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ON THE CONNECTION BETWEEN THE BEURLING-MALLIAVIN DENSITY AND THE ASYMPTOTIC DENSITY

Rita Giuliano[1] — Georges Grekos[2]

[1]Dipartimento di Matematica, Università di Pisa, Pisa, ITALY

[2]Institut Camille Jordan, Université Jean Monnet, Saint-Étienne Cedex 2, FRANCE

ABSTRACT. We study the notion of Beurling-Malliavin density from the point of view of Number Theory. We prove a general relation between the Beurling-Malliavin density and the upper asymptotic density; we identify a class of sequences for which the two densities coincide; this class contains the arithmetic progressions. Last, by means of an alternative definition of Beurling-Malliavin density, we study the connection with the asymptotic density for another kind of sequences that again generalizes the arithmentic progressions.

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1. Introduction

Let $\Lambda = \big(\lambda_n\big)_{n\in\mathbb{Z}}$ be a sequence of real numbers. In order to solve the problem of finding the radius of completeness of Λ (denoted by $\mathcal{R}(\Lambda)$), A. Beurling and P. Malliavin introduced for the first time in the paper [2] the quantity $b(\Lambda)$, defined as follows: if $\big(I_n\big)_{n\in\mathbb{Z}}$ is a sequence of disjoint intervals on $\mathbb{R}$, call it *short* if

$$\sum_{n\in\mathbb{Z}} \frac{|I_n|^2}{1+\operatorname{dist}^2(0,I_n)} < \infty$$

and *long* otherwise (where $\operatorname{dist}(0,I_n) := \inf\{|x|, x\in I_n\}$);

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then define

$$b(\Lambda) := \sup\{a : \exists \text{ long } (I_n)_{n\in\mathbb{Z}} \quad \text{such that} \quad \#(\Lambda \cap I_n) > a|I_n|, \forall n \in \mathbb{Z}\}$$

(the particular formulation used above comes from [12] and [14]). The celebrated Theorem of [2] states that the radius of completeness of Λ is connected with $b(\Lambda)$ by the formula

$$\mathcal{R}(\Lambda) = 2\pi b(\Lambda).$$

Later the same quantity $b(\Lambda)$ was studied by other authors, for instance [8], [9], and used in several papers; we cite [7], [10] and [15]; for an extensive study and generalizations see [13], [12], [1] (see [14] for a complete list of references); in particular the author of [15] discovers an alternative equivalent definition, which will be stated and used later (see Definition 2.1).

In the whole literature on the subject the quantity $b(\Lambda)$ is called the "density" of Λ; in the sequel we call it "Beurling-Malliavin density" (or "BM-density" for short); anyway, to the best of our knowledge, no paper addresses the problem of investigating whether it "deserves" (so to say) the name of "density", in the sense in which this term is used in Number Theory. Just to give a trivial example, we have not found any paper stating that the BM-density of the set of multiples of the positive number p is $\frac{1}{p}$.

The most natural notion of density used in Number Theory is the asymptotic density, which is mostly used for sequences of integers; here we extend this notion to general sequences of positive real numbers converging to $+\infty$. Precisely, we say that $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ has "asymptotic density" $d(\Lambda)$ if the limit

$$\lim_{n\to\infty} \frac{F_\Lambda(n)}{n} =: d(\Lambda)$$

exists, where F_Λ is the *counting function* of Λ, defined as

$$F_\Lambda(t) = \begin{cases} 0, & t = 0, \\ \#\{k \in \mathbb{N}^* : \lambda_k \le t\}, & t > 0. \end{cases}$$

In Section 3, we shall give also the definitions of "upper" and "lower" asymptotic density.

The Beurling-Malliavin density is defined in [2] for general sequences

$$\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$$

of real numbers; on the other hand, if $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ is a sequence of integers, we may be interested in calculating its asymptotic density. Thus *we emphasize that, in order to compare these two concepts, in what follows we shall confine ourselves to sequences of real numbers (and not only of integers) indexed, by* $n \in \mathbb{N}^*$, *positive, ultimately strictly increasing and such that* $\lim_{n\to\infty} \lambda_n = +\infty$.

In the sequel, by the term *sequence* we always mean a sequence with such properties, unless otherwise specified.

Remark 1.1. Notice that sequences with a finite number of elements are excluded from this discussion, but this is a trivial case (both the BM-density and asymptotic density vanish). Similarly, we are not dealing with sequences with repetitions (see Remark 6.2).

The aim of the present investigation is to survey various equivalent definitions of the BM-density and to study the connection between $b(\Lambda)$ and $d(\Lambda)$.

The paper is structured as follows. Section 2 contains two equivalent definitions of the BM-density; similarly, Section 3 discusses two equivalent definitions of the asymptotic density (upper and lower); in Section 4, we prove a general inequality relating the upper asymptotic density and the BM-density; in Section 5, we prove the coincidence of the asymptotic density and the BM-density for a particular class of sequences, to which arithmetic progressions belong. Last, in Section 6 (i), we show how to use the alternative Definition 2.1 for estimating the BM-density of some kind of sequences that again generalizes arithmetic progressions; (ii) we find a nice characterization of the BM-density (see Theorem 6.2 and Corollary 6.2).

We shall use the standard symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to mean respectively the greatest integer less than or equal to x (*integer part* of x) and the least integer greater than or equal to x.

2. Various equivalent definitions of the Beurling-Malliavin density

The Beurling-Malliavin density $b(\Lambda)$, firstly defined in [2], has been studied later in [9] and in [15] (among others); both the Definitions of [9] and [15] are different but equivalent to the original one. In the very recent paper [6] a further equivalent definition has been given. In the present paper we shall use first the approach of [15] and later the one of [6].

2.1. The definition of [15].

For every sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$, denote by A_Λ the set of numbers $a \geq 0$ for which there exists a sequence of distinct positive integers $N = (n_k)_{k\in\mathbb{N}^*}$ such that

$$\sum_{k=1}^{\infty} \left| \frac{1}{\lambda_k} - \frac{a}{n_k} \right| < \infty.$$

In [15] it is proved that an equivalent definition of the Beurling–Malliavin density is given by the following

Definition 2.1. The Beurling Malliavin density $b(\Lambda)$ of the sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ is the infimum of A_Λ.

In [15] it is shown that the set A_Λ is a right infinite interval in $\mathbb{R}^+$ (i.e., if a belongs to A_Λ and $b > a$, then b belongs to A_Λ as well). If A_Λ is empty, then $b(\Lambda) = +\infty$.

Remark 2.1. If Λ is a sequence of integers, we have obviously $b(\Lambda) \leq 1$ (take $a = 1$ and $N = \Lambda$ in the above definition).

Remark 2.2. Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ be a sequence of real numbers and let $\tilde{\Lambda} = (\lambda_{k_h})_{h\geq 1}$ be a subsequence of Λ. It is evident that $A_\Lambda \subseteq A_{\tilde{\Lambda}}$, which implies that

$$b(\tilde{\Lambda}) \leq b(\Lambda).$$

This relation says that $\Lambda \mapsto b(\Lambda)$ is a monotone set function.

2.2. The definition of [6].

Let $\mathfrak{C}$ be the family of all sequences

$$\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*}$$

of intervals in $(0, +\infty)$ such that $a_n < b_n \leq a_{n+1}$ for all $n \in \mathbb{N}^*$ and

$$\sum_{n=1}^{\infty} \left(\frac{b_n}{a_n} - 1\right)^2 = +\infty.$$

Remark 2.3. These systems of intervals are referred to as long in [12] and [14] (see the Introduction). In [9] they are called substantial.

For every $\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*} \in \mathfrak{C}$, let

$$\ell_{\mathcal{I}} = \liminf_{n\to\infty} \frac{F_\Lambda(b_n) - F_\Lambda(a_n)}{b_n - a_n}. \tag{1}$$

In [6] it is proved that

Proposition 2.1. *The following relation holds true*

$$b(\Lambda) = \sup\{\ell_{\mathcal{I}},\, \mathcal{I} \in \mathfrak{C}\}.$$

We shall be interested in the subset of $\mathfrak{C}$ defined as

$$\mathfrak{C}_{>1} = \left\{\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*} \in \mathfrak{C} : \limsup_{n\to\infty} \frac{b_n}{a_n} > 1\right\}.$$

Accordingly, we shall denote

$$b_{>1}(\Lambda) = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}_{>1}\}. \tag{2}$$

From Proposition 2.1 it follows that

$$b_{>1}(\Lambda) \leq b(\Lambda). \tag{3}$$

3. Two equivalent definitions of the asymptotic density

Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ be a sequence of positive numbers, with $\lim_{n\to\infty} \lambda_n = +\infty$.

Definition 3.1.

(i) The *lower* (resp. *upper*) *asymptotic density of lambda* is

$$\underline{d}(\Lambda) = \liminf_{n\to\infty} \frac{F_\Lambda(n)}{n}, \quad \text{resp. } \overline{d}(\Lambda) = \limsup_{n\to\infty} \frac{F_\Lambda(n)}{n}.$$

(ii) We say that $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ has *asymptotic density* $d(\Lambda)$ if

$$\underline{d}(\Lambda) = \overline{d}(\Lambda) \ \big(= d(\Lambda)\big).$$

It is well known that, besides the definition, there is another way for calculating the upper and lower asymptotic densities of Λ. Precisely we have

Proposition 3.1. *The set of limit points of the sequence* $\left(\frac{k}{\lambda_k}\right)_{k\in\mathbb{N}^*}$ *coincides with the set of limit points of the sequence* $\left(\frac{F_\Lambda(n)}{n}\right)_{n\in\mathbb{N}^*}$. *In particular the following relations hold true*

$$\underline{d}(\Lambda) = \liminf_{k\to\infty} \frac{k}{\lambda_k}; \qquad \overline{d}(\Lambda) = \limsup_{k\to\infty} \frac{k}{\lambda_k}.$$

Moreover, $d(\Lambda)$ *exists if either of the two limits*

$$\lim_{n\to\infty} \frac{F_\Lambda(n)}{n}, \qquad \lim_{k\to\infty} \frac{k}{\lambda_k}$$

exists; in this case

$$d(\Lambda) = \lim_{n\to\infty} \frac{F_\Lambda(n)}{n} = \lim_{k\to\infty} \frac{k}{\lambda_k}.$$

P r o o f. In fact, for any integer n, let k_n (depending on n) be defined as

$$k_n = \max\{k \in \mathbb{N} : \lambda_k \leq n\}.$$

In other words, k_n is such that $\lambda_{k_n} \leq n < \lambda_{k_n+1}$; moreover we have $F_\Lambda(n) = k_n$, and

$$\frac{k_n}{\lambda_{k_n+1}} < \frac{F_\Lambda(n)}{n} \leq \frac{k_n}{\lambda_{k_n}}. \tag{4}$$

Notice that $\lim_{n\to\infty} k_n = \infty$ and let ℓ be a limit point for the sequence

$$\left(\frac{F_\Lambda(n)}{n}\right)_{n\in\mathbb{N}^*}.$$

Then there exists a subsequence $(n_r)_{r\in\mathbb{N}^*}$ converging to ∞ such that

$$\frac{F_\Lambda(n_r)}{n_r} \to \ell \quad \text{as} \quad r \to \infty.$$

Relation (4) implies that

$$\overline{\kappa} := \limsup_{r\to\infty} \frac{k_{n_r}}{\lambda_{k_{n_r}}} = \limsup_{r\to\infty} \frac{k_{n_r}}{\lambda_{k_{n_r}+1}} \le \ell \le \liminf_{r\to\infty} \frac{k_{n_r}}{\lambda_{k_{n_r}}} =: \underline{\kappa}\,.$$

In other words every limit point for $\left(\frac{F_\Lambda(n)}{n}\right)_{n\in\mathbb{N}^*}$ is a limit point for $\left(\frac{k}{\lambda_k}\right)_{k\in\mathbb{N}^*}$. To go the inverse direction, notice that $F_\Lambda(\lambda_k) = k$; thus, for $n_k \le \lambda_k < n_k + 1$,

$$\frac{F_\Lambda(n_k)}{n_k+1} \le \frac{k}{\lambda_k} \le \frac{F_\Lambda(n_k+1)}{n_k};$$

now argue as in the first part of this proof, using the fact that

$$\lim_{k\to\infty} n_k = \infty. \qquad \square$$

4. The general inequality

Theorem 4.1. *For every sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$, the relation*

$$\overline{d}(\Lambda) \le b(\Lambda) \tag{5}$$

holds true.

P r o o f. Recall that the Beurling–Malliavin Theorem states that, putting

$$\mathcal{E}_\Lambda = \{\mathrm{e}^{\pm \mathrm{i}\lambda_n x}\}, \quad \mathcal{R}(\Lambda) = \sup\{a \in \mathbb{R}^+ : \mathcal{E}_\Lambda \text{ is complete in } L^2(0,a)\},$$

we have

$$\mathcal{R}(\Lambda) = 2\pi b(\Lambda).$$

Hence it suffices to prove that, for every $\epsilon > 0$, $\mathcal{E}_\Lambda$ is complete in

$$L^2\big(0, 2\pi(\overline{d}(\Lambda) - \epsilon)\big).$$

Let f be a function in $L^2\big(0, 2\pi(\overline{d}(\Lambda) - \epsilon)\big)$ and assume that

$$\int_0^{2\pi(\overline{d}(\Lambda)-\epsilon)} \mathrm{e}^{\pm \mathrm{i}\lambda_n x} f(x)\mathrm{d}x = 0, \qquad n = 1, 2, 3, \dots$$

By the change of variable $y = \frac{x}{\overline{d}(\Lambda)-\epsilon} - \pi$, the above integral becomes

$$\big(\overline{d}(\Lambda) - \epsilon\big)\mathrm{e}^{\pm\mathrm{i}\lambda_n(\overline{d}(\Lambda)-\epsilon)\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\pm\mathrm{i}\lambda_n(\overline{d}(\Lambda)-\epsilon)y} f\Big(\big(\overline{d}(\Lambda) - \epsilon\big)(y+\pi)\Big)\mathrm{d}y.$$

In [3] the following result is proved (see Theorem XXVIII, p. 84):

Theorem 4.2. *Let* $0 < m_1 < m_2 < \cdots$ *and let*

$$\limsup_{n\to\infty} \frac{n}{m_n} > 1.$$

Then, if $f \in L^2(-\pi,\pi)$ *and*

$$\int_{-\pi}^{\pi} \mathrm{e}^{\pm\mathrm{i}m_n x} f(x)\mathrm{d}x = 0, \qquad n = 1, 2, 3, \dots,$$

then $f(x) = 0$ *for almost every* $x \in (-\pi,\pi)$.

Setting $m_n = \big(\overline{d}(\Lambda) - \epsilon\big)\lambda_n$, we obtain that

$$\int_{-\pi}^{\pi} \mathrm{e}^{\pm\mathrm{i}m_n y} g(y)\mathrm{d}y = 0,$$

where g is the function $y \in (-\pi,\pi) \mapsto f\big((\overline{d}(\Lambda) - \epsilon)(y+\pi)\big)$. Observe that

$$\limsup_{n\to\infty} \frac{n}{m_n} = \limsup_{n\to\infty} \frac{n}{(\overline{d}(\Lambda) - \epsilon)\lambda_n} = \frac{\overline{d}(\Lambda)}{\overline{d}(\Lambda) - \epsilon} > 1.$$

Hence Theorem 4.2 is in force and we deduce that g vanishes almost everywhere on $(-\pi,\pi)$. This is obviously equivalent to saying that f vanishes almost everywhere on $2\pi\big(\overline{d}(\Lambda) - \epsilon\big)$, i.e., that $\mathcal{E}_\Lambda$ is complete in $L^2\big(0, 2\pi(\overline{d}(\Lambda) - \epsilon)\big)$. □

Remark 4.1. Theorem 4.1 is stated but not proved in [13]. In any case, [13] does not formulate it in terms of densities; see slide 3 in [13].

The following example shows that the inequality (5) may be strict.

Example 4.1. Let p be an integer with $1 \le p \le 9$ and denote by $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ the - strictly increasing - sequence formed by the positive integers having first digit equal to p. It is well known (see, e.g. [5]) that $\overline{d}(\Lambda) = \frac{10}{9(p+1)} < 1$, and now we prove that $b(\Lambda) = 1$. Recall that the BM-density of any set of integers is ≤ 1. By formula (2) and relation (3), it suffices to prove that there exists a family of intervals $\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*}$ such that

$$\limsup_{n\to\infty} \frac{b_n}{a_n} > 1 \quad \text{and} \quad \ell_{\mathcal{I}} = \liminf_{n\to\infty} \frac{F_\Lambda(b_n) - F_\Lambda(a_n)}{b_n - a_n} \ge 1.$$

Take

$$a_n = p10^n, \quad b_n = (p+1)10^n - 1.$$

We have $b_n - a_n = 10^n - 1$ and

$$F_\Lambda(b_n) = \sum_{k=1}^{n} \sum_{j=p10^k}^{(p+1)10^k-1} 1 = \sum_{k=1}^{n} \left\{(p+1)10^k - p10^k\right\} = \sum_{k=1}^{n} 10^k = \frac{10}{9}(10^n - 1);$$

$$F_\Lambda(a_n) = F_\Lambda(b_{n-1}) + 1 = \frac{10}{9}(10^{n-1} - 1) + 1 = \frac{1}{9}(10^n - 1).$$

Thus

$$\ell_{\mathcal{I}} = \liminf_{n\to\infty} \frac{F_\Lambda(b_n) - F_\Lambda(a_n)}{b_n - a_n} = \liminf_{n\to\infty} \frac{\frac{10}{9}(10^n - 1) - \frac{1}{9}(10^n - 1)}{10^n - 1} = 1.$$

Remark 4.2. It is easy to check that for

$$a_n = (p+1)10^n - 1 \quad \text{and} \quad b_n = a_{n+1} = (p+1)10^{n+1} - 1$$

we have

$$\ell_{\mathcal{I}} = \overline{d}(\Lambda) = \frac{10}{9(p+1)}.$$

Remark 4.3. Let $\overline{\Lambda} = \mathbb{N} \setminus \Lambda$, where Λ is the sequence described in Example 4.1. Similar calculations as in Example 4.1 prove that $b(\overline{\Lambda}) = 1$. Hence the map $\Lambda \mapsto b(\Lambda)$ (with domain $\mathcal{P}(\mathbb{N})$) is not additive.

5. A general result concerning the equality

The aim of this Section is to prove the following result, in which a class of sequences Λ is identified for which $d(\Lambda) = b(\Lambda)$.

Theorem 5.1. *Assume that $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ is a sequence with the following property: there exists an integer $p \geq 1$ and non-negative constants $a_0, a_1, \ldots, a_{p-1}$ such that*

$$\lim_{k\to\infty} \{\lambda_{kp+j+1} - \lambda_{kp+j}\} = a_j, \quad \forall j = 0, 1, 2, \ldots p-1.$$

In addition assume that $\sum_{j=0}^{p-1} a_j > 0$. Then

$$d(\Lambda) = b(\Lambda) = \frac{p}{\sum_{j=0}^{p-1} a_j}.$$

For the proof of this result, we need some lemmas. Lemma 5.1 is obvious (see the two lines after the statement); the proofs of Lemmas 5.2 and 5.3 are postponed in the Appendix at the end of the paper.

Lemma 5.1. *Let* $(x_n)_{n\in\mathbb{N}^*}$ *and* $(y_n)_{n\in\mathbb{N}^*}$ *be two sequences of numbers, with*

$$\lim_{n\to\infty}(y_n - x_n) = +\infty. \tag{6}$$

Then

$$\lim_{n\to\infty}\frac{\lfloor y_n\rfloor - \lfloor x_n\rfloor}{y_n - x_n} = 1.$$

This Lemma follows from the elementary fact that both numerator and denominator tend to infinity and they differ by a number not greater than 1.

Lemma 5.2. *Assume that* $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ *verifies the conditions of Theorem 5.1. Then*

$$\lim_{n\to\infty}\frac{\lambda_n}{n} = \frac{\sum_{j=0}^{p-1} a_j}{p}.$$

Denote by $\overline{\lambda}$ the interpolated linear function relative to Λ, i.e., the function defined by

$$\overline{\lambda}(x) = \big(\lambda_{k+1} - \lambda_k\big)(x-k) + \lambda_k, \qquad k \le x < k+1, \qquad k = 1, 2\ldots$$

or equivalently

$$\overline{\lambda}(x) = \big(\lambda_{\lceil x\rceil} - \lambda_{\lfloor x\rfloor}\big)\big(x - \lfloor x\rfloor\big) + \lambda_{\lfloor x\rfloor}.$$

From [4] we recall the

Definition 5.1. The function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *regularly varying with exponent* $\rho > 0$ if, for every $t > 0$, we have

$$\lim_{x\to+\infty}\frac{f(tx)}{f(x)} = t^\rho.$$

Lemma 5.3. *Assume that* $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ *is such that*

$$\lim_{n\to\infty}\frac{\lambda_n}{n} = \ell > 0. \tag{7}$$

Then

(i) $\lim_{x\to+\infty}\frac{\overline{\lambda}(x)}{x} = \ell$;

(ii) $\overline{\lambda}$ *is regularly varying with exponent 1;*

(iii) *for* $\beta > 0$ *let* $S_\beta = \{(x,y) \in (\mathbb{R}^+)^2, \beta x < y\}$*; then, for every* $\beta > 1$,

$$\lim_{\substack{x^2+y^2\to+\infty\\ (x,y)\in S_\beta}}\frac{\overline{\lambda}(y) - \overline{\lambda}(x)}{y-x} = \ell.$$

Now we are in a position for proving Theorem 5.1.

P r o o f o f T h e o r e m 5.1. .

Recall that the asymptotic density $d(\Lambda)$ of the sequence Λ is the $\lim_{n\to\infty}\frac{n}{\lambda_n}$, if this limit exists (see Proposition 3.1). Hence by Lemma 5.2 we have $d(\Lambda)=\frac{p}{\sum_{j=0}^{p-1}a_j}$, and now we shall prove that

$$\frac{p}{\sum_{j=0}^{p-1}a_j}\le b_{>1}(\Lambda). \tag{8}$$

Recalling (2) and the formula preceding (2), let $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$ be such that $\lim_{n\to\infty}a_n=+\infty$ and

$$\limsup_{n\to\infty}\frac{b_n}{a_n}\ge\liminf_{n\to\infty}\frac{b_n}{a_n}=\alpha>1;$$

we are interested in calculating $\frac{F_\Lambda(b_n)-F_\Lambda(a_n)}{b_n-a_n}$ (see relation (1)).

For every $x\in\mathbb{R}^+$ with $\lambda_k\le x<\lambda_{k+1}$ we have $F_\Lambda(x)=k$; since $k\le\left(\overline{\lambda}\right)^{-1}(x)<k+1$, we can write $F_\Lambda(x)=\lfloor\left(\overline{\lambda}\right)^{-1}(x)\rfloor$.

Thus

$$\begin{aligned}\frac{F_\Lambda(b_n)-F_\Lambda(a_n)}{b_n-a_n}&=\frac{\lfloor\left(\overline{\lambda}\right)^{-1}(b_n)\rfloor-\lfloor\left(\overline{\lambda}\right)^{-1}(a_n)\rfloor}{b_n-a_n}\\&=\frac{\lfloor y_n\rfloor-\lfloor x_n\rfloor}{\overline{\lambda}(y_n)-\overline{\lambda}(x_n)}=\frac{y_n-x_n}{\overline{\lambda}(y_n)-\overline{\lambda}(x_n)}\cdot\frac{\lfloor y_n\rfloor-\lfloor x_n\rfloor}{y_n-x_n},\end{aligned}$$

where

$$x_n=\left(\overline{\lambda}\right)^{-1}(a_n),\qquad y_n=\left(\overline{\lambda}\right)^{-1}(b_n).$$

Notice that $x_n\to+\infty$. Then (8) follows immediately from (3), Lemma 5.3 (iii) and Lemma 5.1 (see (6)) if we prove that there exists $\beta>1$ such that $y_n>\beta x_n$ for every sufficiently large n.

Let $\epsilon>0$ be fixed with $\epsilon<\alpha-1$. There exists n_0 such that, for $n>n_0$, $(\alpha-\epsilon)a_n\le b_n$, which yields

$$\left(\overline{\lambda}\right)^{-1}\big((\alpha-\epsilon)a_n\big)\le\left(\overline{\lambda}\right)^{-1}(b_n).$$

By Lemma 5.3 (ii) and Theorem 1.5.12 in [4], $(\overline{\lambda})^{-1}$ is regularly varying with exponent 1, hence

$$\frac{(\overline{\lambda})^{-1}((\alpha-\epsilon)a_n)}{(\overline{\lambda})^{-1}(a_n)}\to\alpha-\epsilon,\qquad n\to\infty.$$

It follows that for every $\delta>0$ with $\delta<\alpha-\epsilon-1$, we have

$$(\overline{\lambda})^{-1}(a_n)(\alpha-\epsilon-\delta)\le(\overline{\lambda})^{-1}\big((\alpha-\epsilon)a_n\big)\le(\overline{\lambda})^{-1}(b_n)$$

or $y_n\ge\beta x_n$ with $\beta=\alpha-\epsilon-\delta>1$.

Now an application of Theorem 4.1 concludes the proof. □

Example 5.1. Arithmetic progressions satisfy the assumption of Theorem 5.1. For a less trivial example consider

$$\lambda_n = \frac{3}{2}n + \frac{1}{2} + \big(1 + (-1)^n\big)\left(\frac{1}{4} + \epsilon_n\right),$$

where $(\epsilon_n)_{n\in\mathbb{N}^*}$ is any sequence converging to 0 as $n \to \infty$. Hence

$$b(\Lambda) = d(\Lambda) = \frac{2}{3}.$$

Notice that the case $\epsilon_n \equiv 0$ is again a trivial one, since the sequence becomes

$$\lambda_n = \frac{3}{2}n + \frac{3 + (-1)^n}{4},$$

which is the union of the two arithmetic progressions $1, 4, 7, \dots$ and $2, 5, 8, \dots$ of common difference 3; hence its BM-density can be calculated more simply by applying Theorem 5.1 to these two progressions and by using the additivity of the asymptotic density.

6. Some results that follow from the definition of [15]

In this Section, we show how to use Definition 2.1 to get information about $b(\Lambda)$ and its connection with $d(\Lambda)$.

Proposition 6.1. *Assume that there exist $\ell > 0$ and a sequence $(\psi_n)_{n\in\mathbb{N}^*}$ such that*

$$\lambda_n = \ell n + \psi_n.$$

If either

(i) *$(\psi_n)_{n\in\mathbb{N}^*}$ is non-decreasing (ultimately) or*
(ii) *$\kappa := \liminf_{n\to\infty} \left|\ell + \frac{\psi_n}{n}\right| > 0$ and $\sum_n \frac{|\psi_n|}{n^2} < +\infty$,*

then

$$b(\Lambda) \le \frac{1}{\ell}.$$

Moreover, in both cases, we have $\alpha := \liminf_{n\to\infty} \frac{\psi_n}{n} \ge 0$. In particular, if $\alpha = 0$, then

$$b(\Lambda) = \overline{d}(\Lambda) = \frac{1}{\ell}.$$

P r o o f. We prove that, in both cases (i) and (ii), the number $\frac{1}{\ell}$ verifies the property stated in Definition 2.1.

(i) Let $n_k = k + \lfloor \frac{\psi_k}{\ell} \rfloor$. This sequence of integers is made of distinct integers since

$$n_{k+1} - n_k = \left\lfloor \frac{\psi_{k+1}}{\ell} \right\rfloor - \left\lfloor \frac{\psi_k}{\ell} \right\rfloor + 1 \geq 1.$$

Moreover,

$$\sum_k \left| \frac{1}{\lambda_k} - \frac{\frac{1}{\ell}}{n_k} \right| = \frac{1}{\ell} \sum_k \left| \frac{1}{k + \frac{\psi_k}{\ell}} - \frac{1}{k + \lfloor \frac{\psi_k}{\ell} \rfloor} \right| = \frac{1}{\ell} \sum_k \frac{\left| \frac{\psi_k}{\ell} - \lfloor \frac{\psi_k}{\ell} \rfloor \right|}{(k + \frac{\psi_k}{\ell})(k + \lfloor \frac{\psi_k}{\ell} \rfloor)}$$
$$\leq \frac{1}{\ell} \sum_k \frac{1}{k^2 (1 + \frac{\psi_k}{\ell k})(1 + \frac{1}{k} \lfloor \frac{\psi_k}{\ell} \rfloor)}.$$

Since $(\psi_n)_{n \in \mathbb{N}^*}$ is non-decreasing, there exists an integer k_0 and a constant α such that for $k \geq k_0$,

$$1 + \frac{\alpha}{k} > 0, \qquad 1 + \frac{\psi_k}{\ell k} \geq 1 + \frac{\alpha}{k}, \qquad 1 + \frac{1}{k} \left\lfloor \frac{\psi_k}{\ell} \right\rfloor \geq 1 + \frac{\alpha}{k}.$$

Thus the series $\sum_k \frac{1}{k^2 (1 + \frac{\psi_k}{\ell k})(1 + \frac{1}{k} \lfloor \frac{\psi_k}{\ell} \rfloor)}$ is majorized by

$$\text{const.} + \sum_k \frac{1}{k^2 (1 + \frac{\alpha}{k})^2},$$

which behaves like $\sum_k \frac{1}{k^2} < +\infty$.

(ii) Take $n_k = k$. We have

$$\sum_k \left| \frac{1}{\lambda_k} - \frac{\frac{1}{\ell}}{k} \right| = \frac{1}{\ell^2} \sum_k \frac{|\psi_k|}{k^2 |1 + \frac{\psi_k}{\ell k}|}$$

which, by the assumption $\kappa > 0$, behaves like $\sum_k \frac{|\psi_k|}{k^2} < +\infty$.

Concerning $\overline{d}(\Lambda)$, we have

$$\overline{d}(\Lambda) = \limsup_{n \to \infty} \frac{n}{\lambda_n} = \limsup_{n \to \infty} \frac{1}{\ell + \frac{\psi_n}{n}} = \frac{1}{\ell + \limsup_{n \to \infty} \frac{\psi_n}{n}} = \frac{1}{\ell + \alpha}.$$

The inequality $\alpha \geq 0$ is obvious in case (i); in case (ii) it follows from the basic inequality (5). If $\alpha = 0$, the above discussion implies that

$$b(\Lambda) \leq \frac{1}{\ell} = \overline{d}(\Lambda),$$

and we get the equality once more by (5). □

Example 6.1. Any arithmetic progression $\lambda_n = cn + a$ verifies both the assumptions (i) and (ii) of Proposition 6.1; in particular the set of multiples of the integer p has Beurling-Malliavin density equal to $\frac{1}{p}$.

The sequence of Example 5.1 verifies (ii) of Proposition 6.1.

Remark 6.1. There are sequences which satisfy the assumptions of Proposition 6.1 but not those of Theorem 5.1. For instance take

$$\lambda_n = \ell n + \sin n.$$

By Proposition 6.1 (ii)

$$b(\Lambda) = d(\Lambda) = \frac{1}{\ell}$$

($d(\Lambda)$ exists since $\lim_{n\to\infty} \frac{\sin n}{n} = 0$). Theorem 5.1 cannot be applied since, for every p, the sequence $\big(\sin(n+p) - \sin n\big)_{n\in\mathbb{N}}$ does not have any limit (see Remark 7.1).

The following result is a consequence of the discussion in Section 2.2. of [6].

Proposition 6.2. *Assume that* $\liminf_{n\to\infty}(\lambda_{n+1} - \lambda_n) =: \ell > 0$. *Then*

$$b(\Lambda) \leq \frac{1}{\ell}.$$

The preceding Proposition and Theorem 4.1 yield immediately the following result, which is a particular case of (actually it has been a motivation for) Theorem 5.1:

Corollary 6.1. *If the limit*

$$\ell := \lim_{n\to\infty}(\lambda_{n+1} - \lambda_n)$$

exists (as an extended number) and is strictly positive, then

$$d(\Lambda) = b(\Lambda) = \frac{1}{\ell},$$

where we adopt the convention $\frac{1}{0} = +\infty$.

Remark 6.2. For the sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ with $\lambda_n = \log n$ we have $\ell = 0$ and as a consequence $b(\Lambda) = +\infty$. The same happens for $M = (\mu_n)_{n\in\mathbb{N}^*}$ with $\mu_n = \lfloor \log n \rfloor$, as well as for sequences with "too many" repetitions. This follows from the Definition in Section 2.2 of [6].

Proposition 6.3. *Assume that* $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ *is a sequence such that*

$$\sum_k \frac{1}{\lambda_k} < +\infty.$$

Then $d(\Lambda) = b(\Lambda) = 0$.

Remark 6.3. This Proposition was proved in [16], but only in the part concerning $d(\Lambda)$.

P r o o f. Recall Olivier's Theorem (also known as Abel's or Pringsheim's Theorem), proved in [11]:

Theorem 6.1. *Let $(a_n)_{n\in\mathbb{N}^*}$ be a non–increasing sequence of positive numbers such that the corresponding series $\sum_{n=1}^{\infty} a_n$ is convergent. Then*

$$\lim_{n\to\infty} na_n = 0.$$

Applying it with $a_n = \frac{1}{\lambda_n}$ we have

$$d(\Lambda) = \lim_{n\to\infty} \frac{n}{\lambda_n} = 0.$$

Moreover, by Definition 2.1, $b(\Lambda) = 0$. □

Example 6.2. This result applies for instance to the set of powers $(k^p)_{k\in\mathbb{N}^*}$ ($p \in \mathbb{N}^*$, $p \geq 2$). More generally, it applies to any increasing sequence of the form $(L(k)k^p)_{k\in\mathbb{N}^*}$, where $(L_k)_{k\in\mathbb{N}^*}$ is a slowly varying sequence: recall that a slowly varying sequence $(L_k)_{k\in\mathbb{N}^*}$ has the property that, for any $\epsilon > 0$, ultimately we have $k^{-\epsilon} < L_k$ (this is an easy consequence of Theorem 1.3.1 in [4]), which implies that

$$\sum_k \frac{1}{\lambda_k} < \sum_k \frac{1}{k^{p-\epsilon}} < +\infty \quad \text{if } \epsilon < p - 1.$$

Last, here below is the announced characterization of the BM-density:

Theorem 6.2. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ be a sequence with $b(\Lambda) \in (0, +\infty)$; assume that there exist a strictly positive number ℓ and another sequence $M = (\mu_n)_{n\in\mathbb{N}^*}$ with $b(M) \in (0, +\infty)$ such that*

$$\sum_k \left| \frac{1}{\lambda_k} - \frac{1}{\ell\mu_k} \right| < +\infty.$$

Then ℓ is unique; furthermore,

$$b(M) = \ell\, b(\Lambda).$$

P r o o f. Denote by

$$\mathcal{A} = \left\{ \ell > 0 : \sum_k \left| \frac{1}{\lambda_k} - \frac{1}{\ell\mu_k} \right| < +\infty \right\}$$

and

$$\sigma = \inf \mathcal{A}, \qquad \tau = \sup \mathcal{A}.$$

Let $\epsilon > 0$ and $N = (n_k)_{k\in\mathbb{N}^*}$ a sequence of distinct integers such that

$$\sum_k \left| \frac{1}{\mu_k} - \frac{b(M)+\epsilon}{n_k} \right| < +\infty.$$

Then, for every $\ell \in \mathcal{A}$,

$$\sum_k \left| \frac{1}{\lambda_k} - \frac{b(M)+\epsilon}{\ell n_k} \right| \leq \sum_k \left| \frac{1}{\lambda_k} - \frac{1}{\ell \mu_k} \right|$$
$$+ \frac{1}{\ell} \sum_k \left| \frac{1}{\mu_k} - \frac{b(M)+\epsilon}{n_k} \right| < +\infty.$$

This yields

$$b(\Lambda) \leq \frac{b(M)+\epsilon}{\ell}$$

and by optimizing in ℓ and ϵ we conclude that

$$b(\Lambda) \leq \frac{b(M)}{\tau}.$$

Relation (6) together with the assumption $b(\Lambda) > 0$, implies that $\tau < \infty$.

Now interchange the roles of Λ and M; we obtain that

$$b(M) \leq \ell(b(\Lambda)+\epsilon),$$

and optimizing

$$b(M) \leq \sigma b(\Lambda). \tag{9}$$

From relation (9) and the assumption $b(M) > 0$ we deduce that $\sigma > 0$; and now, putting together (6) and (9), we get

$$b(\Lambda) \leq \frac{b(M)}{\tau} \leq \frac{\sigma}{\tau} b(\Lambda);$$

this proves the unicity of ℓ and the relation (6.2). □

We put in evidence the following particular case (see also Proposition 6.3 for the case $\ell = 0$):

Corollary 6.2. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ be a sequence with $b(\Lambda) < +\infty$; if there exists a positive number ℓ such that*

$$\sum_k \left| \frac{1}{\lambda_k} - \frac{\ell}{k} \right| < \infty,$$

then $\ell = b(\Lambda)$.

7. Appendix

Here we prove Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. We shall prove that, for every sequence

$$(\lambda_{kp+j})_{k\in\mathbb{N}^*}, \qquad j = 0, 1, \dots, p-1,$$

we have

$$\lim_{k\to\infty} \frac{\lambda_{kp+j}}{kp+j} = \frac{\sum_{j=0}^{p-1} a_j}{p}\,.$$

First, by the Cesaro theorem,

$$\begin{aligned}\lim_{k\to\infty} \frac{\lambda_{kp+j}}{kp+j} &= \lim_{k\to\infty} \frac{\lambda_{(k+1)p+j} - \lambda_{kp+j}}{(k+1)p+j-(kp+j)}\\ &= \lim_{k\to\infty} \frac{\lambda_{(k+1)p+j} - \lambda_{kp+j}}{p}\,.\end{aligned}$$

Now,

$$\begin{aligned}\lim_{k\to\infty}\{\lambda_{(k+1)p+j} - \lambda_{kp+j}\} &= \sum_{r=j}^{p+j-1} \lim_{k\to\infty}\{\lambda_{kp+r+1} - \lambda_{kp+r}\}\\ &= \sum_{r=j}^{p-1} \lim_{k\to\infty}\{\lambda_{kp+r+1} - \lambda_{kp+r}\}\\ &\quad + \sum_{r=p}^{p+j-1} \lim_{k\to\infty}\{\lambda_{kp+r+1} - \lambda_{kp+r}\}\\ &= \sum_{r=j}^{p-1} a_r + \sum_{s=0}^{j-1} \lim_{k\to\infty}\{\lambda_{(k+1)p+s+1} - \lambda_{(k+1)p+s}\}\\ &= \sum_{r=j}^{p-1} a_r + \sum_{s=0}^{j-1} \lim_{k\to\infty}\{\lambda_{kp+s+1} - \lambda_{kp+s}\}\\ &= \sum_{r=j}^{p-1} a_r + \sum_{s=0}^{j-1} a_s\\ &= \sum_{r=0}^{p-1} a_r\,.\end{aligned}$$

□

Remark 7.1. Lemma 5.2 implies that any sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ satisfying the assumptions of Theorem 5.1 has the form

$$\lambda_n = \frac{\sum_{j=0}^{p-1} a_j}{p} n + \psi_n \,,$$

where

$$\lim_{n\to\infty} \frac{\psi_n}{n} = 0. \tag{10}$$

Besides (10), a necessary condition for Λ to satisfy the assumptions of Theorem 5.1 is that

$$\lim_{n\to\infty} \psi_{n+p} - \psi_n = 0\,.$$

This is easily seen:

$$\begin{aligned}\psi_{n+p} - \psi_n &= \lambda_{n+p} - \frac{\sum_{j=0}^{p-1} a_j}{p}(n+p) \\ &\quad - \lambda_n + \frac{\sum_{j=0}^{p-1} a_j}{p} n \\ &= \lambda_{n+p} - \lambda_n - \sum_{j=0}^{p-1} a_j \to 0, \qquad n\to\infty\end{aligned}$$

since, for $n = kp + j$, $j = 0, \dots, p-1$, we have

$$\lim_{k\to\infty} \{\lambda_{kp+j+p} - \lambda_{kp+j}\} = \sum_{j=0}^{p-1} a_j \,,$$

as it has been proven in Lemma 5.2.

Proof of Lemma 5.3. First, from the assumption (7) and the relations

$$\frac{\lfloor x\rfloor}{\lceil x\rceil}\cdot\frac{\overline{\lambda}(\lfloor x\rfloor)}{\lfloor x\rfloor} = \frac{\overline{\lambda}(\lfloor x\rfloor)}{\lceil x\rceil} \le \frac{\overline{\lambda}(\lfloor x\rfloor)}{x} \le \frac{\overline{\lambda}(\lceil x\rceil)}{\lfloor x\rfloor} \tag{11}$$

and

$$\frac{\overline{\lambda}(\lceil x\rceil)}{\lceil x\rceil} \le \frac{\overline{\lambda}(\lceil x\rceil)}{x} \le \frac{\overline{\lambda}(\lceil x\rceil)}{\lfloor x\rfloor} = \frac{\overline{\lambda}(\lceil x\rceil)}{\lceil x\rceil}\cdot\frac{\lceil x\rceil}{\lfloor x\rfloor}, \tag{12}$$

(where we have used also the fact that $\overline{\lambda}$ is non-decreasing) and by the relation $\lim_{x\to+\infty} \frac{\lceil x\rceil}{\lfloor x\rfloor} = 1$, we obtain

$$\lim_{x\to+\infty} \frac{\overline{\lambda}(\lfloor x\rfloor)}{x} = \lim_{x\to+\infty} \frac{\overline{\lambda}(\lceil x\rceil)}{x} = \ell, \tag{13}$$

and (i) is obvious, since

$$\frac{\overline{\lambda}(\lfloor x\rfloor)}{x} \le \frac{\overline{\lambda}(x)}{x} \le \frac{\overline{\lambda}(\lceil x\rceil)}{x}.$$

For (ii), we have to prove that for every $t > 0$,

$$\lim_{x\to\infty} \frac{\overline{\lambda}(tx)}{\overline{\lambda}(x)} = t,$$

and this can be obtained by writing

$$\frac{\overline{\lambda}(tx)}{\overline{\lambda}(x)} = t \cdot \frac{\overline{\lambda}(tx)}{tx} \cdot \frac{x}{\overline{\lambda}(x)}$$

and using point (i).

Now we prove point (iii). Write

$$\begin{aligned}\frac{\overline{\lambda}(y) - \overline{\lambda}(x)}{y-x} &= \frac{(\lambda_{\lceil y\rceil} - \lambda_{\lfloor y\rfloor})(y - \lfloor y\rfloor)}{y-x} \\ &\quad + \frac{(\lambda_{\lceil x\rceil} - \lambda_{\lfloor x\rfloor})(x - \lfloor x\rfloor)}{y-x} \\ &\quad + \frac{\lambda_{\lfloor y\rfloor} - \lambda_{\lfloor x\rfloor}}{y-x}.\end{aligned}$$

Notice that

$$\limsup_{x\to+\infty} (\lambda_{\lceil x\rceil} - \lambda_{\lfloor x\rfloor})(x - \lfloor x\rfloor) < +\infty.$$

This is because

$$x - \lfloor x\rfloor = \{x\}$$

(fractional part of x) is between 0 and 1; furthermore

$$\limsup_{x\to+\infty} \left(\lambda_{\lceil x\rceil} - \lambda_{\lfloor x\rfloor}\right) = \max_{0\le j\le p-1} a_j.$$

(Hence the first two summands above converge to 0, since

$$y - x > x(\beta - 1) \to +\infty \quad \text{as} \quad x \to +\infty.$$

Concerning the last summand, we write it as

$$\frac{\lambda_{\lfloor y\rfloor}}{y} + \left(\frac{\lambda_{\lfloor y\rfloor}}{y} - \frac{\lambda_{\lfloor x\rfloor}}{x}\right) \cdot \frac{x}{y-x}$$

which allows to obtain the claimed result by (13) and the fact that

$$0 \le \frac{x}{y-x} \le \frac{1}{\beta - 1}. \qquad \square$$

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Rita Giuliano
Dipartimento di Matematica,
Università di Pisa
Largo Bruno Pontecorvo 5
I-56127 Pisa
ITALY
E-mail: rita.giuliano@unipi.it

Georges Grekos
Institut Camille Jordan
Université Jean Monnet
20 rue Dr. Rémy Annino
FR-42023 Saint-Étienne Cedex 2
FRANCE
E-mail: grekos@univ-st-etienne.fr