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uniform distribution theory

DOI: 10.2478/UDT-2024-0008
Unif. Distrib. Theory **19** (2024), no.1, 97–120

UNIFORM DUAL APPROXIMATION TO VERONESE CURVES IN SMALL DIMENSION

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ABSTRACT. We refine upper bounds for the classical exponents of uniform approximation for a linear form on the Veronese curve in dimension from 3 to 9. For dimension three, this in particular shows that a bound previously obtained by two different methods is not sharp. Our proof involves parametric geometry of numbers and investigation of geometric properties of best approximation polynomials. Slightly stronger bounds have been obtained by Poels with a different method contemporarily. In fact, we obtain his bounds as a conditional result.

Communicated by Friedrich Pillichshammer

1. Introduction

1.1. New results

Davenport and Schmidt [5], in the course of investigating approximation to real numbers by algebraic integers related to the famous open problem of Wirsing [19], implicitly studied uniform exponents of approximation on the Veronese curve in dimension n defined as $\{(\xi, \xi^2, \dots, \xi^n) : \xi \in \mathbb{R}\}$. Two variants of these exponents were addressed in [5], one for simultaneous approximation to successive powers of ξ and one for small values of a linear form (degree n polynomial). Both types of exponents are indeed closely linked to Wirsing's problem and variants of it, see besides [5] for example also [2]. In this paper, we refine upper bounds for the uniform exponents with respect to the latter polynomial setting.

2020 Mathematics Subject Classification: 11J13.
Keywords: Veronese curve, exponents of approximation, best approximations.

For ξ a real number and $n \geq 1$ an integer, let us denote them by $\widehat{w}_n(\xi)$ which are defined as the supremum of w so that

$$H_P \leq X, \qquad 0 < |P(\xi)| < X^{-w}$$

has a solution in an integer polynomial $P = P(X)$ of degree at most n, for all large X. Here H_P is the height of P, that is the maximum modulus among its coefficients. Let us directly define the associated ordinary exponent of approximation $w_n(\xi)$ as well, given as supremum of w so that

$$0 < |P(\xi)| < H_P^{-w}$$

has infinitely many solutions in integer polynomials P of degree at most n. Clearly, $w_n(\xi) \geq \widehat{w}_n(\xi)$ by choosing $X = H_P$, and moreover

$$w_1(\xi) \leq w_2(\xi) \leq \cdots, \qquad \widehat{w}_1(\xi) \leq \widehat{w}_2(\xi) \leq \cdots$$

hold for any real number ξ. Moreover, Dirichlet's Theorem shows the lower bounds

$$w_n(\xi) \geq \widehat{w}_n(\xi) \geq n.$$

A well-known consequence of the subspace theorem is that for ξ any real algebraic number of degree d we have $w_n(\xi) = \widehat{w}_n(\xi) = \min\{n, d-1\}$, so we may only consider transcendental numbers ξ below. It is well-known that $\widehat{w}_1(\xi) = 1$ for all irrational real numbers ξ, see Khintchine [6], so we may restrict to $n \geq 2$. As indicated above, upper bounds for $\widehat{w}_n(\xi)$ have first been studied by Davenport and Schmidt [5], whose result shows in our notation that

$$\widehat{w}_n(\xi) \leq 2n - 1, \qquad n \geq 2, \tag{1}$$

holds for any real ξ. For $n = 2$, they proved a stronger bound of the form

$$\widehat{w}_2(\xi) \leq \frac{3+\sqrt{5}}{2} = 2.6180\ldots, \tag{2}$$

which surprisingly turned out to be optimal as shown by Roy [12]. For $n \geq 3$ the optimal bound remains unknown. It took almost 50 years for the first small improvements to (1) in [3], where an upper bound of order $2n - \frac{3}{2} + o(1)$ as $n \to \infty$ with positive error term for each n was established. The method also reproved the optimal upper bound (2) for $n = 2$. In fact, as noticed in [15], the method in [3] when combined with the later proved optimal ratio for ordinary and uniform exponents by Marnat and Moshchevitin [8], directly yields

$$\widehat{w}_n(\xi) \leq \alpha_n := \max\{2n-2, \sigma_n\} = \begin{cases} \sigma_n, & 2 \leq n \leq 9, \\ 2n-2, & n \geq 10, \end{cases} \tag{3}$$

where σ_n is the real solution to

$$\frac{(n-1)x}{x-n} - x + 1 = \left(\frac{n-1}{x-n}\right)^n$$

in the interval $x \in (n, 2n-1)$. The case $n = 2$ indeed recovers (2). We have $\sigma_n = 2n - C + o(1)$ as $n \to \infty$, where $C = 2.25\ldots$ is explicitly computable, see [14] for details. In case of strict inequality $w_n(\xi) > w_{n-1}(\xi)$, the term $2n-2$ in (3) can be ignored, thereby implying the bound $\widehat{w}_n(\xi) \leq \sigma_n$ which is stronger than (3) for $n \geq 10$. We remark that a closely related classical exponent, denoted $\widehat{w}_n^*(\xi)$ (see [3] for a definition), satisfies $\widehat{w}_n^*(\xi) \leq \sigma_n$ unconditionally. However, the latter exponent is always bounded above by $\widehat{w}_n(\xi)$, hence giving a weaker claim.

In a later paper the author [15] introduced another method, involving parametric geometry of numbers. It turned out that in [15] the same bound for $n=3$ as in (3) that reads $\widehat{w}_3(\xi) \leq 3 + \sqrt{2} = 4.4142\ldots = \alpha_3$ was obtained, however for larger n the bounds became slightly weaker than α_n. Some significantly stronger but conditional results were stated in [15, § 2] as well. In a very recent preprint that appeared just days before the current paper, Poels [10] improved on the estimates (3) that had been obtained in [3]. He established a stronger bound of the form

$$\widehat{w}_n(\xi) \leq 2n-2, \quad (n \geq 4), \qquad \widehat{w}_3(\xi) \leq 2 + \sqrt{5} = 4.23\ldots \tag{4}$$

Moreover, for large enough n, a bound of the form

$$\widehat{w}_n(\xi) \leq 2n - \frac{1}{3} n^{1/3}$$

was obtained in [10].

In this paper, we refine the method from the latter paper [15] to improve the bound $\alpha_n = \sigma_n$ from (3) in the range $3 \leq n \leq 9$. Unfortunately, our bounds in this range will be weaker than the very recent estimates (4). We want to point out however that our method is considerably different from the one in [10] and may be of independent interest for future improvements. Our main result reads as follows.

Theorem 1.1. *For any $n \geq 2$ and any real number ξ we have*

$$\widehat{w}_n(\xi) \leq \beta_n ,$$

where β_n is the root of the monic quartic polynomial

$$Q_n(T) = T^4 + a_3 T^3 + a_2 T^2 + a_1 T + a_0$$

in the interval $(2n-2, 2n-1)$, where

$$a_3 = 4 - 4n, \qquad a_2 = 5n^2 - 12n + 8,$$
$$a_1 = -2n^3 + 11n^2 - 18n + 7, \qquad a_0 = -2n^3 + 6n^2 - 4n.$$

Any polynomial Q_n has four distinct real roots and β_n is the largest among them. For $n = 2$ we again obtain the optimal bound $\beta_2 = 2.6180\ldots$ from (2) obtained by Davenport and Schmidt [5] and independently in [3] and [15]. For $n = 3$ we get

$$\widehat{w}_3(\xi) \leq \beta_3 = 4.3234\cdots < 4.4142\cdots = 3 + \sqrt{2} = \alpha_3. \tag{5}$$

The value α_3 is from (3) and was obtained with an alternative proof in [15] as well. It turns out that similarly, for $3 \leq n \leq 9$ we get an improvement to (3).

We further provide stronger conditional results. For brevity we delay the definition of best approximation polynomials and refer to § 2.1 below.

Definition 1.2. We say an integer $k \geq 2$ is *good* for n, ξ if the triple of consecutive best approximation polynomials $\{P_{k-1}, P_k, P_{k+1}\}$ defined in Definition 2.1 below associated to n, ξ is linearly independent.

It is well-known and follows from Lemma 2.5 below that there are infinitely many good k for each pair n, ξ. Assuming refinements yield improvements as follows.

Theorem 1.3. *Let $n \geq 2$ be an integer and ξ be a real number.*

(i) *Assume that infinitely often, the integers $k-1$ and k are both good for n, ξ. Then*
$$\widehat{w}_n(\xi) \leq \gamma_n,$$
where γ_n is the root of the monic cubic polynomial
$$R_n(T) := T^3 - (4n-4)T^2 + (5n^2 - 11n + 6)T + (-2n^3 + 8n^2 - 10n + 3).$$
in the interval $(2n-2, 2n-1)$.

(ii) *Assume all sufficiently large integers k are good for n, ξ. Then*
$$\widehat{w}_n(\xi) \leq \rho_n := \max\left\{\frac{\sqrt{5}+1}{2}n - \frac{\sqrt{5}-1}{2},\ 2n-2\right\} = \begin{cases} \frac{\sqrt{5}+3}{2}, & n = 2, \\ \sqrt{5}+2, & n = 3, \\ 2n-2, & n \geq 4. \end{cases}$$

We point out that the conditional bounds ρ_n coincide with those obtained unconditionally by Poels in (4).

Again all roots of R_n are real and γ_n is the largest. The condition in (i) can be relaxed to assuming that on a subsequence of good k, we have
$$\frac{\log P_{\ell-1}}{\log P_k} \to 1, \qquad k \to \infty$$
(understood inside the subsequence) with $\ell = \ell(k) \geq k+1$ chosen minimal so that $\{P_{k-1}, P_k, P_\ell\}$ are linearly independent. A similar relaxation of the hypothesis in (ii) can be stated. Conversely, a stronger assumption then in (i) is four

consecutive best approximation polynomials being linearly independent infinitely often. In view of [9], it is questionable if either condition is true for all real numbers ξ, see the comments in § 2.2 below.

The quality of the bounds are illustrated in the Table 1 with decimal expansions cut off after four digits. As in [15], as an artefact of the method, our bound satisfies the asymptotics

$$\beta_n = 2n - 2 + o(1), \qquad n \to \infty,$$

with positive error terms for each n, however the limit being approached faster than in [15]. The same holds for γ_n, ρ_n. Since $\alpha_n = 2n - 2$ for $n \geq 10$ by (3), our largest value $n = 9$, where an improvement is obtained in either theorem above is a natural barrier, and surpassing it would probably require significant new ideas in the method.

TABLE 1.

n	new bound β_n	bound $\alpha_n = \sigma_n$ derived from [3]	cond. bd. γ_n	cond. bd. ρ_n
2	2.6180	2.6180	2.6180	2.6180
3	4.3234	4.4142	4.3028	4.2360
4	6.1592	6.2875	6.1451	6
5	8.0865	8.2010	8.0791	8
6	10.0528	10.1382	10.0488	10
7	12.0352	12.0906	10.0328	12
8	14.0251	14.0532	14.0236	14
9	16.0187	16.0231	16.0177	16

We emphasize again that especially the improvement compared to (5) for $n = 3$ is remarkable, as it shows that the previously best known bound $3+\sqrt{2}$ that had been obtained from two different methods in [3,15], is not sharp. The new bounds β_n of Theorem 1.1 are not optimal for any $n \geq 3$ due to Poels [10], possibly the same is true for the bounds ρ_n he obtained (unconditionally). Indeed, conversely, if $n \geq 3$ it is not known if the exponent $\widehat{w}_n$ can take a value larger than n for any real number ξ. If this is not the case for some $n \geq 3$, this would imply an affirmative answer to Wirsing's problem (and some variants) for this value of n, see [5, Lemma 1].

1.2. Ideas of the proof and related exponents

The foundation of our method will be similar to [15], using an approach inspired by parametric geometry of numbers, however with several technical twists. Firstly, we need a claim on consecutive best approximation polynomials (see Definition 2.1) lying in two-dimensonal subspaces (Lemma 2.5, only needed for Theorem 1.1). Moreover, some step from the proof in [15] is simplified (via Lemma 2.3), which is necessary for our method here. Furthermore, some technical improvement in the treatment of parametric geometry of numbers is obtained (Lemma 3.1). These new elements will be combined in a concise way to beat the bounds from [15].

While we restrict ourselves to the linear form exponents here, we want to briefly remark on the according exponents usually denoted $\widehat{\lambda}_n(\xi)$ for the dual problem of simultaneous approximation to consecutive powers of a real number, also closely connected to Wirsing's problem and variants. There, similar progress in form of (rather small) improvements of the original bound by Davenport and Schmidt [5] have been made, some very recently. The first improvement for odd n was due to Laurent [7], and a stronger bound for $n = 3$ due to Roy [13]. For even n, the author improved the bound from [5] in [16, 17]. Then in a very recent paper Badziahin [1] improved on the previous results for $n \geq 4$, which in turn has been refined by Poels and Roy [11] to constitute the currently best known bounds for $\widehat{\lambda}_n(\xi)$. In private communication, D. Roy pointed out to me that he recently obtained a very small improvement on his result [13] for $n = 3$, in a paper in preparation. However, as in the case of linear form exponents, all obtained refinements compared to the original result by Davenport and Schmidt [5] are rather small, moreover again it remains unclear if the minimum value $1/n$ is exceeded for any $n \geq 3$ and any real number ξ. In summary, both types of exponents remain rather poorly understood for $n \geq 3$.

2. On best approximation polynomials

2.1. Definition and a linear independence result

An important object for the study of exponents of Diophantine approximation are integer minimal points, used for example in [4, 5], which are polynomials in our case. Let us first define this sequence of best approximation polynomials $(P_k)_{k\geq 1}$ as in [15, Definition 2.1].

Definition 2.1. For $n \geq 1$ an integer and ξ a real number, an integer polynomial P of degree at most n will be called *best approximation polynomial associated to* (n, ξ) if it minimizes $|P(\xi)|$ among all non identically zero integer polynomials of degree at most n and height (maximum of absolute values of coefficients) at most H_P. Any pair n, ξ with ξ not algebraic of degree at most n thus gives rise to a uniquely determined (up to sign) infinite sequence of best approximation polynomials. We denote it by $(P_k)_{k\geq 1}$ and the height of P_k by H_k.

Recall that they satisfy

$$|P_1(\xi)| > |P_2(\xi)| > \cdots, \qquad H_1 < H_2 < \cdots \tag{6}$$

We derive for $k \geq 2$ the quantities

$$\mu_k := -\frac{\log |P_{k-1}(\xi)|}{\log H_k}, \qquad v_k := -\frac{\log |P_k(\xi)|}{\log H_k}. \tag{7}$$

and see the relation to the classical exponents is given by

$$\widehat{w}_n(\xi) = \liminf_{k\to\infty} \mu_k, \qquad w_n(\xi) = \limsup_{k\to\infty} v_k. \tag{8}$$

We comprise some results that were implicitly derived in [15], see more precisely the beginning of the proof of [15, Theorem 1.1] and [15, § 3.2]. Following notation of [15], let

$$\mathscr{V}_k = \{P_k, TP_k, \ldots, T^{n-2}P_k\}, \qquad k \geq 1, \tag{9}$$

consisting of $n-1$ integer polynomials of degree at most $2n-2$.

Lemma 2.2 ([15]). *Let $n \geq 2$ be an integer and ξ a real number that satisfies*

$$\widehat{w}_n(\xi) > 2n-2. \tag{10}$$

Then for all large enough k, the polynomial P_k from Definition 2.1 is irreducible of degree exactly n. Consequenctly, for any large k, the set $\mathscr{V}_{k-1} \cup \mathscr{V}_k$ is linearly independent and thus spans a hyperplane in the space of polynomials of degree at most $2n-2$. Moreover, we have

$$\frac{w_n(\xi)}{\widehat{w}_n(\xi)} \leq \frac{n-1}{\widehat{w}_n(\xi)-n}. \tag{11}$$

Some claims of the lemma originate in results from [3]. We aim to prove Theorem 1.1 by contradiction, assuming throughout we had

$$\widehat{w}_n(\xi) > \beta_n \tag{12}$$

for some ξ and showing that it is impossible. Therefore, as $\beta_n > 2n-2$, the condition (10) is not restrictive. The same applies to Theorem 1.3. It will occur frequently.

The next, new lemma extends the claim about the dimension of unions of consecutive $\mathscr{V}_k$ from Lemma 2.2. Thereby it avoids case 2 from the proof of [15, Theorem 1.1] and gives us more flexibility. The proof uses similar arguments. It can be interpreted as a partial result towards proving the conditional bounds from [15, § 2], however per se it does not lead to an improvement in this framework.

Lemma 2.3. *Let $n \geq 2$ be an integer and ξ a real number. Assume* (10) *holds. For all large enough, good k in the sense of Definition 1.2, the polynomials from the union*

$$\mathscr{R}_k := \mathscr{V}_{k-1} \cup \mathscr{V}_k \cup \mathscr{V}_{k+1}$$

span the space of polynomials of degree at most $2n-2$.

Remark 1. The linear independence assumption of the lemma, that is k being good, is clearly also needed, else the union spans a space of dimension at most $2n-2$ (with equality if (10) holds).

Remark 2. The proof below shows that the claim holds for any three linearly independent irreducible polynomials of exact degree n, in particular upon (10) by Lemma 2.2 for any linearly independent triple of best approximation polynomials.

P r o o f. By Lemma 2.2, it suffices to show that some polynomial from $\mathscr{V}_{k+1}$ does not belong to the span of $\mathscr{V}_{k-1} \cup \mathscr{V}_k$. If $P_{k+1} \in \mathscr{V}_{k+1}$ is not contained in this space, then we are done. So assume it does belong to the span. Then we can write

$$P_{k+1} = Q_1 P_{k-1} + Q_2 P_k \tag{13}$$

for rational polynomials $Q_i = Q_{i,k} \in \mathbb{Q}[T]$ of degrees at most

$$m := \max_{i=1,2} \deg Q_i \leq n-2.$$

On the other hand, $m \geq 1$ follows from the linear independence assumption of the lemma. Without loss of generality, assume $\deg Q_1 = m$, the other case works analogously. Then consider $T^{n-m-1} P_{k+1}$ which has degree at most $2n-2$ so it lies in $\mathscr{V}_{k+1}$. Again if it does not lie in the span of $\mathscr{V}_{k-1} \cup \mathscr{V}_k$ we are done. So we can assume it does, which again means that there is some identity

$$T^{n-m-1} P_{k+1} = B_1 P_{k-1} + B_2 P_k \tag{14}$$

with rational polynomials $B_i \in \mathbb{Q}[T]$ of degrees

$$\deg B_i \leq n-2, \qquad i = 1, 2. \tag{15}$$

On the other hand, by (13) we can write

$$A_1 P_{k-1} + A_2 P_k = T^{n-m-1} P_{k+1}, \tag{16}$$

where $A_i \in \mathbb{Q}[T]$, $i = 1, 2$, are rational polynomials given as $A_i = T^{n-m-1} Q_i$. Note that $\deg A_1 = (n-m-1) + m = n-1$. Hence by (15) also

$$\deg(A_1 - B_1) = n - 1. \tag{17}$$

Now from (14) and (16) we have an identity

$$B_1 P_{k-1} + B_2 P_k = T^{n-m-1} P_{k+1} = A_1 P_{k-1} + A_2 P_k.$$

This yields an identity over $\mathbb{Q}[T]$ given as

$$(A_1 - B_1)P_{k-1} = -(A_2 - B_2)P_k,$$

and multiplying with the common denominator we get an equality

$$(\tilde{A}_1 - \tilde{B}_1)P_{k-1} = -(\tilde{A}_2 - \tilde{B}_2)P_k$$

over integer polynomials $\tilde{A}_i, \tilde{B}_i \in \mathbb{Z}[T]$. So since P_{k-1}, P_k are distinct and irreducible of degree exactly n by Lemma 2.2, we must have that P_k divides $\tilde{A}_1 - \tilde{B}_1$ over $\mathbb{Z}[T]$, but since the latter has degree $n-1$ by (17), this is impossible. The lemma is proved. □

2.2. On consecutive best approximations in two-dimensional subspaces

As alluded by (8), for studying uniform exponents, it is important to understand how fast consecutive best approximations occur. Hence we define

$$\tau_k = \frac{\log H_k}{\log H_{k-1}} > 1, \qquad k \geq 2, \tag{18}$$

that is

$$H_k = H_{k-1}^{\tau_k}.$$

All error terms below will be understood as $k \to \infty$. Then, upon assuming (10), by (11) and a well-known argument (see for example [18, Lemma 1]) we have

$$1 < \tau_k \leq \frac{w_n(\xi)}{\widehat{w}_n(\xi)} + o(1) \leq \frac{n-1}{\widehat{w}_n(\xi) - n} + o(1). \tag{19}$$

Note that only the most right estimate of (19) requires a condition on ξ. Furthermore as a consequence of (8) we have

$$H_k^{-w_n(\xi)/\tau_k - o(1)} = H_{k-1}^{-w_n(\xi)-o(1)} \ll |P_{k-1}(\xi)| \ll H_k^{-\widehat{w}_n(\xi)+o(1)}. \tag{20}$$

The next lemma, only required for the proof of Theorem 1.1, estimates for how long a set of consecutive best approximation polynomials can lie in a two-dimensional subspace. The proof follows closely ideas of Davenport and Schmidt [4] and refines it in a quantitative way. As in [4], it is not specific to points on the Veronese curve and can be formulated for linear forms in any n real variables that are $\mathbb{Q}$-linearly independent together with $\{1\}$.

Definition 2.4. Given n, ξ, for $k \geq 2$ an integer, let $\ell = \ell(k) \geq k+1$ be the maximal integer so that $P_{k-1}, P_k, \dots, P_{\ell-1}$ from Definition 2.1 lie in a two--dimensional space (spanned by P_{k-1}, P_k).

Note that k is good in sense of Definition 1.2 if and only if $\ell(k) = k+1$.

Lemma 2.5. *Let $n \geq 2$ be an integer, ξ be a transcendental real number, and let $(P_j)_{j\geq 1}$ be the sequence of best approximation polynomials for degree n as in Definition 2.1. For $k \geq 2$ an integer, assume*

$$H_k > 2 \cdot H_{k-1}. \tag{21}$$

Let $\ell = \ell(k) \geq k+1$ be as in Definition 2.4 and let v_{k-1} be as in (7)*. Then we have*

$$\frac{\log H_{\ell-1}}{\log H_k} \leq \frac{\frac{v_{k-1}}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} + o(1), \qquad k \to \infty. \tag{22}$$

In particular,

$$\frac{\log H_{\ell-1}}{\log H_k} \leq \frac{\frac{w_n(\xi)}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} + o(1), \qquad k \to \infty. \tag{23}$$

Hence P_{k-1}, P_k, P_ℓ are linearly independent and

$$\frac{\log H_{\ell}}{\log H_k} \leq \frac{\frac{w_n(\xi)}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} \cdot \tau_\ell + o(1), \qquad k \to \infty. \tag{24}$$

Remark 3. The factor two in (21) can be replaced by any value larger than 1. The claim probably remains true without condition (21) at all, however for us this condition will not cause major problems below.

Remark 4. If (10) holds, then we may use (11) to eliminate $w_n(\xi)$ and express the right hand sides in terms of $n, \widehat{w}_n(\xi), \tau_k, \tau_\ell$ only. For example (24) becomes

$$\frac{\log H_{\ell}}{\log H_k} \leq \frac{(n-1)\widehat{w}_n(\xi) - \tau_k(\widehat{w}_n(\xi) - n)}{\tau_k(\widehat{w}_n(\xi) - 1)(\widehat{w}_n(\xi) - n)} \cdot \tau_\ell + o(1), \qquad k \to \infty.$$

This will be used in the proof of Theorem 5.1 below.

We will only explicitly need (24) below. It implies that we cannot have that all but finitely many best approximations lie in a two-dimensional space, a claim from [4, §4]. The latter is false for three-dimensional subspaces and general real vectors of any dimension $n \geq 3$ (possibly none of the exceptions lie on the Veronese curve though), see [9]. Note further that the ratio in (23) decreases to 1 as τ_k approaches its upper bound from (19).

Proof. Following the method of Davenport and Schmidt [4, § 4] stated for $n = 2$ only but which generalizes to any n by the same argument, we see the following: Whenever consecutive best approximations $P_{k-1}, P_k, \ldots, P_{\ell-1}$ lie in a two-dimensional space, denoting $x_j > 0$ the leading coefficient of P_j we have

$$|x_{k-1}P_k(\xi) - x_k P_{k-1}(\xi)| = |x_{\ell-2}P_{\ell-1}(\xi) - x_{\ell-1}P_{\ell-2}(\xi)|. \tag{25}$$

The choice of the leading coefficient is not critical, the according identity remains true when choosing the coefficient of any other power (the same power for all P_j) by the same underlying determinant argument, not explicitly carried out in [4]. Hence we may assume

$$x_k = H_k$$

as otherwise we choose instead the coefficient for the power that induces H_k and the argument below works analogously. Now by (21), (6) and as $x_{k-1} \leq H_{k-1}$ is obvious, this leads to

$$\begin{aligned} |x_{k-1}P_k(\xi) - x_kP_{k-1}(\xi)| &\geq |\, x_kP_{k-1}(\xi)| - |\, x_{k-1}P_k(\xi)| \\ &\geq |H_kP_{k-1}(\xi)| - |H_{k-1}P_k(\xi)| \\ &\geq |H_kP_{k-1}(\xi)| - |(H_k/2)P_k(\xi)| \\ &\geq |H_kP_{k-1}(\xi)| - |(H_k/2)P_{k-1}(\xi)| \\ &= \frac{1}{2} \cdot H_k|P_{k-1}(\xi)| \\ &= \frac{1}{2}H_k \cdot H_k^{-v_{k-1}/\tau_k} \\ &= \frac{1}{2}H_k^{1-v_{k-1}/\tau_k}. \end{aligned} \tag{26}$$

On the other hand (20), (6) and again $x_j \leq H_j$ for all j imply

$$\begin{aligned} |x_{\ell-2}P_{\ell-1}(\xi) - x_{\ell-1}P_{\ell-2}(\xi)| &\leq |x_{\ell-2}P_{\ell-1}(\xi)| + |x_{\ell-1}P_{\ell-2}(\xi)| \\ &\leq 2\max\{x_{\ell-2}, x_{\ell-1}\}|P_{\ell-2}(\xi)| \\ &\leq 2\max\{H_{\ell-2}, H_{\ell-1}\}|P_{\ell-2}(\xi)| \\ &= 2H_{\ell-1}|P_{\ell-2}(\xi)| \\ &\ll H_{\ell-1}^{1-\widehat{w}_n(\xi)+o(1)}. \end{aligned} \tag{27}$$

Combining the three claims (25), (26), (27) gives the estimate (22). Since

$$v_{k-1} \leq w_n(\xi) + o(1) \qquad \text{as } k \to \infty \text{ by (8), we infer (23).}$$

The last claim (24) follows in turn from (23) via

$$\frac{\log H_\ell}{\log H_k} = \frac{\log H_{\ell-1}}{\log H_k} \cdot \frac{\log H_\ell}{\log H_{\ell-1}} \leq \frac{\frac{w_n(\xi)}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} \cdot \tau_\ell + o(1), \qquad k \to \infty.$$

The lemma is proved. □

3. Parametric geometry of numbers: Introduction and a lemma

Our final prerequisite lemma will be formulated in the language of parametric geometry of numbers, which will also be used in the proofs of the main claims Theorem 4.1, 5.1 below. We only give a brief summary of the most important notation. We consider the classical lattice point problem induced by linear form (polynomial) approximation to $(\xi, \xi^2, \dots, \xi^{2n-2})$, as in [15]. Concretely,
we consider the successive minima functions $\kappa_j(Q), 1 \le j \le 2n-1$, with respect to the lattice and parametrised family of convex bodies of constant volume

$$\Lambda_\xi = \left\{ \left(a_1, \dots, a_{2n-2}, a_0 + a_1\xi + \dots + a_{2n-2}\xi^{2n-2}\right) \in \mathbb{R}^{2n-1} : a_i \in \mathbb{Z} \right\},$$

$$K(Q) = \left[-Q^{\frac{1}{2n-2}},\, Q^{\frac{1}{2n-2}}\right]^{2n-2} \times \left[-Q^{-1},\, Q^{-1}\right] \subseteq \mathbb{R}^{2n-1}, \qquad Q > 1,$$

in $\mathbb{R}^{2n-1}$ and we derive the parametric functions

$$L_j(q) = \log \kappa_j(e^q), \qquad q > 0,\ 1 \le j \le 2n-1.$$

These are piecewise linear with slopes among $\{-1/(2n-2), 1\}$, as locally they are realised by the trajectory of some integer polynomial P of degree at most $2n-2$ and height H_P, defined as

$$L_P(q) = \max \left\{ \log H_P - \frac{q}{2n-2},\ \log |P(\xi)| + q \right\}. \tag{28}$$

In other words, for each $q \ge 0$ there are linearly independent polynomials

$$P^{(1)}, \dots, P^{(2n-1)}$$

as above (depending on q) with integer coefficients and of heights

$$H^{(1)}, \dots, H^{(2n-1)},$$

so that

$$L_j(q) = L_{P^{(j)}}(q) = \max \left\{ \log H^{(j)} - \frac{q}{2n-2},\ \log |P^{(j)}(\xi)| + q \right\}, \qquad 1 \le j \le 2n-1. \tag{29}$$

The polynomial $P^{(1)}$ minimizes L_P over all relevant choices of P, hence

$$L_1(q) = \min L_P(q) = \min \max \left\{ \log H_P - \frac{q}{2n-2},\ \log |P(\xi)| + q \right\}$$

with the minimum taken over all non-zero integer polynomials P of degree at most $2n-2$.

In Lemma 3.1 as well as in Theorems 4.1, 5.1 below, we will consider the trajectories (28) for $P = P_j$ the best approximation polynomials of degree at most n (with their natural inclusion in the set of polynomials of degree at most $2n-2$) as in Definition 2.1. Hence

$$L_{P_j}(q) = \max\left\{\log H_j - \frac{q}{2n-2},\ \log|P_j(\xi)| + q\right\}, \quad j \geq 1. \tag{30}$$

More generally, we will consider $L_P(q)$ for $P \in \mathscr{V}_k$ as defined in (9). We recall that Minkowski's Second Convex Body Theorem is equivalent to

$$\left|\sum_{j=1}^{2n-1} L_j(q)\right| = O(1). \tag{31}$$

The constant depends on n only. We recall that the quality of approximation $|P(\xi)|$ induced by some integer polynomial P is essentially encoded by the quotient $L_P(q)/q$ at its unique global minimum point q where the rising and decaying part of the trajectory L_P in (28) coincide. Now we can finally state our lemma.

Lemma 3.1. *Let $n \geq 2$ be an integer and ξ be a real number satisfying* (10). *Let P_{k-1}, P_k be two consecutive minimal polynomials with respect to approximation to $(\xi, \xi^2, \dots, \xi^n)$ as in Definition 2.1. Consider the combined graph with respect to $(\xi, \xi^2, \dots, \xi^{2n-2})$, with induced piecewise linear functions L_{P_j} as in* (30) *and parametric functions $L_1, \dots, L_{2n-1}$. Let q_k be the unique point, where $L_{P_{k-1}}(q_k) = L_{P_k}(q_k)$ and $s_k < q_k$ be the place where $L_{P_{k-1}}$ is minimized, i.e., so that*

$$L_{P_{k-1}}(s_k) = \min_{q>0} L_{P_{k-1}}(q).$$

Then

$$L_{2n-1}(q_k) \geq -(2n-2) \cdot L_{P_{k-1}}(q_k) + \frac{2n^2-5n+2}{2n-2}(q_k - s_k) - O(1).$$

The gain compared to the method in [15, Theorem 1.1] is the non-negative expression $\big((2n^2-5n+2)/(2n-2)\big) \cdot (q_k - s_k)$. If τ_k exceeds some $\theta > 1$ strictly, then the estimate $q_k - s_k \gg_\theta q_k$ can be verified, leading to an improvement for $n \geq 3$. On the other hand, the identity $2n^2 - 5n + 2 = 0$ for $n = 2$ naturally agrees with the bound (2) for $n = 2$ from [5] (also obtained with different proofs in [3] and [15]) being optimal.

The proof of Lemma 3.1 is rather straightforward.

P r o o f. Since the slope of any trajectory is at least $-1/(2n-2)$, we clearly have

$$L_{2n-1}(q_k) \geq L_{2n-1}(s_k) - \frac{q_k - s_k}{2n-2}. \tag{32}$$

On the other hand, it is clear that in the interval (s_k, q_k) the function $L_{P_{k-1}}$ increases with slope 1, whereas L_{P_k} has slope $-1/(2n-2)$. Thus

$$L_{P_{k-1}}(q_k) + L_{P_k}(q_k) \geq L_{P_{k-1}}(s_k) + L_{P_k}(s_k) + \left(1 - \frac{1}{2n-2}\right)(q_k - s_k). \quad (33)$$

Since we assume (10), we may apply Lemma 2.2 to see that $\mathscr{V}_{k-1} \cup \mathscr{V}_k$ span a space of dimension $2n-2$ (hyperplane). We may assume $\xi \in (0,1)$ so that

$$|P_j(\xi)| = \max |P(\xi)|$$

with maximum taken over $P \in \mathscr{V}_j$, for all j. Since the heights of all $P \in \mathscr{V}_j$ coincide (all equal to H_j), via (28) in particular this implies $L_{P_j}(q)$ maximise $L_P(q)$ on $(0,\infty)$ among all $P \in \mathscr{V}_j$, for all $j \geq 1$, i.e.,

$$L_{P_j}(q) = \max_{P \in \mathscr{V}_j} L_P(q), \qquad q > 0,\ j \geq 1. \quad (34)$$

Then (33) implies

$$\begin{aligned}
\sum_{j=1}^{2n-2} L_j(s_k) \leq \sum_{P \in \mathscr{V}_{k-1} \cup \mathscr{V}_k} L_P(s_k) &\leq (n-1)\big(L_{P_{k-1}}(s_k) + L_{P_k}(s_k)\big) \\
&\leq (n-1)\left(L_{P_{k-1}}(q_k) + L_{P_k}(q_k) - \left(1 - \frac{1}{2n-2}\right)(q_k - s_k)\right) \\
&= (n-1)\left(2L_{P_k}(q_k) - \left(1 - \frac{1}{2n-2}\right)(q_k - s_k)\right) \\
&= (2n-2)L_{P_k}(q_k) - \left(n - \frac{3}{2}\right)(q_k - s_k).
\end{aligned}$$

Thus by (31)

$$\begin{aligned}
L_{2n-1}(s_k) &\geq -\sum_{j=1}^{2n-2} L_j(s_k) - O(1) \\
&\geq -(2n-2)L_{P_k}(q_k) + \left(n - \frac{3}{2}\right)(q_k - s_k) - O(1). \quad (35)
\end{aligned}$$

Now combining (32), (35) gives

$$\begin{aligned}
L_{2n-1}(q_k) &\geq -(2n-2)L_{P_{k-1}}(q_k) + \left(n - \frac{3}{2} - \frac{1}{2n-2}\right)(q_k - s_k) - O(1) \\
&= -(2n-2)L_{P_{k-1}}(q_k) + \frac{2n^2 - 5n + 2}{2n-2}(q_k - s_k) - O(1),
\end{aligned}$$

the claimed inequality. □

4. A bound for $\widehat{w}_n(\xi)$ increasing in τ_k

The method of the proof of [15, Theorem 1.1] essentially implicitly gives the following estimate.

Theorem 4.1. *Assume $n \geq 2$ is an integer and ξ is a real number. Let $k \geq 2$ be an integer and τ_k be defined as in* (18) *and μ_k as in* (7). *Then for all good k in terms of Definition 1.2, we have*

$$\tau_{k+1} \geq \mu_k - (2n-3).$$

By (8) *we conclude that for any $\varepsilon > 0$ and all good $k \geq k_0(\varepsilon)$ we have*

$$\tau_{k+1} \geq \widehat{w}_n(\xi) - (2n-3) - \varepsilon. \tag{36}$$

Proof. We may assume (10) as otherwise the claim is obvious by $\tau_k > 1$. We use the parametric geometry of numbers setup introduced in § 3. Very similarly to the proof of [15, Theorem 1.1], at the position q_k, where the trajectories $L_{P_{k-1}}$ and L_{P_k} meet we have

$$L_{2n-2}(q_k) \leq L_{P_k}(q_k) = \frac{2n-2-\mu_k}{(2n-2)(1+\mu_k)} \cdot q_k. \tag{37}$$

The left inequality stems from Lemma 2.2. Indeed, it provides $2n-2$ linearly independent integer polynomials P one of them being P_k, each of which have small evaluations in absolute value at ξ. By definition of L_{2n-2}, this means that $L_{2n-2}(q_k)$ is at most the maximum of the $L_P(q_k)$ taken over the $2n-2$ polynomials P in question. Assuming without loss of generality $\xi \in (0,1)$, then by $L_{P_{k-1}}(q_k) = L_{P_k}(q_k)$, we see that $L_{P_k}(q_k)$ maximizes the numbers $L_P(q_k)$ in question. Then the inequality is immediate. For the latter identity in (37), notice that it is clear that L_{P_k} decreases with slope $-1/(2n-2)$ up to this point q_k whereas $L_{P_{k-1}}$ increases with slope 1 on some left neighborhood of q_k. Hence by (30) we have

$$\log|P_{k-1}(\xi)| + q_k = L_{P_{k-1}}(q_k) = L_{P_k}(q_k) = \log H_k - \frac{q_k}{2n-2} \tag{38}$$

and the identity can be derived with some calculation and the definition of μ_k. We remark that by (8), we may deduce

$$L_{2n-2}(q_k) \leq L_{P_k}(q_k) \leq \left(\frac{2n-2-\widehat{w}_n(\xi)}{(2n-2)(1+\widehat{w}_n(\xi))} + o(1) \right) \cdot q_k, \qquad k \to \infty,$$

which is precisely [15, (49)]. Combining (37) with the rightmost identity of (38) we get

$$\log H_k \leq \left(\frac{2n-2-\mu_k}{(2n-2)(1+\mu_k)} + \frac{1}{2n-2} \right) \cdot q_k. \tag{39}$$

On the other hand, again very similar to [15] from (31), (37) we get

$$L_{2n-1}(q_k) \geq -(2n-2)L_{2n-2}(q_k) - O(1) \geq \frac{\mu_k - 2n + 2}{1 + \mu_k} \cdot q_k.$$

Let P of height H_P denote the integer polynomial realizing the last minimum at q_k, i.e., $L_P(q_k) = L_{2n-1}(q_k)$. Since L_P is easily seen to decrease until q_k with slope $-1/(2n-2)$ (see the proof of [15, Theorem 1.1] for details) by (28) we get

$$\begin{aligned} \log H_P &= L_P(q_k) + \frac{q_k}{2n-2} = L_{2n-1}(q_k) + \frac{q_k}{2n-2} \\ &\geq \left(\frac{\mu_k - 2n + 2}{1 + \mu_k} + \frac{1}{2n-2} \right) \cdot q_k. \end{aligned} \tag{40}$$

Using the linear independence assumption and Lemma 2.3, we will now show that

$$H_{k+1} \geq H_P. \tag{41}$$

Assume otherwise $H_{k+1} < H_P$. By Lemma 2.3 we have $T^j P_{k+1}(T)$ not in the span of $\mathscr{V}_{k-1} \cup \mathscr{V}_k$ for some $0 \leq j \leq n-2$. We may assume $j = 0$ the other cases work analogously. Then since both $L_{P_{k+1}}$ and L_P clearly decrease until q_k, from (28) we get

$$L_{P_{k+1}}(q_k) = \log H_{k+1} - \frac{q_k}{2n-2} < \log H_P - \frac{q_k}{2n-2} = L_P(q_k) = L_{2n-1}(q_k),$$

contradicting the linear independence of

$$\mathscr{V}_{k-1} \cup \mathscr{V}_k \cup \{P_{k+1}\}.$$

So we have proved (41). Combining the estimates (39), (40), (41), we get

$$\tau_{k+1} = \frac{\log H_{k+1}}{\log H_k} \geq \frac{\log H_P}{\log H_k} \geq \frac{\frac{\mu_k - 2n+2}{1+\mu_k} + \frac{1}{2n-2}}{\frac{2n-2-\mu_k}{(2n-2)(1+\mu_k)} + \frac{1}{2n-2}}.$$

Finally, the right hand side can be simplified to the desired expression

$$\mu_k - (2n-3).$$ □

Remark 5. If we combine (36) with (19) for index $k+1$, we get the upper bound for $\widehat{w}_n(\xi)$ from [15, Theorem 1.1]. Recall this bound is weaker than α_k from § 1.1.

Note that we only used Lemma 2.2 and Lemma 2.3, the latter can even be omitted upon introducing a more complicated argument, see Case 2 from the proof of [15, Theorem 1.1].

From τ_k given in (18), derive

$$\overline{\tau} := \limsup_{k \to \infty} \tau_k. \tag{42}$$

For the concern of Theorem 1.1, we will only apply Theorem 4.1 in form of the following corollary.

Corollary 4.2. *Assume $n \geq 2$ is an integer and ξ is a real number satisfying* (10). *Then*

$$\widehat{w}_n(\xi) \leq \overline{\tau} + 2n - 3.$$

Proof. Since infinitely many k are good (as follows from Lemma 2.5, or Davenport and Schmidt's method [4] extended to general n), considering such a subsequence, the estimate follows directly from Theorem 4.1. □

5. A bound for $\widehat{w}_n(\xi)$ decreasing in τ_k, τ_ℓ

The main new contribution to improve the upper bound for $\widehat{w}_n(\xi)$ is the following slightly technical result whose proof uses all three new lemmas from § 2 and § 3.

Theorem 5.1. *Let $n \geq 2$ be an integer and ξ be a real number satisfying* (10). *For any $\varepsilon > 0$, there exists $k_0(\varepsilon) > 0$ such that for any integer $k \geq k_0(\varepsilon)$ satisfying* (21), *if $\ell = \ell(k)$ is as in Definition 2.4 and with $\tau_.$ defined in* (18), *we have*

$$\Theta_n = \Theta_n(\widehat{w}_n(\xi), \tau_k, \tau_\ell) \leq \varepsilon,$$

where

$$\Theta_n = d_3 \widehat{w}_n(\xi)^3 + d_2 \widehat{w}_n(\xi)^2 + d_1 \widehat{w}_n(\xi) + d_0$$

with $d_j = d_j(n, \tau_k, \tau_\ell)$ given by

$$\begin{aligned}
d_3 &= \tau_k, \\
d_2 &= -(2n\tau_k + n - 2), \\
d_1 &= \tau_k \tau_\ell + (n^2 + n - 1)\tau_k + (1 - n)\tau_\ell + n^2 - n - 2, \\
d_0 &= -n \cdot \big((n - 1 + \tau_\ell)\tau_k + n - 2\big).
\end{aligned}$$

Remark 6. Again the assumption (21) is probably not necessary.

We prove the theorem.

Proof of Theorem 5.1. We again consider the combined graph with respect to approximation to $(\xi, \xi^2, \ldots, \xi^{2n-2})$, with notation as in § 3. Take a sequence of k satisfying (21). Recall q_k is the first coordinate of the meeting point of $L_{P_{k-1}}$ and L_{P_k}. Define $\Omega_k < 0$ by

$$\Omega_k := \frac{L_{P_{k-1}}(q_k)}{q_k} = \frac{L_{P_k}(q_k)}{q_k}. \tag{43}$$

As in the proof of Theorem 4.1, again by [15, (49)] we have

$$\Omega_k \leq \frac{2n-2-\widehat{w}_n(\xi)}{(2n-2)(1+\widehat{w}_n(\xi))} + o(1), \qquad k \to \infty. \tag{44}$$

From Lemma 3.1 similarly to the proof of Theorem 4.1 we get

$$\begin{aligned} L_{2n-1}(q_k) &\geq -(2n-2)L_{2n-2}(q_k) + (q_k - s_k)\frac{2n^2-5n+2}{2n-2} - O(1) \\ &\geq -(2n-2)\Omega_k \cdot q_k + (q_k - s_k)\frac{2n^2-5n+2}{2n-2} - O(1) \\ &= \left(\frac{2n^2-5n+2}{2n-2} - (2n-2)\Omega_k\right) q_k - \frac{2n^2-5n+2}{2n-2} s_k - O(1). \end{aligned} \tag{45}$$

We next provide an estimate for s_k in terms of q_k and τ_k. Notice first that the definition of q_k implies that L_{P_k} decays with slope $-1/(2n-2)$ on the interval $(0, q_k)$, so (28) and (43) imply

$$\log H_k = L_{P_k}(q_k) + \frac{q_k}{2n-2} = q_k \cdot \left(\frac{1}{2n-2} + \Omega_k\right). \tag{46}$$

Since $L_{P_{k-1}}$ decreases with slope $-1/(2n-2)$ up to s_k and increases with slope 1 from s_k to q_k by (28), we get

$$\begin{aligned} L_{P_{k-1}}(q_k) &= \log H_{k-1} - \frac{s_k}{2n-2} + (q_k - s_k) \\ &= \frac{\log H_k}{\tau_k} - \frac{s_k}{2n-2} + (q_k - s_k) \\ &\leq \left(\frac{\frac{1}{2n-2} + \Omega_k}{\tau_k}\right) q_k - \frac{s_k}{2n-2} + (q_k - s_k) \\ &= \left(\frac{\frac{1}{2n-2} + \Omega_k}{\tau_k} + 1\right) q_k - \frac{2n-1}{2n-2} s_k. \end{aligned} \tag{47}$$

On the other hand, by (43) we have

$$L_{P_k}(q_k) = L_{P_{k-1}}(q_k) = \Omega_k q_k. \tag{48}$$

Comparing (47), (48) and solving for s_k we get

$$s_k \leq \frac{2n-2}{2n-1} \cdot \left(\frac{\frac{1}{2n-2} + \Omega_k}{\tau_k} + 1 - \Omega_k\right) \cdot q_k. \tag{49}$$

Note that the bound for s_k becomes q_k if $\tau_k = 1$, an intuitively expected result. Inserting (49) in (45) and dividing by q_k we get

$$\begin{aligned}\frac{L_{2n-1}(q_k)}{q_k} &\geq \frac{2n^2-5n+2}{2n-2} - (2n-2)\Omega_k \\ &\quad - \frac{2n^2-5n+2}{2n-2}\cdot\frac{2n-2}{2n-1}\cdot\left(\frac{\frac{1}{2n-2}+\Omega_k}{\tau_k}+1-\Omega_k\right) \\ &= \frac{2n^2-5n+2}{2n-2} - (2n-2)\Omega_k \\ &\quad - \frac{2n^2-5n+2}{2n-1}\cdot\left(\frac{\frac{1}{2n-2}+\Omega_k}{\tau_k}+1-\Omega_k\right).\end{aligned}$$

Now by differentiating the bound with respect to Ω_k and observing

$$(2n-2)(2n-1) = 4n^2-6n+2 \geq 2n^2-5n+2 \quad \text{for } \ n \geq 1,$$

we readily check that this lower bound decreases in the variable Ω_k. Hence we may assume asymptotic equality in (44) as $k \to \infty$. Inserting, after rearrangements this leads to the estimate

$$\frac{L_{2n-1}(q_k)}{q_k} \geq \frac{(2n-2)\tau_k\cdot\widehat{w}_n(\xi) - (2n^2-3n+2)\tau_k - (2n^2-5n+2)}{(2n-2)\tau_k(1+\widehat{w}_n(\xi))} - \frac{\epsilon}{2}, \quad (50)$$

for $\epsilon > 0$ and any large enough $k \geq k_0(\epsilon)$.

We now aim to find a reverse upper bound for $L_{2n-1}(q_k)/q_k$. With $\ell = \ell(k)$ as in Definition 2.4, note that P_{k-1}, P_k, P_ℓ are linearly independent and so by Lemma 2.3 the polynomials $\mathscr{V}_{k-1} \cup \mathscr{V}_k \cup \mathscr{V}_\ell$ span the space of polynomials of degree at most $2n-2$ (Lemma 2.3 holds for any linearly independent triple of P_j, see Remark 2). Thus $T^j P_\ell(T)$ does not lie in the span of $\mathscr{V}_{k-1} \cup \mathscr{V}_k$ for some $0 \leq j \leq n-2$. We may assume for simplicity $j = 0$ so that $P_\ell \in \mathscr{V}_\ell$ does not belong to the span of $\mathscr{V}_{k-1} \cup \mathscr{V}_k$, the other cases work very similarly. So $\mathscr{V}_{k-1} \cup \mathscr{V}_k \cup \{P_\ell\}$ are linearly independent. Now

$$L_{P_\ell}(q_k) \ > \ 0 \ > \max_{P\in\mathscr{V}_{k-1}\cup\mathscr{V}_k} L_P(q_k) = L_{P_k}(q_k)$$

is easily verified where the last identity comes from (34) and $L_{P_{k-1}}(q_k) = L_{P_k}(q_k)$ by definition of q_k. Moreover, again by (28), the function L_{P_ℓ} obviously decays with slope $-1/(2n-2)$ in the interval $(0, q_k)$ as its minimum is taken at some $q_\ell > q_k$. Combining all these facts we get

$$L_{2n-1}(q_k) \leq \max_{P\in\mathscr{V}_{k-1}\cup\mathscr{V}_k\cup\{P_\ell\}} L_P(q_k) = L_{P_\ell}(q_k) = \log H_\ell - \frac{q_k}{2n-2}. \qquad (51)$$

To bound the right hand side we next estimate H_ℓ. Note that we may apply Lemma 2.5 by assumption (21). By applying in this order (24) of Lemma 2.5, (46), (44), and (11), we get

$$\begin{aligned}
\log H_\ell &\leq \left(\frac{\frac{w_n(\xi)}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} \cdot \tau_\ell + o(1) \right) \cdot \log H_k \\
&\leq \left(\frac{\frac{w_n(\xi)}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} \cdot \tau_\ell \left(\frac{1}{2n-2} + \Omega_k \right) + o(1) \right) \cdot q_k \\
&\leq \left(\frac{\frac{w_n(\xi)}{\tau_k} - 1}{\widehat{w}_n(\xi) - 1} \cdot \tau_\ell \cdot \frac{2n-1}{(2n-2)(1+\widehat{w}_n(\xi))} + o(1) \right) \cdot q_k \\
&\leq \left(\frac{(n-1)\widehat{w}_n(\xi) - \tau_k(\widehat{w}_n(\xi) - n)}{\tau_k(\widehat{w}_n(\xi) - 1)(\widehat{w}_n(\xi) - n)} \cdot \frac{(2n-1)\tau_\ell}{(2n-2)(1+\widehat{w}_n(\xi))} + o(1) \right) \cdot q_k.
\end{aligned} \tag{52}$$

See also Remark 4. Inserting in (51) and dividing by q_k yields

$$\frac{L_{2n-1}(q_k)}{q_k} \leq \Delta - \frac{1}{2n-2} + \frac{\epsilon}{2} \tag{53}$$

where

$$\Delta = \frac{(n-1)\widehat{w}_n(\xi) - \tau_k(\widehat{w}_n(\xi) - n)}{\tau_k(\widehat{w}_n(\xi) - 1)(\widehat{w}_n(\xi) - n)} \cdot \frac{(2n-1)\tau_\ell}{(2n-2)(1+\widehat{w}_n(\xi))},$$

for $\epsilon > 0$ and $k \geq k_1(\epsilon)$ again.

Comparing the lower bound (50) and the upper bound (53) obtained for $L_{2n-1}(q_k)/q_k$ gives a relation involving n, τ_k, τ_ℓ and $\widehat{w}_n(\xi)$ of the form

$$\frac{(2n-2)\tau_k \cdot \widehat{w}_n(\xi) - (2n^2-3n+2)\tau_k - (2n^2-5n+2)}{(2n-2)\tau_k(1+\widehat{w}_n(\xi))} \leq$$

$$\frac{(n-1)\widehat{w}_n(\xi) - \tau_k(\widehat{w}_n(\xi) - n)}{\tau_k(\widehat{w}_n(\xi) - 1)(\widehat{w}_n(\xi) - n)} \cdot \frac{(2n-1)\tau_\ell}{(2n-2)(1+\widehat{w}_n(\xi))} - \frac{1}{2n-2} + \epsilon,$$

for $k \geq \max\{k_0(\epsilon), k_1(\epsilon)\}$. Subtracting the right side from the left, multiplying

with the common denominator and some rearrangements lead to the estimate in Theorem 5.1 upon modifying ϵ. □

6. Deduction of Theorem 1.1 from Theorems 4.1, 5.1

Recall that we want to prove the claim indirectly, assuming (12) holds for some ξ and leading it to a contradiction. Hence, as $\beta_n > 2n-2$, we can assume the condition (10) of Theorem 5.1 is satisfied. Recall $\overline{\tau}$ from (42). Notice this quantity depends on ξ, n only. We distinguish two cases.

Case 1 ($\overline{\tau} = 1.$). Then Corollary 4.2 directly implies

$$\widehat{w}_n(\xi) \leq \overline{\tau} + 2n - 3 = 2n - 2 < \beta_n.$$

Case 2 ($\overline{\tau} > 1.$). We may assume the sequence $\overline{\tau} < \infty$ in view of (19), otherwise again the strengthened bound $\widehat{w}_n(\xi) \leq 2n-2 < \beta_n$ follows immediately. Choose a sequence of values k converging to $\overline{\tau}$. For any large enough k in this sequence, by assumption of Case 2 we have that $\tau_k > \theta > 1$ is strictly bounded away from 1, in particular condition (21) of Lemma 2.5 and Theorem 5.1 holds. Thus for such k we can apply Theorem 5.1. Note that its bound Θ_n increases as a function in he third argument τ_ℓ (but decreases in the argument τ_k). As $\tau_k \to \overline{\tau}$ and clearly $\tau_\ell \leq \overline{\tau} + o(1)$ for $\ell = \ell(k)$ as in Definition 2.4 as $k \to \infty$, we may put $\tau_k = \tau_\ell = \overline{\tau}$ in the inequality claim of Theorem 5.1 upon modifying ϵ. In the limit we may ignore this error term ϵ and we obtain

$$\Theta_n\big(\widehat{w}_n(\xi), \overline{\tau}, \overline{\tau}\big) \leq 0. \tag{54}$$

It can be checked that (54) induces a bound

$$\widehat{w}_n(\xi) \leq F_n(\overline{\tau})$$

for some decreasing function $F_n : (1, \infty) \to \mathbb{R}$. Combined with Corollary 4.2 we get

$$\widehat{w}_n(\xi) \leq \min\{\overline{\tau} + 2n - 3, F_n(\overline{\tau})\}.$$

Now the left bound increases while the right bound decreases in $\overline{\tau}$, hence the maximum (worst case) of the above minimum is obtained for the equilibrium $\overline{\tau}$ where the expressions coincide. Hence we can put

$$\overline{\tau} = \widehat{w}_n(\xi) - 2n + 3$$

in the defining equation (54) of F_n and solve for equality

$$\Theta_n(\widehat{w}_n(\xi), \overline{\tau}, \overline{\tau}) = \Theta_n(\widehat{w}_n(\xi), \widehat{w}_n(\xi) - 2n + 3, \widehat{w}_n(\xi) - 2n + 3) = 0$$

in the variable $\widehat{w}_n(\xi)$. After simplifications, this gives a quartic polynomial equation $Q_n(\widehat{w}_n(\xi)) = 0$ for $Q_n(T)$ precisely the polynomial in Theorem 1.1. Its largest root β_n is checked to be in the interval $(2n-2, 2n-1)$ and so Theorem 1.1 is proved.

7. Sketch of the proof of Theorem 1.3

We establish a variant of Theorem 5.1. Assume first hypothesis (i) of Theorem 1.3 holds and fix any large good k. Then $\ell = k+1$ in Definition 2.4, hence in (52) within the proof of Theorem 5.1 the factor

$$\big(w_n(\xi)/\tau_k - 1\big)/\big(\widehat{w}_n(\xi) - 1\big)$$

derived from Lemma 2.5 can be omitted (replaced by 1). Proceeding as in the proof of Theorem 5.1 this results in an estimate

$$\tilde{\Theta}_n = \tilde{\Theta}_n(\widehat{w}_n(\xi), \tau_k, \tau_\ell) \leq \epsilon, \tag{55}$$

for $\epsilon > 0$ provided that $k \geq k_0(\epsilon)$ is good, where

$$\tilde{\Theta}_n := (2n-2)\tau_k \widehat{w}_n(\xi) - (2n^2 - 3n + 2)\tau_k - \\ (2n^2 - 5n + 2) - \tau_k\big((2n-1)\tau_\ell - 1 - \widehat{w}_n(\xi)\big).$$

This again defines an inequality $\widehat{w}_n(\xi) \leq \tilde{F}_n(\tau_k, \tau_\ell) + o(1)$ with some function $\tilde{F}_n$ decreasing in τ_k and increasing in τ_ℓ. Now we estimate τ_ℓ as in (19) in terms of $\widehat{w}_n(\xi)$ to derive a bound $\widehat{w}_n(\xi) \leq \tilde{G}_n(\tau_k) + o(1)$ for some decreasing one--parameter function $\tilde{G}_n$. Since linear independence is assumed for P_{k-2}, P_{k-1}, P_k as well, on the other hand, we may apply Theorem 4.1 for index $k-1$ and combining we get

$$\widehat{w}_n(\xi) \leq \min\{\tau_k + 2n - 3, \tilde{G}_n(\tau_k)\} + o(1).$$

We let $k \to \infty$ within our subsequence to make the error term disappear, and the worst case scenario happens when the bounds coincide, which in particular means $\tau_k = \widehat{w}_n(\xi) - 2n + 3$. Plugging this into $\tilde{\Theta}_n$ yields

$$\tilde{\Theta}_n \left(\widehat{w}_n(\xi), \widehat{w}_n(\xi) - 2n + 3, \frac{n-1}{\widehat{w}_n(\xi) - n} \right) \leq 0$$

and the claimed bound γ_n follows after some rearrangements.

Now as in (ii) assume the linear independence hypothesis holds for all large k. For the same reasons as in § 6 we may again assume the sequence $(\tau_k)_{k\geq 1}$ is bounded from above. Then choosing a subsequence of k with $\tau_k \to \overline{\tau} < \infty$, by (55) and as $\tilde{\Theta}$ increases in τ_ℓ we obtain

$$\tilde{\Theta}_n = \tilde{\Theta}_n\big(\widehat{w}_n(\xi), \overline{\tau}, \overline{\tau}\big) \leq \epsilon_1, \qquad k \geq k_1(\epsilon_1). \tag{56}$$

This results in a bound of the form $\widehat{w}_n(\xi) \leq \tilde{H}_n(\overline{\tau}) + o(1)$ for the function

$$\tilde{H}_n(x) = x + n - 1 + \frac{n-2}{x}.$$

This function has a local minimum at $x=\sqrt{n-2}$ and decreases on $x\in(1,\sqrt{n-2})$, if non-empty, and increases for $x>\sqrt{n-2}$. Moreover, $\tilde{H}_n(x)$ can be checked to be smaller than $x+2n-3$ from Theorem 4.1 for all $x>1$. By these observations and since $\overline{\tau}\geq 1$, it follows from (19) that

$$\widehat{w}_n(\xi)\leq\max\left\{\tilde{H}_n(1),\tilde{H}_n\left(\frac{n-1}{\widehat{w}_n(\xi)-n}\right)\right\}=\max\left\{2n-2,\tilde{H}_n\left(\frac{n-1}{\widehat{w}_n(\xi)-n}\right)\right\},$$

where the left bound applies if $\overline{\tau}\leq\sqrt{n-2}$ and the right bound applies if $\overline{\tau}>\sqrt{n-2}$. In case that the right hand side value is the maximum, rearrangements lead to the bound

$$\widehat{w}_n(\xi)\leq\frac{\sqrt{5}+1}{2}n-\frac{\sqrt{5}-1}{2}.$$

Finally, it is checked that precisely for integers $n\geq 4$ this is smaller than $2n-2$.

Remark 7. Assume a slight generalization of Lemma 2.3, namely that the polynomials need not be irreducible of exact degree n (which was implied by assumption (10) via Lemma 2.2) for the implication. Then the proof above shows that, for $n\geq 4$ we could improve the bound $\rho_n=2n-2$ in the setting (ii) if we can settle a non-trivial lower bound for $\overline{\tau}$. A related estimate for simultaneous approximation is obtained in [8], however the result for linear forms is not implied immediately. Thus for $n\geq 10$ we would improve the best known bounds α_n in (3) from [3].

Acknowledgement. The paper originates in the author's visit to the University of Sydney in the International Visitor Program in 2023. The author thanks the University of Sydney and his host Dmitry Badziahin for the hospitality. Further, the author thanks the referee for the fast and careful reading of the article and for providing several suggestions, mostly improving linguistics.

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Received October 7, 2024
Accepted October 17, 2024

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