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SUMSETS WITH PRESCRIBED LOWER AND UPPER ASYMPTOTIC DENSITIES II

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ABSTRACT. We show that for natural numbers h, k, such that $k \geq 2$, and for real numbers α_1, α_2, such that $0 < \alpha_1 \leq \alpha_2 < 1$, there exists a subset A of $\mathbb{N} = \{0, 1, 2, \ldots\}$, such that for all finite subsets B of $\mathbb{N}$ such that $\max B - \min B = h$, the lower and upper asymptotic densities of $kA + B$ are α_1 and α_2, respectively.

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1. Introduction

Let $\mathbb{N}$ denote the set of natural numbers $\{0, 1, 2, \ldots\}$. For any subset $A \subseteq \mathbb{N}$, we define

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n\}|}{n}$$

to be the lower asymptotic density of A and

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n\}|}{n}$$

to be the upper asymptotic density of A. If $\underline{d}(A) = \overline{d}(A) = \alpha$, then we put $d(A) = \alpha$ and say that A has natural density α. For two sets $A, B \subseteq \mathbb{N}$, we define the sumset

$$A + B := \{a + b : a \in A, b \in B\}.$$

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For any k subsets $A_1, \dots, A_k$ of $\mathbb{N}$, we define

$$A_1 + \dots + A_k := \{n_1 + \dots + n_k : n_i \in A_i,\ \forall 1 \le i \le k\}.$$

If $A = A_1 = A_2 = \dots = A_k$, then we put

$$kA = A_1 + \dots + A_k = \{n_1 + \dots + n_k : n_i \in A,\ \ \forall 1 \le i \le k\}.$$

In [1], it was proved that for any real number $\alpha \in [0,1]$ and positive integer k, there exists a subset of natural numbers A such that kA is of natural density α. In the same paper, seven questions were raised.

In [2], it was proved that for all real numbers α_1, α_2 such that $0 \le \alpha_1 \le \alpha_2 \le 1$, there exists a subset of natural numbers such that the lower and upper asymptotic density of kA are α_1 and α_2, respectively. This solves the question in [1, Q5].

In this paper, we address the following question that was asked in [1].

[1, Q6]: Given a subset B of $\mathbb{N}$, finite or of zero asymptotic density, a real number α, $0 \le \alpha \le 1$, and an integer $k \ge 2$, is there a set $A \subseteq \mathbb{N}$ such that $d(B + kA) = \alpha$? Search for "thin" and for "thick" such sets A.

In this paper, we deal with finite subsets B. We prove the following theorem.

Theorem 1. *For any two real numbers $0 < \alpha_1 \le \alpha_2 < 1$ and natural numbers h and k, with $k \ge 2$, there exists a set A such that for all finite sets B such that $\max B - \min B = h$ we have*

$$\underline{d}(kA + B) = \alpha_1 \quad \textit{and} \quad \overline{d}(kA + B) = \alpha_2.$$

For the approach in [1], the construction of A such that $d(A + B) = \alpha$ depends heavily on the choice of the set B. Our present construction is much more general: B can be any finite set satisfying very general conditions. Concretely, to construct A, we only need that the quantity $\max B$–$\min B$ is fixed. It is the quantity h in the theorem. Thus B can be any subset of $\{0, 1, \dots, h\}$ containing 0 and h, or any translation of such a set.

2. The proofs

We need some lemmas to prove Theorem 1. For any real numbers a, b such that $a \le b$, let $[\![a, b]\!]$ denote the set $[a, b] \cap \mathbb{Z}$. The proof of the following lemma is similar to the proof of [2, Lemma 1].

Lemma 2.1. *Let p, q, h, k be natural numbers such that $k \geq 2$, $p \geq 1$, and $q \geq k(p-1)+h+1$. Let α, β be real numbers such that $\alpha > k\beta$ and $\beta > 2$ and let $A_n = [\alpha^n, \beta\alpha^n] \cap (q.\mathbb{N} + [\![0, p-1]\!])$ and $A = \bigcup_{n=1}^{\infty} A_n$. Then for any finite set of natural numbers B such that $\max B - \min B = h$ we have*

$$\underline{d}(kA+B) = \frac{(k\beta-1)(k(p-1)+1+h)}{(\alpha-1)q}$$

and

$$\overline{d}(kA+B) = \frac{\alpha(k\beta-1)(k(p-1)+1+h)}{k\beta(\alpha-1)q}.$$

P r o o f. Let $\min B = m$. Since

$$\underline{d}(kA+B) = \underline{d}\big(kA+(B-m)\big) \quad \text{and} \quad \overline{d}(kA+B) = \overline{d}\big(kA+(B-m)\big),$$

without loss of generality we can assume $\min B = 0$ and $\max B = h$. Let

$$K_n = \bigcup_{n_1,\dots,n_{k-1} \leq n} (A_n + A_{n_1} + \cdots + A_{n_{k-1}}).$$

We have $kA = \bigcup_{n=0}^{\infty} K_n$. As $\{0, h\} \subset B \subset [\![0, h]\!]$, we have

$$\bigcup_{n=0}^{\infty} (K_n + \{0, h\}) \subset kA + B \subset \bigcup_{n=0}^{\infty} (K_n + [\![0, h]\!]).$$

As

$$A_n = [\alpha^n, \beta\alpha^n] \cap (q.\mathbb{N} + [\![0, p-1]\!]),$$

each element of $K_n + B = A_n + A_{n_1} + \cdots + A_{n_{k-1}} + B$ is greater than or equal to α^n, for all sufficiently large n. Additionally, each element of A_{n_i} is less than or equal to $\beta\alpha^n$ and each element of B is less than or equal to h, so we have that each element of $K_n + B$ is less than or equal to $k\beta\alpha^n + h$, for all sufficiently large n. This implies that

$$K_n + B \subset [\alpha^n, k\beta\alpha^n + h].$$

Each of the sets $A_n, A_{n_1}, \dots, A_{n_{k-1}}$ is a subset of $q.\mathbb{N} + [\![0, p-1]\!]$ and $B \subset [\![0, h]\!]$ which implies

$$K_n + B \subset q.\mathbb{N} + [\![0, k(p-1)+h]\!].$$

Hence

$$K_n + B \subset [\alpha^n, k\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big). \tag{1}$$

Now consider the least natural number t such that there exists an $s \in \mathbb{N}$ such that $[\![sq, sq+p-1]\!] \subset A_t$. Now for any $n \geq t$ we have

$$\begin{aligned} K_n + B &= \bigcup_{n_1,\ldots,n_{k-1} \leq n} \big(A_n + A_{n_1} + \cdots + A_{n_{k-1}} + B\big) \\ &\supset \bigcup_{i=1}^{k} \big(iA_n + (k-i)A_t + \{0,h\}\big) \\ &\supset \bigcup_{i=1}^{k} \big(iA_n + (k-i)[\![sq, sq+(p-1)]\!] + \{0,h\}\big). \end{aligned}$$

We have

$$\begin{aligned} &iA_n + (k-i)[\![sq, sq+(p-1)]\!] + \{0,h\} \\ &= iA_n + \big[\!\big[(k-i)sq, (k-i)\big(sq+(p-1)\big)\big]\!\big] + \{0,h\} \\ &= i\big([\alpha^n, \beta\alpha^n] \cap (q.\mathbb{N} + [\![0, p-1]\!])\big) \\ &\quad + \big[\!\big[(k-i)sq, (k-i)\big(sq+(p-1)\big)\big]\!\big] + \{0,h\}. \end{aligned}$$

Observe that, except for $O(1)$ exceptions with respect to n, $iA_n + (k-i)[\![sq, sq+(p-1)]\!]$ contains all elements of $[i\alpha^n, i\beta\alpha^n] \cap \big(q.\mathbb{N} + [\![0, k(p-1)]\!]\big)$. Therefore, $iA_n + \big[\!\big[(k-i)sq, (k-i)\big(sq+(p-1)\big)\big]\!\big] + \{0,h\}$, except for $O(1)$ exceptions, contains

$$[i\alpha^n, i\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big).$$

Therefore,

$$K_n + B \supset \bigcup_{i=1}^{k} \Big([i\alpha^n, i\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big) \setminus S_{n,i}\Big)$$

for a set $S_{n,i}$ containing $O(1)$ elements. As $[\alpha^n, \beta\alpha^n + h] \cup [2\alpha^n, 2\beta\alpha^n + h] \cup \cdots \cup [k\alpha^n, k\beta\alpha^n + h] = [\alpha^n, k\beta\alpha^n + h]$ we have

$$K_n + B \supset [\alpha^n, k\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big) \Big\backslash \left(\bigcup_{i=1}^{k} S_{n,i}\right). \tag{2}$$

As $\bigcup_{i=1}^{k} S_{n,i} \subset \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big)$ contains $O(1)$ elements and $K_n + B \subset [\alpha^n, k\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big)$, we can conclude that there exists a positive integer M, independent of n, such that the cardinality of

$$S_n = \Big([\alpha^n, k\beta\alpha^n + h] \cap \big(q \cdot \mathbb{N} + [\![0, k(p-1)+h]\!]\big)\Big) \Big\backslash (K_n + B)$$

is less than or equal to M for all $n \in \mathbb{N}$.

We claim that $S = \bigcup_{n=0}^{\infty} S_n$ is of zero density. Since for any positive integer N, there are $O(\log N)$ values of n, such that $[\alpha^n, k\beta\alpha^n + h] \cap [\![0, N]\!]$ is nonempty, there are $O(\log N)$ values of n, such that $S_n \cap [\![0, N]\!]$ is nonempty. Hence,

$$\frac{|S \cap [\![0,N]\!]|}{N} \leq \sum_{S_n \cap [\![0,N]\!] \neq \emptyset} \frac{|S_n|}{N} \leq M \sum_{S_n \cap [\![0,N]\!] \neq \emptyset} \frac{1}{N} = O\left(\frac{\log N}{N}\right).$$

From $\frac{|S\cap[\![0,N]\!]|}{N} = O\big(\frac{\log N}{N}\big)$, we have $d(S) = \lim_{N\to\infty} \frac{|S\cap[\![0,N]\!]|}{N} = 0$.

Note that it is not hard to see that

$$\underline{d}\left(\bigcup_{n=0}^{\infty} [\alpha^n, k\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big)\right) = \frac{(k\beta-1)(k(p-1)+1+h)}{(\alpha-1)q}$$

and

$$\overline{d}\left(\bigcup_{n=0}^{\infty} [\alpha^n, k\beta\alpha^n + h] \cap \big(q.\mathbb{N} + [\![0, k(p-1)+h]\!]\big)\right) = \frac{\alpha(k\beta-1)(k(p-1)+1+h)}{k\beta(\alpha-1)q}.$$

These can be seen by considering, for example, the subsequence $\alpha^l - 1$, for sufficiently large l, for the lower asymptotic density and the subsequence $k\beta\alpha^l + h - \big(q - k(p-1) - h\big)$, with sufficiently large l, for the upper asymptotic density. By (1), (2) and the fact that $d(S) = 0$, we obtain

$$\underline{d}(kA + B) = \frac{(k\beta-1)(k(p-1)+1+h)}{(\alpha-1)q}$$

and

$$\overline{d}(kA + B) = \frac{\alpha(k\beta-1)(k(p-1)+1+h)}{k\beta(\alpha-1)q}. \qquad \square$$

Lemma 2.2. *Let $\theta > 1$ be any irrational number and let $\delta \leq 1$ be any positive real number. Let k and b be non-negative integers such that $k \geq 2$ and $\{b\theta\} + k\delta \leq 1$. Let $A = \big\{\lfloor n\theta \rfloor : n \in \mathbb{N}\setminus\{0\}, \{n\theta\} \in (0,\delta)\big\}$. For every $\epsilon > 0$,*

$$\left\{\lfloor n\theta \rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1} \big(\{b\theta\} + \epsilon + i\delta, \{b\theta\} - \epsilon + (i+1)\delta\big)\right\} \Big\backslash (\lfloor b\theta \rfloor + kA)$$

is a finite set.

P r o o f. It suffices to show that

$$\big\{\lfloor n\theta\rfloor : \{n\theta\} \in \big(\{b\theta\}+\epsilon+j\delta, \{b\theta\}-\epsilon+(j+1)\delta\big)\big\}\setminus(\lfloor b\theta\rfloor + kA)$$

is a finite set for all natural numbers j satisfying $0 \le j \le k-1$. Let j be a natural number such that $0 \le j \le k-1$. If $\epsilon \ge \frac{\delta}{2}$, then the lemma is trivially true (as $\big\{\lfloor n\theta\rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1}\big(\{b\theta\}+\epsilon+i\delta, \{b\theta\}-\epsilon+(i+1)\delta\big)\big\} = \{\}$).

Let us assume $\epsilon < \frac{\delta}{2}$. Let r be any positive integer such that $\frac{j\delta-\epsilon}{k-1} < \{r\theta\} < \min\{\frac{j\delta+\epsilon}{k-1}, \delta\}$. Notice that $\lfloor r\theta\rfloor \in A$. We will show that

$$\big\{\lfloor n\theta\rfloor : \{n\theta\} \in \big(\{b\theta\}+\epsilon+j\delta, \{b\theta\}-\epsilon+(j+1)\delta\big), n > (k-1)r+b\big\} \subset \lfloor b\theta\rfloor + kA$$

which implies

$$\big\{\lfloor n\theta\rfloor : \{n\theta\} \in \big(\{b\theta\}+\epsilon+j\delta, \{b\theta\}-\epsilon+(j+1)\delta\big)\big\}\setminus(\lfloor b\theta\rfloor + kA)$$

is a finite set.

Let n be any integer such that

$$n > (k-1)r+b \quad \text{and} \quad \{n\theta\} \in \big(\{b\theta\}+\epsilon+j\delta, \{b\theta\}-\epsilon+(j+1)\delta\big).$$

We have

$$\begin{aligned}
&\big\lfloor\big(n-(k-1)r-b\big)\theta\big\rfloor + (k-1)\lfloor r\theta\rfloor + \lfloor b\theta\rfloor - \lfloor n\theta\rfloor \\
&= \big(n-(k-1)r-b\big)\theta + (k-1)r\theta + b\theta - n\theta \\
&\quad - \big\{\big(n-(k-1)r-b\big)\theta\big\} - (k-1)\{r\theta\} - \{b\theta\} + \{n\theta\} \\
&= -\big\{\big(n-(k-1)r-b\big)\theta\big\} - (k-1)\{r\theta\} - \{b\theta\} + \{n\theta\} \\
&> -\big\{\big(n-(k-1)r-b\big)\theta\big\} - (j\delta+\epsilon) - \{b\theta\} + \{b\theta\} + \epsilon + j\delta \\
&= -\big\{\big(n-(k-1)r-b\big)\theta\big\} > -1
\end{aligned}$$

and

$$\begin{aligned}
&\big\lfloor\big(n-(k-1)r-b\big)\theta\big\rfloor + (k-1)\lfloor r\theta\rfloor + \lfloor b\theta\rfloor - \lfloor n\theta\rfloor \\
&= -\big\{\big(n-(k-1)r-b\big)\theta\big\} - (k-1)\{r\theta\} - \{b\theta\}\{n\theta\} \le \{n\theta\} < 1.
\end{aligned}$$

Therefore,

$$\big\lfloor\big(n-(k-1)r-b\big)\theta\big\rfloor + (k-1)\lfloor r\theta\rfloor + \lfloor b\theta\rfloor - \lfloor n\theta\rfloor$$

is an integer in $(-1,1)$. Hence

$$\big\lfloor\big(n-(k-1)r-b\big)\theta\big\rfloor + (k-1)\lfloor r\theta\rfloor + \lfloor b\theta\rfloor - \lfloor n\theta\rfloor = 0$$

and

$$\begin{aligned}
\big\{\big(n-(k-1)r-b\big)\theta\big\} &= \{n\theta\} - \{b\theta\} - (k-1)\{r\theta\} \\
&< \{b\theta\} - \epsilon + (j+1)\delta - \{b\theta\} - (k-1)\frac{j\delta+\epsilon}{k-1} = \delta - 2\epsilon < \delta.
\end{aligned}$$

It follows that

$$\lfloor \big(n-(k-1)r-b\big)\theta \rfloor \in A$$

and

$$(k-1)\lfloor r\theta \rfloor + \lfloor \big(n-(k-1)r-b\big)\theta \rfloor + \lfloor b\theta \rfloor = \lfloor n\theta \rfloor \in kA + \lfloor b\theta \rfloor .$$

Thus

$$\big\{ \lfloor n\theta \rfloor : \{n\theta\} \in \big(\{b\theta\}+\epsilon+j\delta, \{b\theta\}-\epsilon+(j+1)\delta\big), n > (k-1)r+b \big\} \subset \lfloor b\theta \rfloor + kA$$

which implies

$$\big\{ \lfloor n\theta \rfloor : \{n\theta\} \in \big(\{b\theta\}+\epsilon+j\delta, \{b\theta\}-\epsilon+(j+1)\delta\big) \big\} \setminus (\lfloor b\theta \rfloor + kA)$$

is a finite set. □

Lemma 2.3. *Let θ be any irrational number greater than 1. Let $I \subset [0,1]$ be any finite union of open intervals. The density of $\big\{ \lfloor n\theta \rfloor : \{n\theta\} \in I \big\} = \frac{\mu(I)}{\theta}$, where $\mu(I)$ is the Lebesgue measure of I.*

Proof. The uniform distribution of the sequence $(\{n\theta\})_{n\in\mathbb{N}}$ when θ is irrational ([3, p. 8], [4, p. 2–72]), implies

$$d\big(\{ n \in \mathbb{N} : \{n\theta\} \in I \}\big) = \mu(I)$$

and so

$$d\big(\{ \lfloor n\theta \rfloor : \{n\theta\} \in I \}\big) = \frac{\mu(I)}{\theta}. \qquad \square$$

Lemma 2.4. *Let $\theta > 1$ be any irrational number and let k be any positive integer. Let $b_1, \dots, b_r$ be natural numbers and δ be any positive number. Let B be the finite set given by $B = \{\lfloor b_1\theta \rfloor, \dots, \lfloor b_r\theta \rfloor\}$. Let $m_1 = \min\{\{b_i\theta\} : 1 \le i \le r\}$, $m_2 = \max\{\{b_i\theta\} : 1 \le i \le r\}$ and let $A = \big\{ \lfloor n\theta \rfloor : n \in \mathbb{N} \setminus \{0\}, \{n\theta\} \in (0,\delta) \big\}$. If $m_2 + k\delta \le 1$ and $m_1 + k\delta \ge m_2$ then the density of $kA+B$ s equal to $\frac{m_2-m_1+k\delta}{\theta}$.*

Proof. First we will prove that $kA+B$ is subset of $S = \big\{ \lfloor n\theta \rfloor : \{n\theta\} \in (m_1, m_2+k\delta), n \in \mathbb{N} \setminus \{0\} \big\}$. For any element $\lfloor n_1\theta \rfloor + \dots + \lfloor n_k\theta \rfloor + \lfloor b_i\theta \rfloor$ of $kA+B$, we have

$$(n_1 + \dots + n_k + b_i)\theta \ge \lfloor n_1\theta \rfloor + \dots + \lfloor n_k\theta \rfloor + \lfloor b_i\theta \rfloor$$

and

$$\begin{aligned}(n_1 + \dots + n_k + b_i)\theta &= \lfloor n_1\theta \rfloor + \dots + \lfloor n_k\theta \rfloor + \lfloor b_i\theta \rfloor \\ &\quad + \{n_1\theta\} + \dots + \{n_i\theta\} + \{b_i\theta\} \\ &< \lfloor n_1\theta \rfloor + \dots + \lfloor n_k\theta \rfloor + \lfloor b_i\theta \rfloor + k\delta + m_2 \\ &\le \lfloor n_1\theta \rfloor + \dots + \lfloor n_k\theta \rfloor + \lfloor b_i\theta \rfloor + 1.\end{aligned}$$

Hence

$$\lfloor (n_1 + \cdots + n_k + b_i)\theta \rfloor = \lfloor n_1\theta \rfloor + \cdots + \lfloor n_k\theta \rfloor + \lfloor b_i\theta \rfloor$$

and

$$\{(n_1 + \cdots + n_k + b_i)\theta\} = \{n_1\theta\} + \cdots + \{n_i\theta\} + \{b_i\theta\} \in (m_1, m_2 + k\delta)$$

and therefore

$$kA + B \subset \{\lfloor n\theta \rfloor : \{n\theta\} \in (m_1, m_2 + k\delta)\} = S.$$

Hence, by Lemma 2.3,

$$\bar{d}(kA + B) \le \bar{d}(S) = \frac{m_2 - m_1 + k\delta}{\theta}. \tag{3}$$

Without loss of generality, assume $\{b_1\theta\} = m_1$ and $\{b_r\theta\} = m_2$. From Lemma 2.2, it follows that for all $\epsilon > 0$, $kA + B$ contains all, except for finitely many exceptions of the set

$$\Bigg\{ \lfloor n\theta \rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1} \big(\{b_1\theta\} + \epsilon + i\delta, \{b_1\theta\} - \epsilon + (i+1)\delta\big) \cup \bigcup_{i=0}^{k-1} \big(\{b_r\theta\} + \epsilon + i\delta, \{b_r\theta\} - \epsilon + (i+1)\delta\big),\ n \in \mathbb{N} \setminus \{0\} \Bigg\}.$$

We have $S = \{\lfloor n\theta \rfloor : \{n\theta\} \in (\{b_1\theta\}, \{b_r\theta\} + k\delta)\}$. Let

$$T_1 = \left\{ \lfloor n\theta \rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1} [\{b_1\theta\} + i\delta, \{b_1\theta\} + i\delta + \epsilon] \right\},$$

$$T_2 = \left\{ \lfloor n\theta \rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1} [\{b_r\theta\} + i\delta, \{b_r\theta\} + i\delta + \epsilon] \right\},$$

$$T_3 = \left\{ \lfloor n\theta \rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1} [\{b_1\theta\} + (i+1)\delta - \epsilon, \{b_1\theta\} + (i+1)\delta] \right\},$$

and

$$T_4 = \left\{ \lfloor n\theta \rfloor : \{n\theta\} \in \bigcup_{i=0}^{k-1} [\{b_r\theta\} + (i+1)\delta - \epsilon, \{b_r\theta\} + (i+1)\delta] \right\}.$$

Except for finitely many exceptions, $kA + B$ contains $S \setminus \bigcup_{i=1}^{4} T_i$. Since $\underline{d}(S) = \frac{m_2 - m_1 + k\delta}{\theta}$ and for all $1 \le i \le 4$ we have $\overline{d}(T_i) \le k\epsilon$. Hence

$$\underline{d}(kA + B) \ge \frac{m_2 - m_1 + k\delta}{\theta} - 4k\epsilon, \tag{4}$$

for all $\epsilon > 0$. From (3) and (4), we have $d(kA + B) = \frac{m_2 - m_1 + k\delta}{\theta}$. □

Lemma 2.5. *Let α be any real number between 0 and 1 and let h, k be any positive integers. There exists a $\delta > 0$, with $\delta < \frac{\alpha}{k}$, $\delta > \frac{h\alpha}{2hk-\alpha k}$ and $\delta > \frac{h\alpha-h+\alpha}{\alpha k}$, such that $\frac{\alpha-k\delta}{h-\alpha}$ is irrational number.*

Proof. As $h < 2h-\alpha$ we have $\frac{h}{2h-\alpha} < 1$ and $\frac{h\alpha}{2hk-\alpha k} < \frac{\alpha}{k}$. As $0 < (\alpha-1)^2$, we have $(\alpha-1)+\alpha < \alpha^2$ and since $(\alpha-1) < 0$ we have $h(\alpha-1)+\alpha \leq (\alpha-1)+\alpha < \alpha^2$. Dividing this inequality by αk on both sides we get $\frac{h\alpha-h+\alpha}{\alpha k} < \frac{\alpha}{k}$. Let $m = \max\{\frac{h\alpha}{2hk-\alpha k}, \frac{h\alpha-h+\alpha}{\alpha k}\}$. From $0 < \frac{h\alpha}{2hk-\alpha k} < \frac{\alpha}{k}$ and $\frac{h\alpha-h+\alpha}{\alpha k} < \frac{\alpha}{k}$ we have $0 < m < \frac{\alpha}{k}$.

Let β be any positive irrational number less than $\frac{\alpha-km}{h-\alpha}$. We can get a δ that satisfies all the conditions by solving the equation $\frac{\alpha-k\delta}{h-\alpha} = \beta$ for δ. This is true since $0 < \frac{\alpha-k\delta}{h-\alpha} = \beta < \frac{\alpha-km}{h-\alpha}$ implies $m < \delta < \frac{\alpha}{k}$. □

Proof of Theorem 1. Without loss of generality assume $0 = \min B$ and $h = \max B$. Let us divide the proof into several cases.

When $\alpha_1 = \alpha_2 = \alpha < 1$. For a $\delta > 0$, with $\delta < \frac{\alpha}{k}$, $\delta > \frac{h\alpha}{2hk-\alpha k}$ and $\delta > \frac{h\alpha-h+\alpha}{\alpha k}$, such that $\frac{\alpha-k\delta}{h-\alpha}$ is irrational (there exists such a δ from Lemma 2.5), let $\theta = 1 + \frac{\alpha-k\delta}{h-\alpha}$. We have $B \subset \{0,\dots,h\}$ and for all $b \in B$ we have $\lfloor b\theta \rfloor = b + \lfloor \frac{b\alpha-bk\delta}{h-\alpha} \rfloor = b$ since $0 \leq \frac{b\alpha-bk\delta}{h-\alpha} \leq \frac{h\alpha-hk\delta}{h-\alpha} < 1$. Hence $B = \{\lfloor b\theta \rfloor : b \in B\}$. We have $m_1 = \min\{\{b\theta\} : b \in B\} = 0$ and $m_2 = \max\{\{b\theta\} : b \in B\} = \max\{b\theta - \lfloor b\theta \rfloor : b \in B\} = \max\{b\theta - b : b \in B\} = h(\theta-1)$. We have $\delta > \frac{h\alpha-h+\alpha}{\alpha k}$ implies $m_2 + k\delta = h\frac{(\alpha-k\delta)}{(h-\alpha)} + k\delta = \frac{h\alpha-\alpha k\delta}{h-\alpha} < \frac{h\alpha-(h\alpha-h+\alpha)}{(h-\alpha)} = 1$ and $\delta > \frac{h\alpha}{2hk-\alpha k}$ implies $m_1 + k\delta - m_2 = k\delta - h\frac{(\alpha-k\delta)}{(h-\alpha)} = \frac{2hk\delta-\alpha k\delta-h\alpha}{h-\alpha} > 0$. Hence $m_2 + k\delta \leq 1$ and $m_1 + k\delta \geq m_2$. From Lemma 2.4, the density of $kA+B$ is equal to $\frac{m_2-m_1+k\delta}{\theta} = \frac{h\left(\frac{\alpha-k\delta}{h-\alpha}\right)+k\delta}{1+\frac{\alpha-k\delta}{h-\alpha}} = \alpha$.

When $0 < \alpha_1 < \alpha_2 < 1$. Choose positive integers p and q such that $q = k(p-1)+1+h$ if $2k\alpha_2 - \alpha_1 > 2k-1$ and $\frac{(2k-1)(k(p-1)+1+h)}{(2k\alpha_2-\alpha_1)} < q < \frac{k(p-1)+1+h}{\alpha_2}$ if $2k\alpha_2 - \alpha_1 \leq 2k-1$. Let $\gamma = \frac{q}{k(p-1)+1+h}$. Setting $\alpha = \frac{(1-\gamma\alpha_1)\alpha_2}{(1-\gamma\alpha_2)\alpha_1}$ and $\beta = \frac{1-\gamma\alpha_1}{k(1-\gamma\alpha_2)}$, we have $\alpha > k\beta$. We claim that $\beta > 2$. Observe that, if $2k\alpha_2 - \alpha_1 > 2k-1$, then $\frac{2k-1}{2k\alpha_2-\alpha_1} < \gamma = \frac{q}{k(p-1)+1+h} = 1 < \frac{1}{\alpha_2}$. If $2k\alpha_2 - \alpha_1 \leq 2k-1$, then from the choice of p and q, we have $\frac{2k-1}{2k\alpha_2-\alpha_1} < \gamma = \frac{q}{k(p-1)+1+h} < \frac{1}{\alpha_2}$. Therefore, in both cases, we have $\frac{2k-1}{2k\alpha_2-\alpha_1} < \gamma < \frac{1}{\alpha_2}$. As $k(1-\gamma\alpha_2)(\beta-2) = -(2k-1) + \gamma(2k\alpha_2 - \alpha_1)$, which is greater than 0 from the inequality $\gamma > \frac{2k-1}{2k\alpha_2-\alpha_1}$. Therefore, $k(1-\gamma\alpha_2)(\beta-2) > 0$, and as $k(1-\gamma\alpha_2) > 0$, we have $\beta > 2$. Let $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = [\alpha^n, \beta\alpha^n] \cap (q.\mathbb{N} + [\![0, p-1]\!])$.

Then from Lemma 2.1, we have

$$\underline{d}(kA+B) = \frac{(k\beta-1)(k(p-1)+1+h)}{(\alpha-1)q} = \frac{k\beta-1}{(\alpha-1)\gamma} = \alpha_1$$

and

$$\overline{d}(kA+B) = \frac{\alpha(k\beta-1)(k(p-1)+1+h)}{k\beta(\alpha-1)q} = \frac{\alpha(k\beta-1)}{k\beta(\alpha-1)\gamma} = \alpha_2. \qquad \square$$

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