalence class is represented in this square.
Proof follows from Andrews theorem + Square lemma.
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Define the Box with parameter \( \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d_+ \) by

\[ \text{Box}(\gamma) = \{ x \in \mathbb{R}^d : 0 \leq x_i \leq \gamma_i, \ i = 1, \ldots, d \}. \]

Lemma (Box lemma)

For every \( P \in \mathcal{P}^d \) there is \( Q \sim P \) and \( \gamma \in \mathbb{Z}_+^d \) such that \( Q \subset \text{Box}(\gamma) \) and \( \text{Vol} \text{Box}(\gamma) = \prod \gamma_i \ll \text{Vol} P. \)

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Lemma (Key lemma)

The number of lattice polytopes contained in $\text{Box}(\gamma)$ is

$$\leq \exp \left( c_d \left( \text{Vol Box}(\gamma) \right)^{\frac{d-1}{d+1}} \right)$$

ingredients:

- Minkowski’s theorem: outer normals to the facets, of lengths equal to the surface area, determine $P$ uniquely (up to translation),
- for a lattice polytope this outer normal vector is in $\frac{1}{(d-1)!} \mathbb{Z}^d$,
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FACT. The key lemma implies Andrews’s theorem

Proof. \( P \in \mathcal{P}^d \), \( V = \text{Vol} \ P \), we assume \( P \subset \text{Box}(\gamma) \) with \( \text{Vol} \ \text{Box}(\gamma) \ll V \). Let \( f_0(P) = n \).

\( \Downarrow \)

there are at least \( 2^n - 1 \) distinct convex lattice polytopes in \( \text{Box}(\gamma) \), the subpolytopes of \( P \)

\( \Downarrow \)

\( 2^n - 1 \leq \exp \left( c_d (\text{Vol} \ \text{Box}(\gamma))^\frac{d-1}{d+1} \right) \)

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better setting: $\mathbb{Z}_t = \frac{1}{t} \mathbb{Z}^d$ where $t$ is large

$K \in \mathcal{K}^d$ is fixed with $\text{Vol } K = 1$, say

$\mathcal{P}(K, t)$ family of all convex $\mathbb{Z}_t$-lattice polytopes contained in $K$

$M(K, t) = \max\{f_0(P) : P \in \mathcal{P}(K, t)\}$,

same as maximal number of points in $\mathbb{Z}_t \cap K$ in convex position

Theorem

Suppose $K \in \mathcal{K}^d$ and $\text{Vol } K = 1$. Then

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OPEN PROBLEM. Does the limit \( \lim_{t \to \infty} t^{-d \frac{d-1}{d+1}} M(K, t) \) exist???

Yes, when \( d = 2 \):

Theorem (B.-Prodromou '06)  
When \( K \subset \mathcal{K}^2 \),  
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\lim_{t \to \infty} t^{-2/3} M(K, t) = \frac{3}{(2\pi)^{2/3}} A^*(K)
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Limit shape
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$$A(n) = \min\{\text{Area } P : P \in \mathcal{P}^d, \ f_0(P) = n\} = V_2(n) \text{ previous notation}$$

**Theorem (B.-Tokushige, '04)**

$$\lim n^{-3} A(n) \text{ exists and equals 0.0185067... most likely.}$$

**FACT.** $C \subset \mathbb{R}^2$ is an 0-symmetric convex body, and $|C \cap P| = n$

there is a unique (up to translation) convex lattice $n$-gon, $P(C)$, with edge set $C \cap P$.

**Proof:** order the vectors in $C \cap P$ by increasing slope...
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\[ C = [-t, t] \times [-1, 1] \text{ with } t \text{ chosen so that } |C \cap P| = n \Rightarrow \]

\[
\text{Area } P(C) = \left( \frac{1}{48} + o(1) \right) n^3 \\
= (0.0204085 \cdots + o(1))n^3
\]
$C = rB^2$ with $r$ chosen so that $|C \cap P| = n$ \implies \\

$$\text{Area } P(rB^2) = \left( \frac{1}{54} + o(1) \right) n^3 = (0.0185185185 \cdots + o(1)) n^3$$
$r B$

$P(rB)$
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\( \min \) is taken over all 0-symmetric \( C \in \mathcal{K}^2 \) with \( |C \cap P| = n \).

**Lemma (Reduction Lemma)**

For even \( n \), \( A(n) = M(n) \).

Let \( C_n \) be a minimizer for \( M(n) \), and \( w_n \) be its lattice width.

**Theorem**

There is a positive constant \( D \) such that

\[ M(n) \geq \left( \frac{1}{54} - D \frac{\log w_n}{w_n} \right) n^3. \]

\( D \approx 5000 \)
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either \( w_n \to \infty \) and then \( \lim M(n)/n^3 = 1/54 \),
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Determining \( M(n) \) with side condition \( w(C) = b \) leads to an extremal problem \( E(b) \) with \( b \) variables
which can be solved by a computer for fixed (not too large) \( b \)
\( b = 8, 9, 14, 15, \ldots \) gives \( M(n)/n^3 < 1/54 \)
Enough to solve \( E(b) \) for \( b \leq 10^{10} \)
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and long axis dictated by $|C_n \cap P| = n$
Arnold’s problem in the plane:
\( N(A) = \text{number of equivalence classes in } \mathcal{P}^2 \text{ of area } \leq A. \)

OPEN PROBLEM. Does \( \lim A^{-1/3} \log N(A) \) exist??