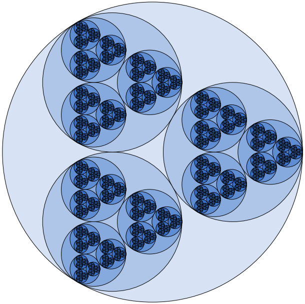
Continued fractions and semigroups of Möbius transformations

Ian Short



Wednesday 8 October 2014

The Open University

Work in preparation

Matthew Jacques
Mairi Walker

A problem about continued fractions

Problem – version I. Determine those finite sets of real numbers X with the property that each continued fraction

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

with coefficients in X converges.

Examples

Problem – version I. Determine those finite sets X with the property that each continued fraction with coefficients in X converges.

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Śleszyński–Pringsheim theorem. If $|b_n| \geqslant 1 + |a_n|$ for each positive integer n, then the continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

converges.

Continued fractions and Möbius transformations

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Continued fractions and Möbius transformations

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$t_1(z) = \frac{1}{b_1 + z}$

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Continued fractions and Möbius transformations

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$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

A problem about sequences of Möbius transformations

Definition. Given a set of (real) Möbius transformations $\mathcal{F}$, a *composition sequence* from $\mathcal{F}$ is a sequence

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n, \quad \text{where} \quad f_i \in \mathcal{F}.$$

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Problem – version II. Determine those finite sets of Möbius transformations $\mathcal{F}$ with the property that every composition sequence from $\mathcal{F}$ converges at 0.

Real Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0$$

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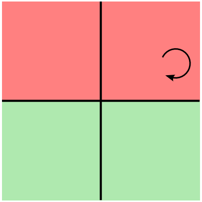

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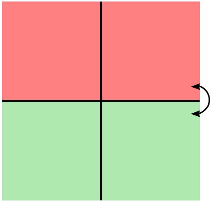

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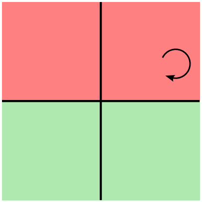

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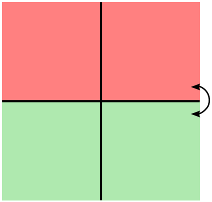

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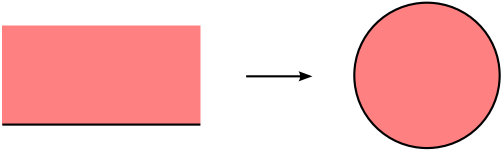

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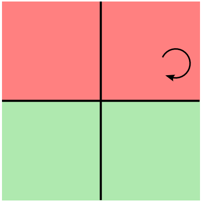

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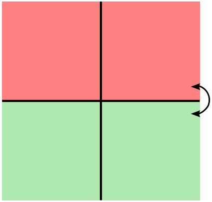

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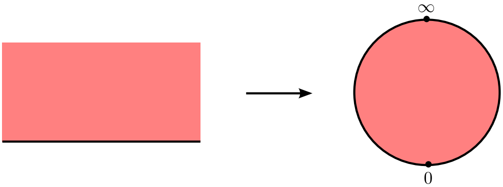

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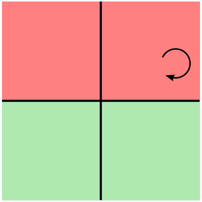

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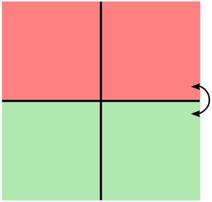

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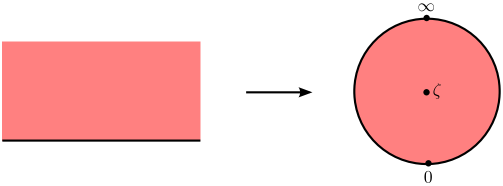
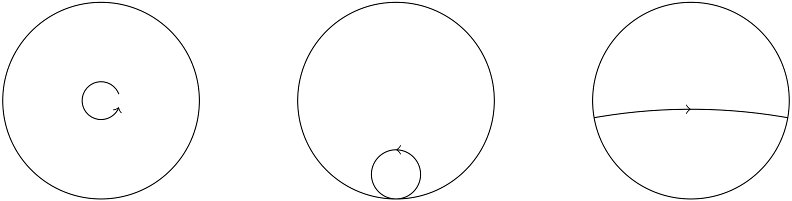


Conjugacy classification of Möbius transformations



elliptic parabolic loxodromic

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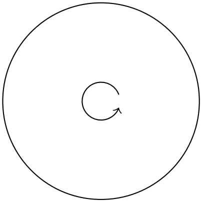
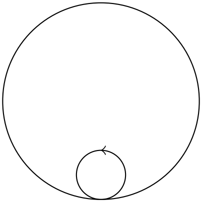


elliptic

one fixed point
inside



parabolic

one fixed point
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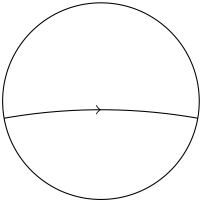

loxodromic

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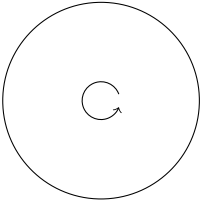
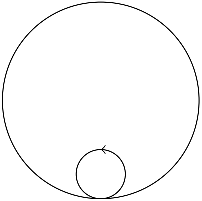
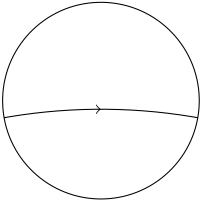
Conjugacy classification of Möbius transformations

		
elliptic	parabolic	loxodromic
one fixed point inside	one fixed point on the boundary	two fixed points on the boundary
hyperbolic rotation	$z \mapsto z + 1$	$z \mapsto \lambda z, \quad \lambda > 1$

Example – the modular group

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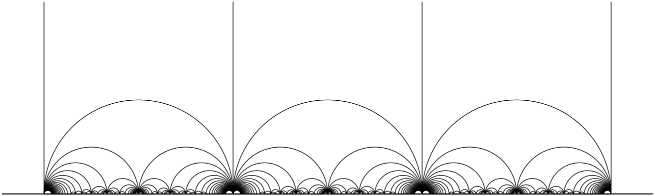
$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} \;:\; a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}$$

Farey graph



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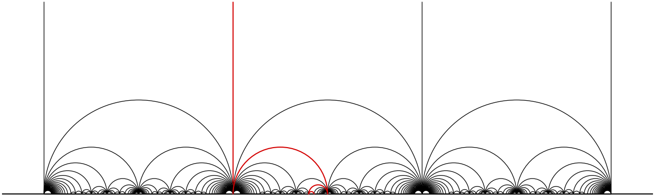

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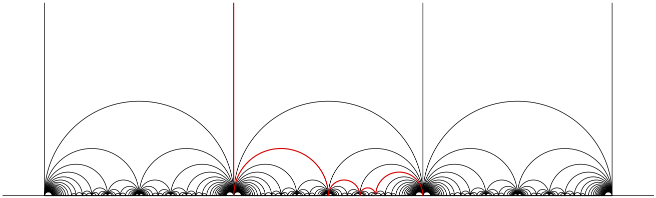

Example – pairs of parabolics

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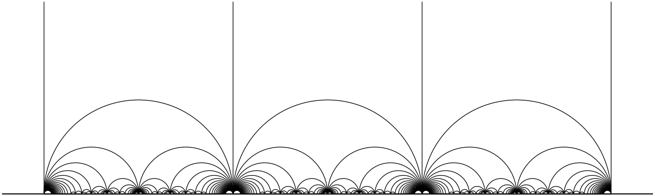
Example – even-integer continued fractions.

$$f(z) = z + 2 \qquad g(z) = \frac{z}{2z + 1}$$

Farey tree



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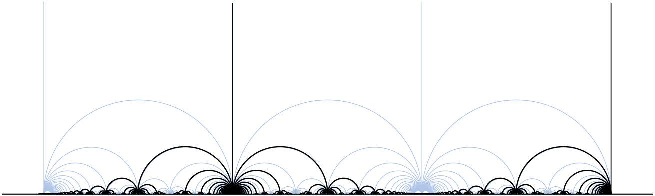

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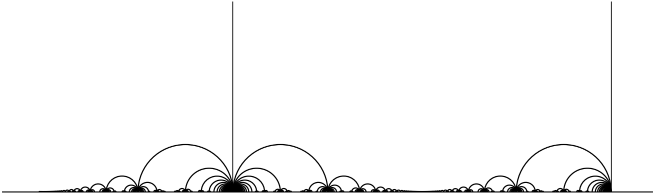
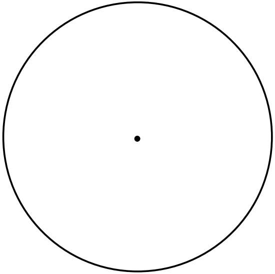


Escaping sequences and general convergence



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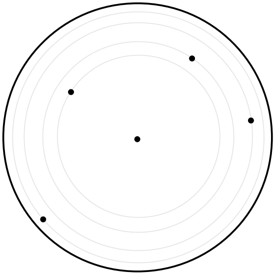

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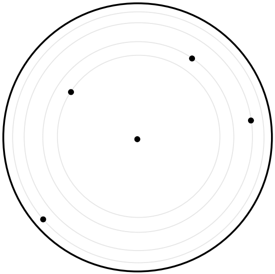
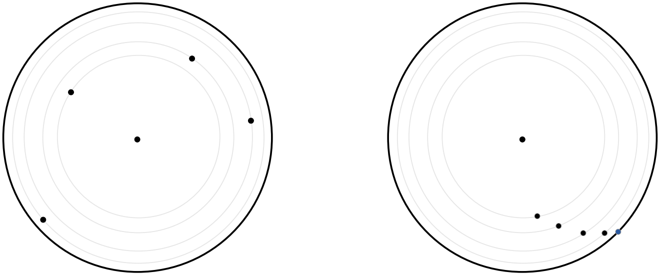


Definition. A sequence F_n of Möbius transformations is an *escaping sequence* if the sequence $F_n(\zeta)$ accumulates only on the unit circle in the Euclidean metric.

Escaping sequences and general convergence



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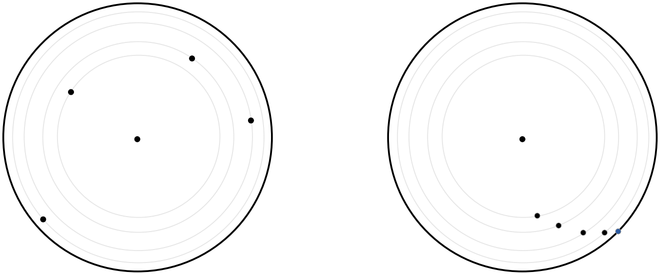
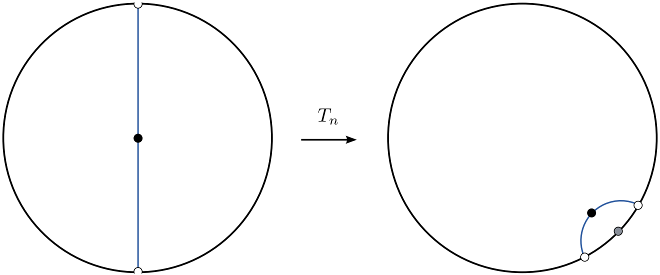


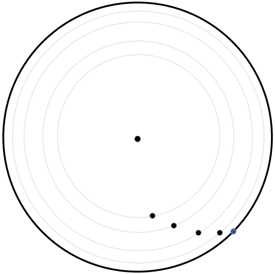
Definition. A sequence F_n of Möbius transformations is an *escaping sequence* if the sequence $F_n(\zeta)$ accumulates only on the unit circle in the Euclidean metric.

An escaping sequence F_n *converges generally* to a point p on the unit circle if $F_n(\zeta) \to p$ as $n \to \infty$ (in the Euclidean metric).

Classical convergence implies general convergence



Backwards limit sets and conical limit sets



F_n

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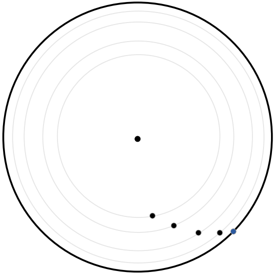

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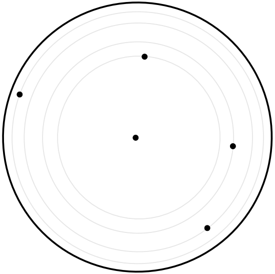

${F_n}^{-1}$

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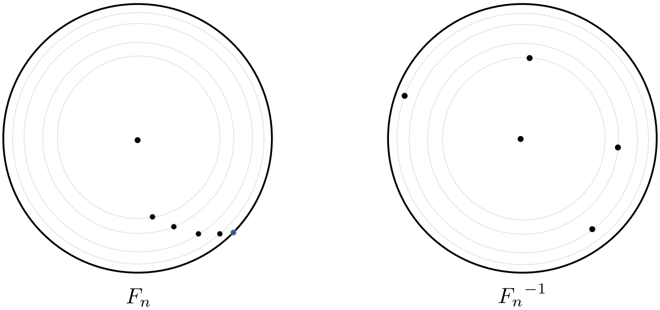

Definition. The *backwards limit set* of an escaping sequence F_n is the set of accumulation points of the sequence $F_n^{-1}(\zeta)$.

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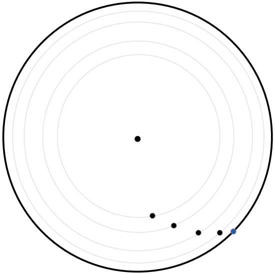

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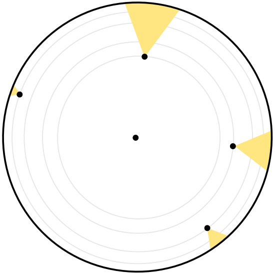

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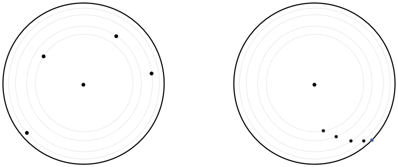

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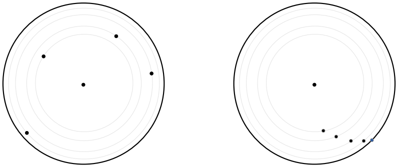

Conjecture. If every composition sequence from $\mathcal{F}$ is an escaping sequence, then every composition sequence converges generally.

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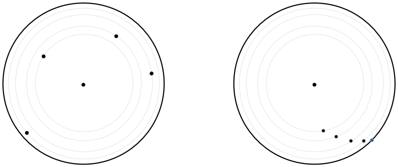

Conjecture. If every composition sequence from $\mathcal{F}$ is an escaping sequence, then every composition sequence converges generally.

Theorem. The conjecture is true if $\mathcal{F}$ has order two.

From Möbius transformations to semigroups

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Lemma. Let $\mathcal{F}$ be a finite set of Möbius transformations, and let S be the semigroup generated by $\mathcal{F}$. The following are equivalent:

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Definition. A semigroup S is *inverse free* if no element of S has an inverse in S.

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Theorem. Let $\mathcal{F}$ be a set of two Möbius transformations, and let S be the semigroup generated by $\mathcal{F}$.

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Theorem. Let $\mathcal{F}$ be a set of two Möbius transformations, and let S be the semigroup generated by $\mathcal{F}$. Then, with one exception, the following are equivalent:

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Discrete semigroups

Lemma. Let $\mathcal{F}$ be a finite set of Möbius transformations, and let S be the semigroup generated by $\mathcal{F}$. The following are equivalent:

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Problem – version IV. Classify the inverse-free discrete semigroups.

Recap

Problem – version I. Determine those finite sets of real numbers X with the property that each continued fraction with coefficients in X converges.

Problem – version II. Determine those finite sets of Möbius transformations $\mathcal{F}$ with the property that every composition sequence from $\mathcal{F}$ converges at 0.

Problem – version III. Determine those finite sets of Möbius transformations $\mathcal{F}$ with the property that every composition sequence from $\mathcal{F}$ converges generally.

Problem – version IV. Classify the inverse-free discrete semigroups.

Selected literature on Möbius semigroups

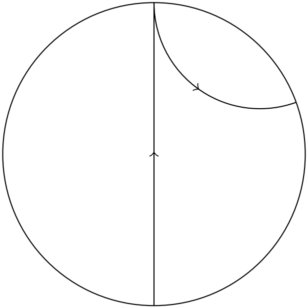
Some Moebius semigroups on the 2-sphere
C.S. Ballantine
J. Math. Mech., 1962

On certain semigroups of hyperbolic isometries
T. Jørgensen and K. Smith
Duke Math. J., 1990

Complex dynamics of Möbius semigroups
D. Fried, S.M. Marotta and R. Stankewitz
Ergodic Theory Dynam. Systems, 2012

Entropie des semi-groupes d'isométrie d'un espace hyperbolique
P. Mercat
To be published

Exceptional semigroup



$f(z) = 2z \qquad g(z) = \frac{1}{2}z + 1$

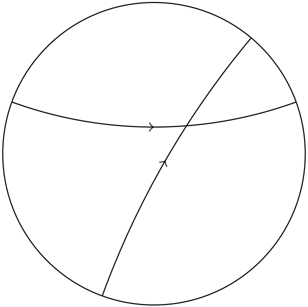
Two-generator Fuchsian groups literature

The classification of discrete 2-generator subgroups of PSL(2, $\mathbf{R}$)
J.P. Matelski
Israel J. Math., 1982

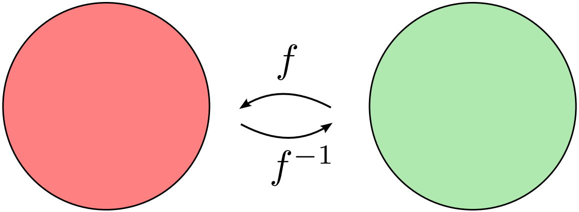
An algorithm for 2-generator Fuchsian groups
J. Gilman and B. Maskit
Michigan Math. J., 1991

Two-generator discrete subgroups of PSL$(2, \mathbf{R})$
J. Gilman
Mem. Amer. Math. Soc., 1995

Two-generator Fuchsian groups



Schottky groups



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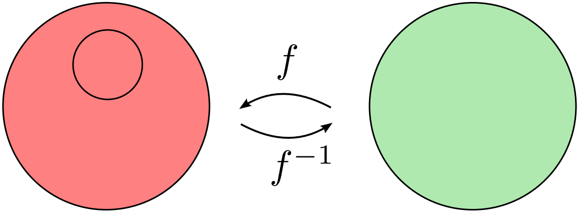

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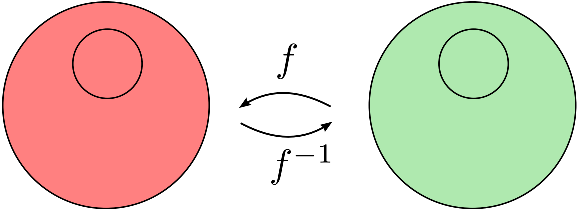

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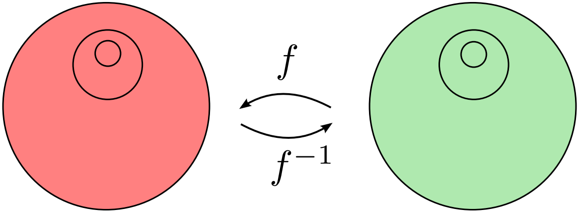

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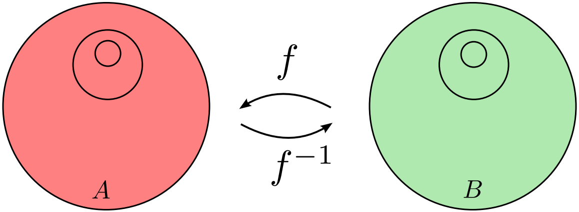

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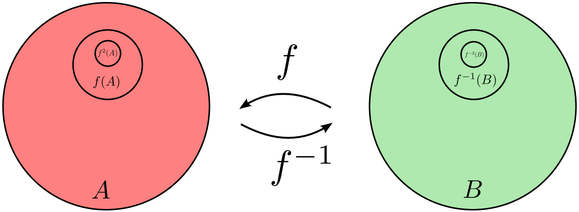

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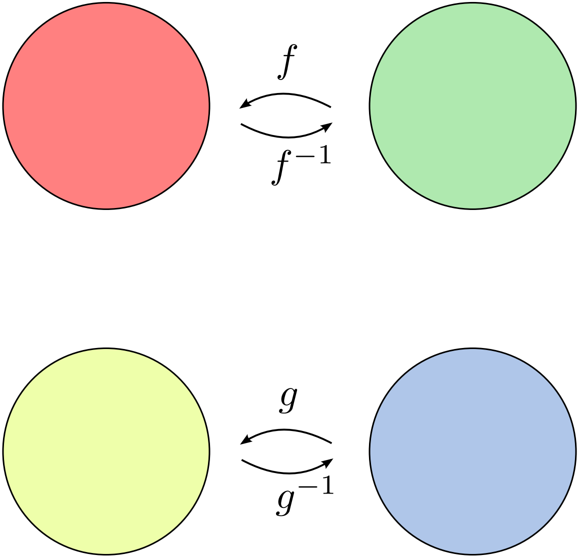

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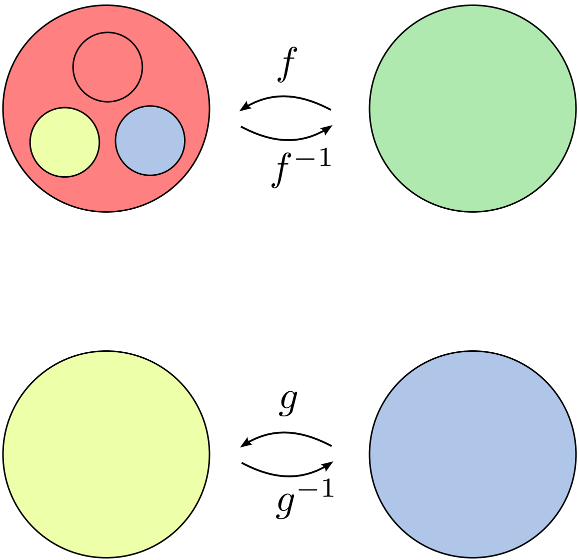

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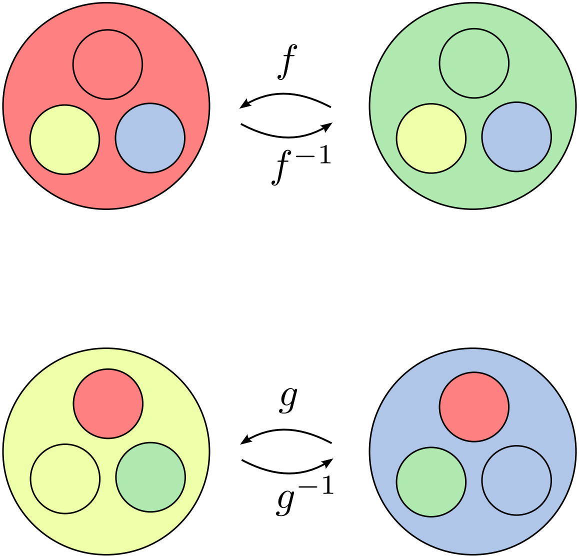

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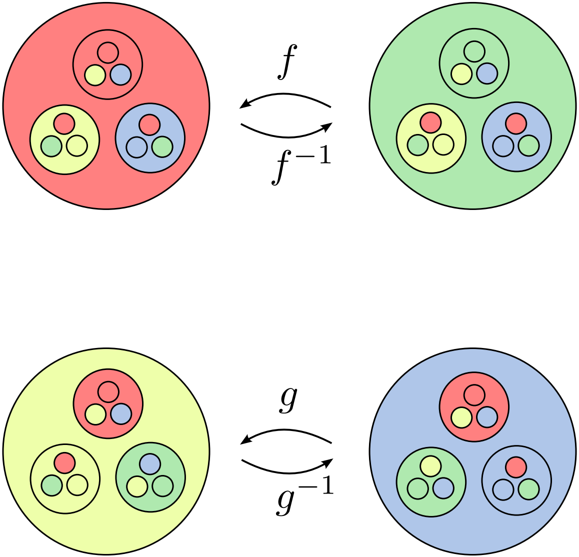

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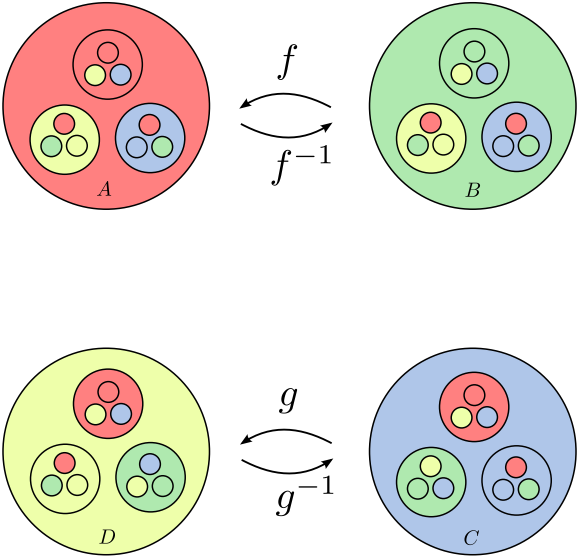
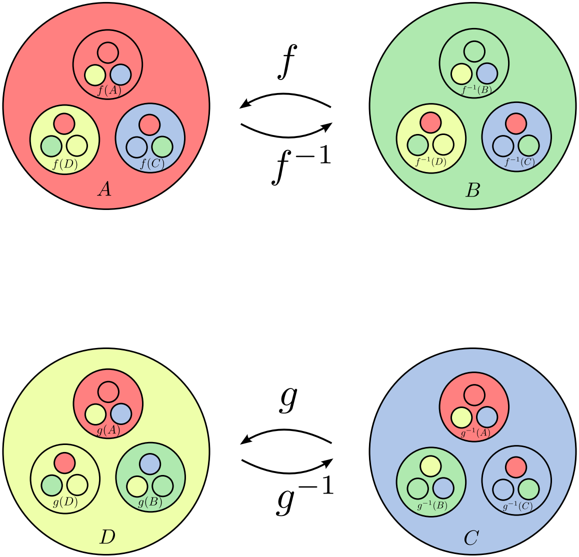
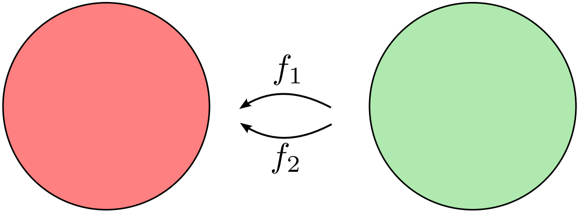


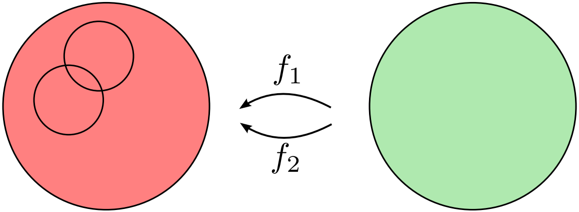
Schottky groups



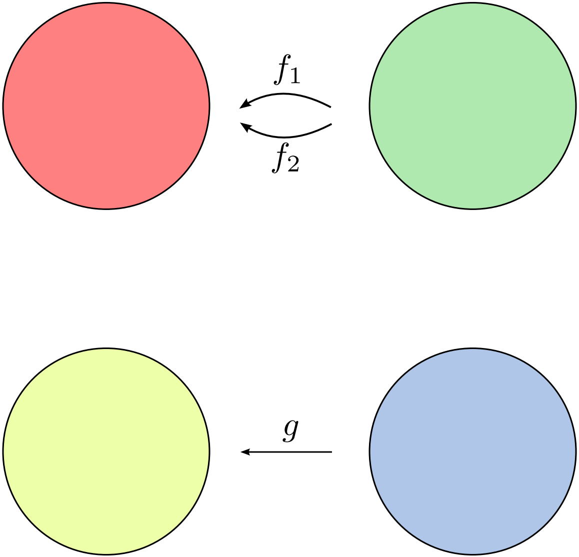
Schottky semigroups



Schottky semigroups



Schottky semigroups



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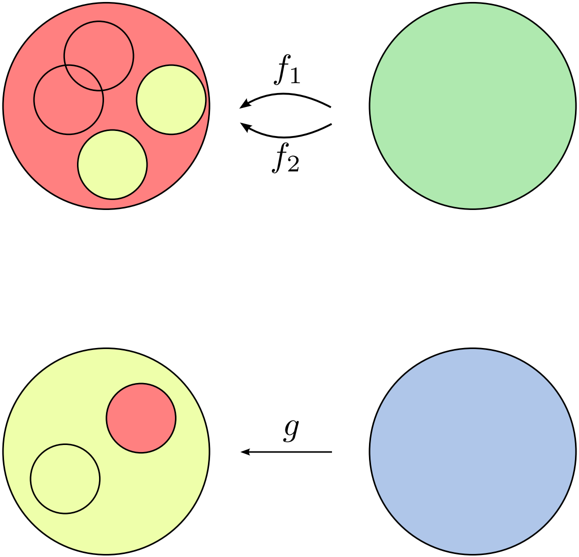
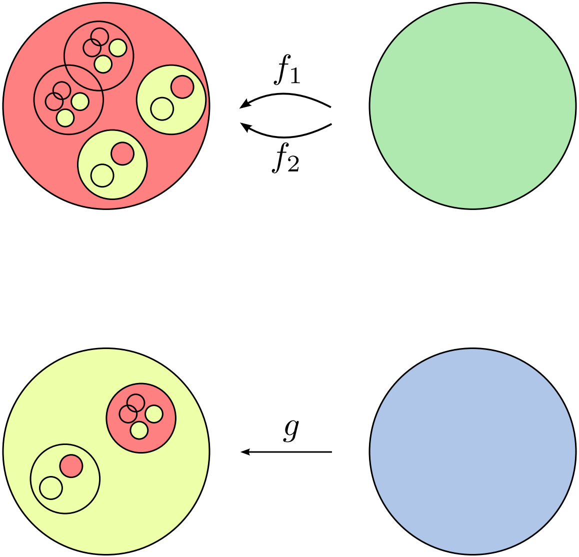
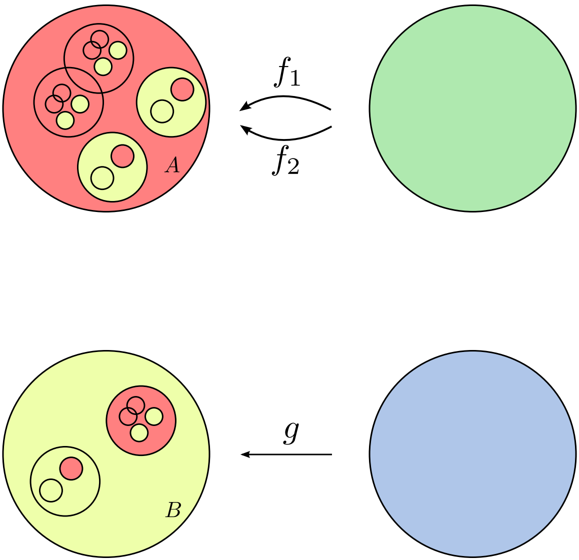


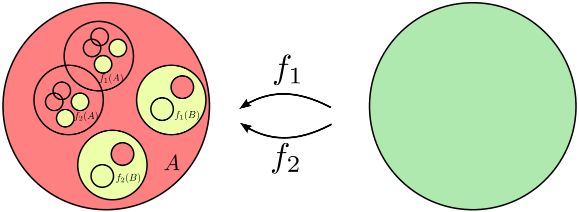
Schottky semigroups

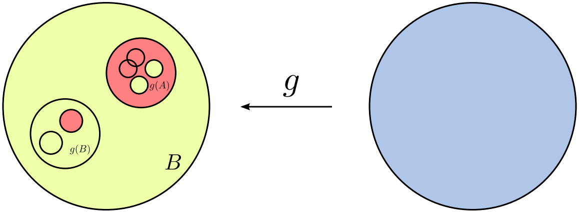


Schottky semigroups

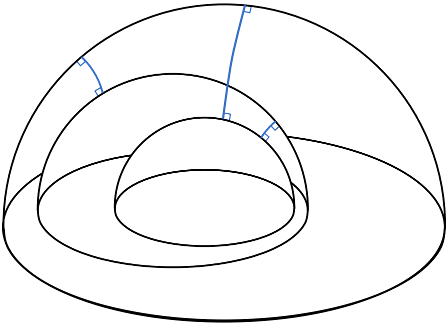


Schottky semigroups





Reverse triangle inequality



Two-generator inverse-free discrete semigroups

elliptic	elliptic
elliptic	parabolic
elliptic	loxodromic
parabolic	parabolic
parabolic	loxodromic
loxodromic	loxodromic

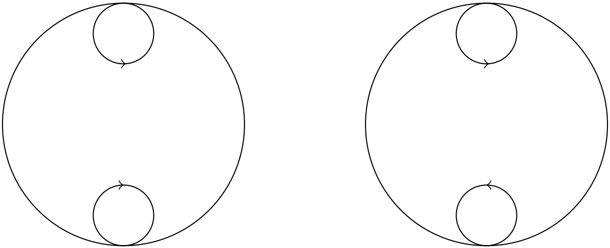
Two-generator inverse-free discrete semigroups

elliptic	elliptic	✗
elliptic	parabolic	✗
elliptic	loxodromic	✗
parabolic	parabolic	
parabolic	loxodromic	
loxodromic	loxodromic	

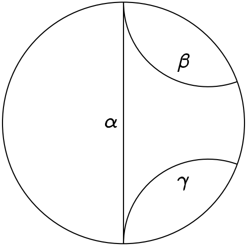
Two-generator inverse-free discrete semigroups

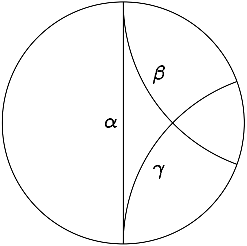
elliptic	elliptic	✘
elliptic	parabolic	✘
elliptic	loxodromic	✘
parabolic	parabolic	✔
parabolic	loxodromic	✔
loxodromic	loxodromic	✔

Pairs of parabolics

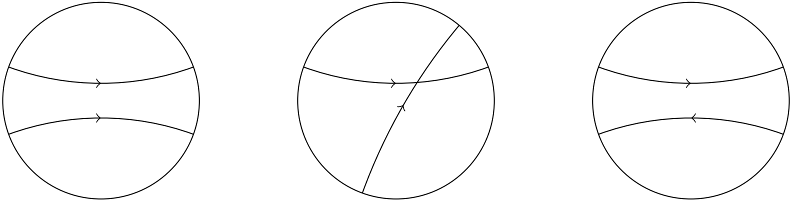


Pairs of parabolics

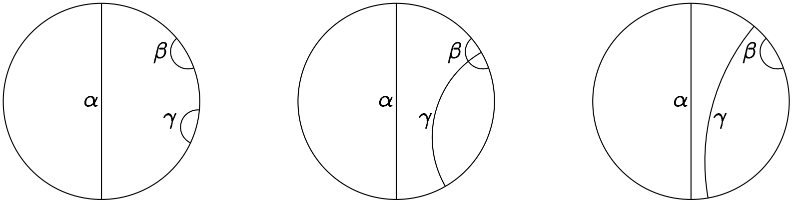




Pairs of loxodromics



Pairs of loxodromics



Solution to Problem III

Theorem. A set $\mathcal{F} = \{f, g\}$ of two Möbius transformations has the property that every composition sequence from $\mathcal{F}$ converges generally if and only if one of the following conditions is satisfied:

(i) f and g are parabolic and fg is not elliptic

(ii) one of f or g is loxodromic and the other is either parabolic or loxodromic, and fg^n and $f^n g$ are not elliptic for any positive integer n.

Return to the original problem

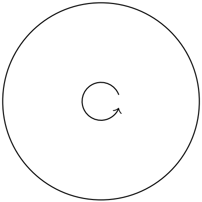
Problem – version I. Determine those finite sets X with the property that each continued fraction with coefficients in X converges.

Return to the original problem

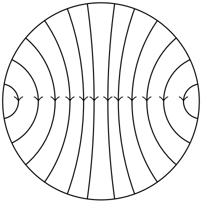
Problem – version I. Determine those finite sets X with the property that each *minus* continued fraction with coefficients in X converges.

Return to the original problem

Problem – version I. Determine those finite sets X with the property that each *minus* continued fraction with coefficients in X converges.



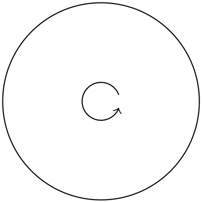
$$-\frac{1}{b+z} \qquad |b| < 2$$



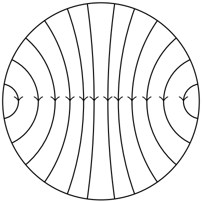
$$-\frac{1}{b+z} \qquad |b| \geqslant 2$$

Return to the original problem

Problem – version I. Determine those finite sets X with the property that each *minus* continued fraction with coefficients in X converges.



$-\dfrac{1}{b+z} \quad |b| < 2$



$-\dfrac{1}{b+z} \quad |b| \geqslant 2$

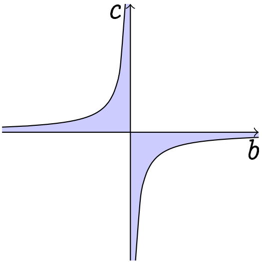
Theorem. A finite set X has the property that every minus continued fraction with coefficients in X converges if and only if $X \subset (-\infty, 2] \cup [2, +\infty)$.

Return to the original problem

Problem – version I. Determine those finite sets X with the property that each continued fraction with coefficients in X converges.

Return to the original problem

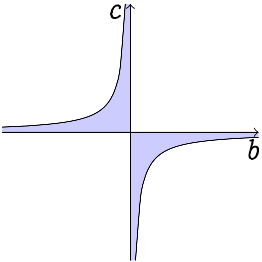
Problem – version I. Determine those finite sets X with the property that each continued fraction with coefficients in X converges.



$bc \in (-4, 0]$

Return to the original problem

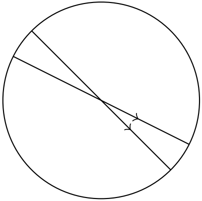
Problem – version I. Determine those finite sets X with the property that each continued fraction with coefficients in X converges.



$bc \in (-4, 0]$

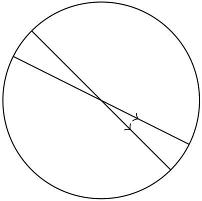
Theorem. A set $\{b, c\}$ of real numbers has the property that every continued fraction with coefficients in $\{b, c\}$ converges if and only if $bc \notin (-4, 0]$.

Examples

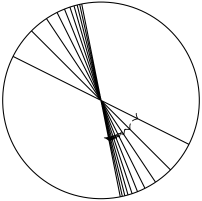


$$\frac{1}{1+z} \qquad \frac{1}{2+z}$$

Examples

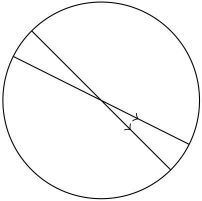


$$\frac{1}{1+z} \qquad \frac{1}{2+z}$$

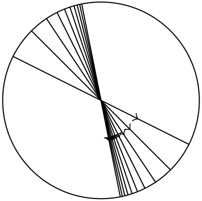


$$\frac{1}{b+z} \qquad b \in \mathbb{N}$$

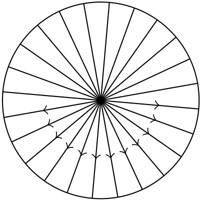
Examples



$\frac{1}{1+z}$ $\frac{1}{2+z}$

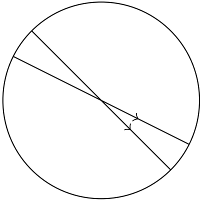


$\frac{1}{b+z}$ $b \in \mathbb{N}$



$\frac{1}{b+z}$ $b \in \mathbb{R}$

Examples



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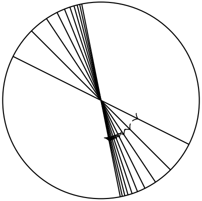
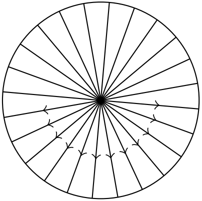
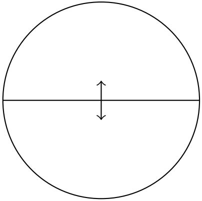


$$\frac{1}{b+z} \quad b \in \mathbb{N}$$

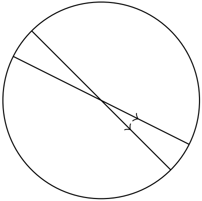


$$\frac{1}{b+z} \quad b \in \mathbb{R}$$

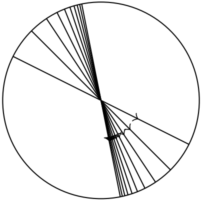


$$\frac{1}{z}$$

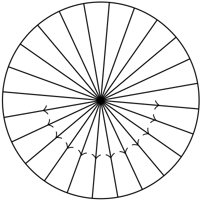
Examples



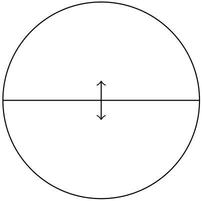
$\dfrac{1}{1+z}$ $\dfrac{1}{2+z}$



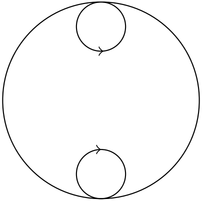
$\dfrac{1}{b+z}$ $b \in \mathbb{N}$



$\dfrac{1}{b+z}$ $b \in \mathbb{R}$

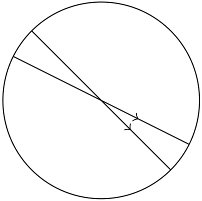


$\dfrac{1}{z}$

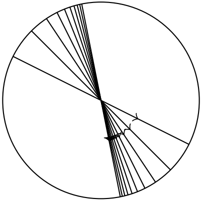


$z+1$ $\dfrac{z}{z+1}$

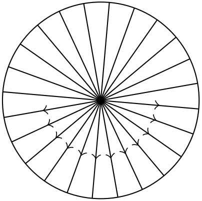
Examples



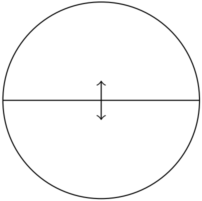
$\frac{1}{1+z}$ $\frac{1}{2+z}$



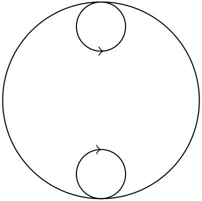
$\frac{1}{b+z}$ $b \in \mathbb{N}$



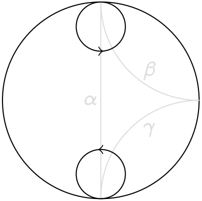
$\frac{1}{b+z}$ $b \in \mathbb{R}$



$\frac{1}{z}$



$z+1$ $\frac{z}{z+1}$



$z+2$ $\frac{z}{-2z+1}$

The unknown

Conjecture. If every composition sequence from $\mathcal{F}$ is an escaping sequence, then every composition sequence converges generally.

The unknown

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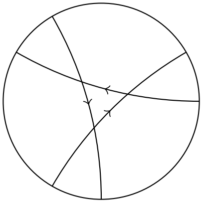
More generators. Classify those sets of transformations $\mathcal{F}$ of size greater than two with the property that every composition sequence from $\mathcal{F}$ converges generally.

The unknown

Conjecture. If every composition sequence from $\mathcal{F}$ is an escaping sequence, then every composition sequence converges generally.

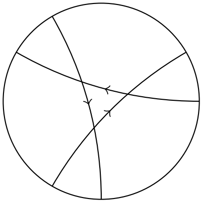
More generators. Classify those sets of transformations $\mathcal{F}$ of size greater than two with the property that every composition sequence from $\mathcal{F}$ converges generally.



The unknown

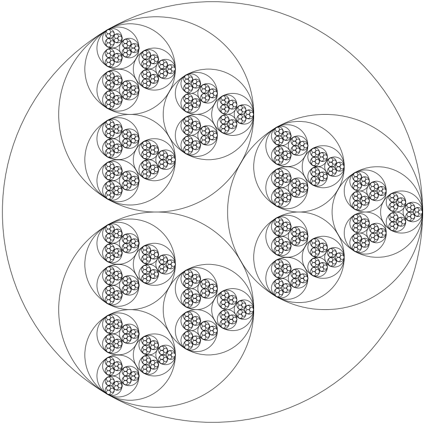
Conjecture. If every composition sequence from $\mathcal{F}$ is an escaping sequence, then every composition sequence converges generally.

More generators. Classify those sets of transformations $\mathcal{F}$ of size greater than two with the property that every composition sequence from $\mathcal{F}$ converges generally.



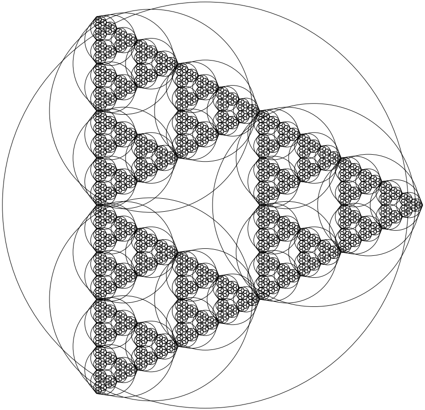
Higher dimensions. Classify those sets of *complex* transformations $\mathcal{F}$ of size two with the property that every composition sequence from $\mathcal{F}$ converges generally.

Gallery[1]



[1] Created using *lim* by Curt McMullen

Gallery[1]



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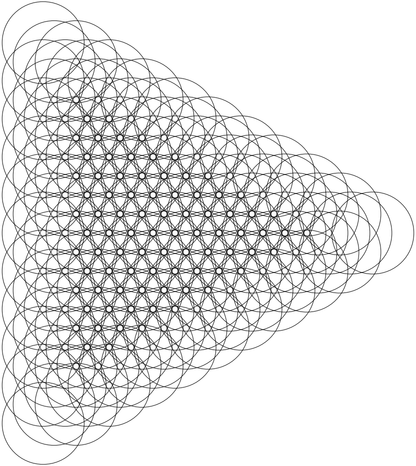
Gallery[1]



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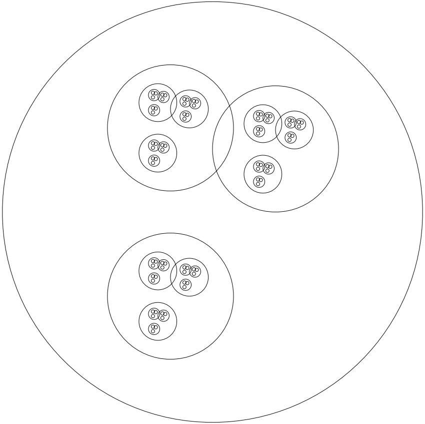
Gallery[1]



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Thank you for your attention.