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uniform distribution theory

DOI: 10.2478/UDT-2024-0005
Unif. Distrib. Theory **19** (2024), no.1, 43–66

THE BEURLING-MALLIAVIN DENSITY, THE PÓLYA DENSITY AND THEIR CONNECTION

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ABSTRACT. In this paper we analyze the Beurling-Malliavin density and some other quantities related to it. Then we consider the upper Pólya density and show how its existence is connected with the concept of subadditivity; moreover a theorem is presented that clarifies the connection between the upper Pólya and the Beurling-Malliavin densities. In the last section we discuss the classical definition of the upper Pólya density and we prove a result which seems to be new.

Communicated by Vladimír Baláž

1. Introduction

Before presenting our work on the Beurling-Malliavin and Pólya densities, a clarification is in order. The Beurling-Malliavin density is defined in [1] for general families $(\lambda_n)_{n\in\mathbb{Z}}$ of real numbers; on the other hand, the Pólya density concerns sequences of real numbers (i.e., indexed by $n \in \mathbb{N}^*$), positive and strictly increasing ($0 \le \lambda_1 < \lambda_2 < \cdots$) and such that $\lim_{n\to\infty} \lambda_n = +\infty$. Thus, in order to compare these two concepts in what follows *we shall confine ourselves to sequences of the second (i.e., Pólya's) kind. We also emphasize*

2020 Mathematics Subject Classification: 11B25, 11B05.
Keywords: Beurling–Malliavin density, upper Pólya density, increasing sequence, substantial interval, counting function, subadditivity, η-covering.

that our study concerns sequences of real numbers and not only of integers. In the sequel, by the term *sequence* we always mean a sequence of Pólya's kind, unless otherwise specified. Notice in particular that we are not dealing with finite sequences and sequences with repetitions.

Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ be such a sequence. We need the following

Definition 1. The *counting function* of $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ is the function

$$F(t) = \begin{cases} 0, & t = 0, \\ \#\{k \in \mathbb{N}^* : \lambda_k \le t\}, & t > 0. \end{cases}$$

For $0 \le a < b$ we have clearly

$$F(b) - F(a) = \#\{k \in \mathbb{N}^* : \lambda_k \in (a, b]\}.$$

The counting function of a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ is non-decreasing, is identified by the sequence itself, and viceversa; since the definitions of the Pólya and BM densities can be formulated in terms of F, in the sequel we shall always adopt the point of view of counting functions in place of that of sequences.

1.1. The Beurling-Malliavin density.

Note. Let $\Lambda = \big(\lambda_n\big)_{n\in\mathbb{Z}}$ be an indexed family of real numbers, without finite accumulation points. Denote by $\mathcal{E}_\Lambda$ the system of characters

$$\mathcal{E}_\Lambda = \left\{ \mathrm{e}^{\pm \mathrm{i}\lambda_n x}, \lambda_n \in \Lambda \right\}$$

and define the *radius of completeness of* Λ as

$$\mathcal{R}(\Lambda) = \sup \left\{ a \in \mathbb{R}^+ : \mathcal{E}_\Lambda \text{is complete in } L^2(0, a) \right\}.$$

In order to quantify the connection between Λ and $\mathcal{R}(\Lambda)$, A. Beurling and P. Malliavin introduced for the first time in the paper [1] the number $b(\Lambda)$, which is now called the "Beurling-Malliavin density"of Λ (BM-density for short). The celebrated Theorem of [1] states that the radius of completeness of Λ is connected with $b(\Lambda)$ by the formula

$$\mathcal{R}(\Lambda) = 2\pi b(\Lambda).$$

The use of the term "density"can be motivated as follows: the idea is to try to express the periodicity of a set of functions generated by a certain set of characters $\mathcal{E}_\Lambda$, and this periodicity is then considered as "richness" of $\mathcal{E}_\Lambda$, hence of Λ.

The BM-density, firstly defined in [1], has been studied by other authors, and various equivalent formulations have been found; see [7] and the recent [3] for exhaustive lists of references. In particular it is extensively discussed in [5]. This book employs a definition which is apparently completely different from the original one and proves the equivalence of the two concepts. Here we introduce the definition of [5] and, for the sake of clarity, we reformulate it in a simpler fashion than the original one.

The definition of the BM-density. Go back to the restricted framework described at the beginning of this Introduction, and let $\mathfrak{C}$ be the family of all sequences $\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*}$ of intervals in $(0,+\infty)$ such that $a_n < b_n \le a_{n+1}$ for all $n \in \mathbb{N}^*$ and

$$\sum_{n=1}^{\infty}\left(\frac{b_n}{a_n}-1\right)^2 = +\infty.$$

In [5] these systems of intervals are called *substantial* (they are nothing but the *l*ong sequences of [6] and [7]). Then put

$$\overline{\delta}(\mathcal{I}) := \limsup_{n\to\infty}\frac{b_n}{a_n}; \qquad \ell_{\mathcal{I}} := \liminf_{n\to\infty}\frac{F(b_n)-F(a_n)}{b_n-a_n}. \tag{1}$$

[5] considers the set

$$\mathfrak{R} = \Big\{R \ge 0 : \exists\, \mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*} \in \mathfrak{C},\ \frac{F(b_n)-F(a_n)}{b_n-a_n} \ge R \text{ for, each sufficiently large } n\Big\}.$$

Reformulating, we see immediately that

$$\mathfrak{R} = \bigcup_{\mathcal{I}\in\mathfrak{C}}[0,\ell_{\mathcal{I}}];$$

Then the *Beurling–Malliavin density* $b(\Lambda)$ is defined as the supremum of $\mathfrak{R}$; in formula

$$b(\Lambda) = \sup\mathfrak{R} = \sup\{\ell_{\mathcal{I}}, \mathcal{I}\in\mathfrak{C}\}. \tag{2}$$

In the present paper we analyze the BM-density according to the definition given above. Precisely, in Section 2, we introduce some new quantities that have connection with $b(\Lambda)$; these quantities will be used later to state the first main result.

1.2. The upper Pólya density.

Another interesting type of density is introduced in [8] with the scope of studying gaps and singularities of power series. It is usually named as "upper Pólya density".

The *u*pper *P*ólya density of Λ is the number

$$\overline{p}(\Lambda) = \lim_{\xi \to 1^-} \limsup_{x \to \infty} \frac{F(x) - F(\xi x)}{x - x\xi}.$$

The existence of the above limit (as an extended number, see Remark 2 (i) here below) is a known fact, proved in [8] (see also [4]). In Section 3, we give a new proof of this fact (see Proposition 8). The aim is to show that $\overline{p}(\Lambda)$ is connected with the notion of subadditivity.

Remark 2.

(i) Let $\Lambda = (\log n)_{n \geq 1}$. From $t - 1 \leq F(t) \leq t + 1$ it follows easily that $\overline{p}(\Lambda) = +\infty$.

(ii) It is also easy to see that $\overline{p}(\Lambda) \in [0, 1]$ if Λ is a sequence of integers.

1.3. The first main result.

Our first main result (Theorem 9) clarifies the connection between the upper Pólya and the BM-densities. Precisely, recall the notation (1) and let $\mathfrak{C}$ be the family of all substantial sequences of intervals. Then put

$$\mathfrak{C}_{(1,+\infty]} = \{\mathcal{I} \in \mathfrak{C} : \overline{\delta}(\mathcal{I}) > 1\}, \qquad \mathfrak{R}_{(1,+\infty]} = \bigcup_{\mathcal{I} \in \mathfrak{C}_{(1,+\infty]}} [0, \ell_{\mathcal{I}}];$$

finally let

$$b_{(1,+\infty]}(\Lambda) = \sup \mathfrak{R}_{(1,+\infty]} = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}_{(1,+\infty]}\}.$$

Theorem 9, in Section 4, states that

$$\overline{p}(\Lambda) = b_{(1,+\infty]}(\Lambda)$$

and, as a consequence of the discussion of Section 2, we have that

$$\overline{p}(\Lambda) \leq b(\Lambda).$$

1.4. The second main result.

In the last section of the paper we prove our second main result (Theorem 12); precisely we show that the inner limit appearing in the definition of the upper Pólya density, which in its original definition in [8] is calculated along the reals, can actually be calculated along the sequence of integers only. This fact seems to have not been noticed anywhere in the past.

2. Analysis of the Beurling-Malliavin density

Let $A \subseteq [1,+\infty]$. In this Section we are interested in the subset of $\mathfrak{C}$ defined as

$$\mathfrak{C}_A = \big\{\mathcal{I} \in \mathfrak{C} : \overline{\delta}(\mathcal{I}) \in A\big\}.$$

Accordingly, we denote

$$\mathfrak{R}_A = \bigcup_{\mathcal{I}\in\mathfrak{C}_A} [0, \ell_{\mathcal{I}}],$$

and finally

$$b_A(\Lambda) = \sup \mathfrak{R}_A = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}_A\}. \tag{3}$$

In the case $A = \{\alpha\}$ (i.e., A is a singleton) we shall simplify this set of notation to $\mathfrak{C}_\alpha$, $\mathfrak{R}_\alpha$ and $b_\alpha(\Lambda)$, respectively.

For $A \subseteq B \subseteq [1,+\infty]$ we point out the obvious relations

$$\mathfrak{C}_A \subseteq \mathfrak{C}_B, \qquad \mathfrak{R}_A \subseteq \mathfrak{R}_B, \qquad b_A(\Lambda) \le b_B(\Lambda). \tag{4}$$

Since $a_n < b_n$ for every substantial sequence of intervals, of course,

$$\overline{\delta}(\mathcal{I}) \in [1,+\infty];$$

thus, according to the preceding notation, we have

$$b(\Lambda) = b_{[1,+\infty]}(\Lambda); \quad \text{hence} \quad b_{(1,+\infty]}(\Lambda) \le b(\Lambda).$$

The aim of the present Section is to give a precise mathematical formulation and a rigorous proof of the intuitive feelings that, in defining the BM-density,

(i) all the sequences of substantial intervals

$$\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*} \quad \text{with} \quad \limsup_{n\to\infty} \frac{b_n}{a_n} > 1$$

have the same "status", so to say (see Proposition 4);

(ii) for identifying the value of $b(\Lambda)$ the only important sequences of substantial intervals $\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*}$ are those with $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$ (see Theorem 3 here below).

Theorem 3. $b(\Lambda) = b_1(\Lambda)$.

For proving Theorem 3 the first step is the following Proposition, which says that in formula (2) we can restrict ourselves to taking the supremum of $\ell_{\mathcal{I}}$ in particular classes $\mathfrak{C}_A$ (i.e., for particular sets A) in place of the whole class $\mathfrak{C}$ (see (3)).

Proposition 4. *For $k > 1$ consider the set $A_k = [1, k]$. Then, for every $k > 1$,*

$$b(\Lambda) = b_{A_k}(\Lambda).$$

The inequality $\geq$ is obvious by the last relation in (4). Thus it is sufficient to show that

Proposition 5. *For fixed $k > 1$ consider the set $A_k = [1, k]$. Then*

(i) *for every $\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*} \in \mathfrak{C}$, there exists $\mathcal{J}_{\mathcal{I}} \in \mathfrak{C}_{A_k}$ such that $\ell_{\mathcal{J}_{\mathcal{I}}} \geq \ell_{\mathcal{I}}$. Hence,*

$$\begin{aligned} b(\Lambda) \;&=\; \sup\{\ell_{\mathcal{I}},\ \mathcal{I} \in \mathfrak{C}\} \;\leq\; \sup\{\ell_{\mathcal{J}_{\mathcal{I}}}, \mathcal{I} \in \mathfrak{C}\} \\ &\leq\; \sup\{\ell_{\mathcal{J}}, \mathcal{J} \in \mathfrak{C}_{A_k}\} \;=\; b_{A_k}(\Lambda). \end{aligned}$$

(ii) *$\mathcal{J}_{\mathcal{I}} = \big((c_n, d_n]\big)_{n\in\mathbb{N}^*}$ can be chosen in such a way that $\lim_{n\to\infty} \frac{d_n}{c_n}$ exists (and belongs to A_k).*

Proof. For any $n \in \mathbb{N}^*$, take $r_n = \big\lfloor \log_k \frac{b_n}{a_n} \big\rfloor + 1$, $\alpha_n = \big(\frac{b_n}{a_n}\big)^{\frac{1}{r_n}}$ and, for $i = 1, 2, \dots, r_n$, define

$$c_{n,i} = \alpha_n^{i-1} a_n, \quad d_{n,i} = \alpha_n^i a_n;$$

noticing that

$$c_1 = a_n, \quad d_{r_n} = b_n, \quad \frac{d_{n,i}}{c_{n,i}} = \alpha_n \leq k,$$

we have split the interval $(a_n, b_n]$ into r_n subintervals $(c_{n,i}, d_{n,i}]$ such that

$$1 \leq \frac{d_{n,i}}{c_{n,i}} \leq k, \qquad i = 1, \dots r_n. \tag{5}$$

There exists $j_n \in \{1, 2, \dots, r_n\}$ such that

$$\frac{F(d_{n,j_n}) - F(c_{n,j_n})}{d_{n,j_n} - c_{n,j_n}} \geq \frac{F(b_n) - F(a_n)}{b_n - a_n}; \tag{6}$$

if not, i.e., if

$$\frac{F(d_{n,i}) - F(c_{n,i})}{d_{n,i} - c_{n,i}} < \frac{F(b_n) - F(a_n)}{\sum_{j=0}^{r_n}(d_{n,i} - c_{n,i})}, \qquad \forall\, i = 0, 1, 2, \dots, r_n,$$

and since the family $\big((c_{n,i}, d_{n,i}]\big)$ covers $(a_n, b_n]$, we obtain

$$\begin{aligned} F(b_n) - F(a_n) &\le \sum_{i=1}^{r_n} \big(F(d_{n,i}) - F(c_{n,i})\big) \\ &< \frac{F(b_n) - F(a_n)}{\sum_{j=0}^{r_n}(d_{n,i} - c_{n,i})} \sum_{i=1}^{r_n}(d_{n,i} - c_{n,i}) \\ &= F(b_n) - F(a_n), \end{aligned}$$

a contradiction.

Now notice that the sequence of numbers

$$\left(\frac{d_{n,j_n}}{c_{n,j_n}}\right)_{n\in\mathbb{N}^*}$$

is bounded by (5), hence, by possibly passing to a subsequence, we can assume that it is convergent. It is clear that the sequence of intervals

$$\mathcal{J} = \big((c_{n,j_n}, d_{n,j_n}]\big)_{n\in\mathbb{N}^*}$$

belongs to $\mathfrak{C}_{A_k}$ and, by (6), verifies the relation $\ell_{\mathcal{J}} \ge \ell_{\mathcal{I}}$. The Proposition is proved. ☐

Proof of Theorem 3. The inequality $\ge$ is obvious by the last relation in (4). Thus, by Proposition 4, it suffices to prove that for each sequence of intervals $\mathcal{I} = \big((a_n, b_n]\big)_{n\in\mathbb{N}^*} \in \mathfrak{C}_{A_2}$ there exists a sequence of intervals $\mathcal{J}_{\mathcal{I}} \in \mathfrak{C}_1$ such that $\ell_{\mathcal{J}_{\mathcal{I}}} \ge \ell_{\mathcal{I}}$. This implies

$$\begin{aligned} b(\Lambda) \;&=\; \sup\{\ell_{\mathcal{I}},\, \mathcal{I} \in \mathfrak{C}\} \;\le\; \sup\{\ell_{\mathcal{J}_{\mathcal{I}}}, \mathcal{I} \in \mathfrak{C}\} \\ &\le\; \sup\{\ell_{\mathcal{J}}, \mathcal{J} \in \mathfrak{C}_1\} \;=\; b_1(\Lambda). \end{aligned}$$

Let $n \in \mathbb{N}^*$ be fixed and put $\eta_n = (\frac{b_n}{a_n})^{\frac{1}{n}}$. Similarly as in the proof of Proposition 5 we can construct an interval

$$J_n = (c_n, d_n] \subseteq (a_n, b_n]$$

such that

$$1 \le \frac{d_n}{c_n} \le \eta_n \tag{7}$$

and

$$\frac{F(d_n) - F(c_n)}{d_n - c_n} \ge \frac{F(b_n) - F(a_n)}{b_n - a_n}.$$

Now notice that, since

$$\limsup_{n\to\infty} \frac{b_n}{a_n} \le 2,$$

the sequence $\left(\frac{b_n}{a_n}\right)_{n\in\mathbb{N}^*}$ is bounded by some constant c, hence $\lim_{n\to\infty}\eta_n = 1$ by the inequalities

$$1 \le \liminf_{n\to\infty}\eta_n \le \limsup_{n\to\infty}\eta_n \le \lim_{n\to\infty}\sqrt[n]{c} = 1.$$

Thus

$$\lim_{n\to\infty}\frac{d_n}{c_n} = 1$$

by (7), which concludes the proof. □

Let A be a subset of $[1, +\infty]$ containing 1. Then, by the last relation in (4) and by Theorem 3,

$$b_1(\Lambda) \le b_A(\Lambda) \le b(\Lambda) = b_1(\Lambda).$$

This proves

Corollary 6. $b_A(\Lambda) = b(\Lambda)$ *if* $1 \in A$.

3. On the existence of the upper Pólya density $\overline{p}(\Lambda)$

As announced in the Introduction (see Subsection 1.2), we wish to show how the existence of the upper Pólya density has to do with the concept of subadditivity. We do this in Proposition 8, which is based on Lemma 7.

Lemma 7 is a generalization of a famous "subadditivity lemma" due to M. Fekete (see [2]). The proof is new and is reported in the Appendix at the end of the paper.

Lemma 7. *Let* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be a function such that there exists a continuous non-decreasing function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following property: for every* $a, b \ge 0$,

$$g(a+b) \le \frac{\phi(a)}{\phi(a+b)}g(a) + \left(1 - \frac{\phi(a)}{\phi(a+b)}\right)g(b). \tag{8}$$

Then $\lim_{x\to 0^+} g(x)$ *exists and*

$$\lim_{x\to 0^+} g(x) = \sup_{x>0} g(x).$$

Proposition 8. *The limit*

$$\overline{p}(\Lambda) = \lim_{\xi\to 1^-} \limsup_{x\to\infty} \frac{F(x) - F(\xi x)}{x - x\xi}$$

exists. Moreover,

$$\lim_{\xi\to 1^-} \limsup_{x\to\infty} \frac{F(x) - F(\xi x)}{x - x\xi} = \sup_{\xi<1} \limsup_{x\to\infty} \frac{F(x) - F(\xi x)}{x - x\xi}. \tag{9}$$

P r o o f. The aim is to apply the previous Lemma to the function

$$g(x) = \limsup_{y\to\infty} \frac{F(y) - F\big(y\mathrm{e}^{-x}\big)}{y - y\mathrm{e}^{-x}}, \quad x > 0,$$

obtaining

$$\begin{aligned}\lim_{\xi\to 1^-} \limsup_{y\to\infty} \frac{F(y) - F(\xi y)}{y - y\xi} &= \lim_{x\to 0^+} \limsup_{y\to\infty} \frac{F(y) - F\big(y\mathrm{e}^{-x}\big)}{y - y\mathrm{e}^{-x}} \\ &= \sup_{x>0} \limsup_{y\to\infty} \frac{F(y) - F(y\mathrm{e}^{-x})}{y - y\mathrm{e}^{-x}} \\ &= \sup_{\xi<1} \limsup_{y\to\infty} \frac{F(y) - F(\xi y)}{y - y\xi}.\end{aligned}$$

So we prove the subadditivity of g. We have

$$\begin{aligned}\frac{F(y) - F(y\mathrm{e}^{-(a+b)})}{y - y\mathrm{e}^{-(a+b)}} &= \frac{F(y) - F(y\mathrm{e}^{-a})}{y - y\mathrm{e}^{-a}} \times \frac{1 - \mathrm{e}^{-a}}{1 - \mathrm{e}^{-(a+b)}} \\ &\quad + \frac{F(y\mathrm{e}^{-a}) - F(y\mathrm{e}^{-(a+b)})}{y\mathrm{e}^{-a}(1 - \mathrm{e}^{-b})} \\ &\quad \times \left(1 - \frac{1 - \mathrm{e}^{-a}}{1 - \mathrm{e}^{-(a+b)}}\right);\end{aligned}$$

passing to the limsup in y we obtain

$$g(a + b) \leq \frac{1 - \mathrm{e}^{-a}}{1 - \mathrm{e}^{-(a+b)}} g(a) + \Big(1 - \frac{1 - \mathrm{e}^{-a}}{1 - \mathrm{e}^{-(a+b)}}\Big) g(b).$$

Since the function $\phi(x) = 1 - \mathrm{e}^{-x}$ is trivially non–decreasing and continuous, the proof is concluded. □

4. Comparison between $\overline{p}(\Lambda)$ and $b_{(1,+\infty]}(\Lambda)$

In this section, we are concerned with

$$\mathfrak{C}_{(1,+\infty]}, \quad \mathfrak{R}_{(1,+\infty]} \quad \text{and} \quad b_{(1,+\infty]}(\Lambda),$$

which we shall denote as $\mathfrak{C}_{>1}$, $\mathfrak{R}_{>1}$ and $b_{>1}(\Lambda)$, respectively, for easier writing. It is known that in general $\overline{p}(\Lambda) \leq b(\Lambda)$ (see [5]). The aim of this section is to prove a more precise relation. In fact

Theorem 9. *We have*

$$\overline{p}(\Lambda) = b_{>1}(\Lambda).$$

We split the proof into two parts, namely, Propositions 10 and 11.

Proposition 10. *The following inequality holds true:*

$$\overline{p}(\Lambda) \leq b_{>1}(\Lambda).$$

Proof. It suffices to show that, for every $R < \overline{p}(\Lambda)$ we have $R \leq b_{>1}(\Lambda)$. By definition of $\overline{p}(\Lambda)$, there exists $\xi < 1$ such that

$$\limsup_{x\to\infty} \frac{F(x) - F(x\xi)}{x - x\xi} > R.$$

By definition of limsup, there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ such that

$$\lim_{n\to\infty} x_n = +\infty, \quad \lim_{n\to\infty} \frac{F(x_n) - F(x_n\xi)}{x_n - x_n\xi} = \limsup_{x\to\infty} \frac{F(x) - F(x\xi)}{x - x\xi} > R. \tag{10}$$

Set

$$n_1 = 1, \quad y_1 = x_{n_1} = x_1 \quad \text{and} \quad n_2 = \min\left\{n > 1 : x_n \geq \tfrac{x_1}{\xi}\right\}$$

(n_2 exists since otherwise the sequence $(x_n)_{n\in\mathbb{N}^*}$ would be bounded). Put

$$y_2 = x_{n_2}, \quad \textit{then} \quad y_2 \geq \frac{y_1}{\xi}.$$

Assume we have constructed

$$n_2, \dots, n_r \quad \text{and} \quad y_2, \dots, y_r \quad \text{such that} \quad y_k \geq \frac{y_{k-1}}{\xi} \quad \text{for each} \quad k = 2, \dots, r.$$

Let

$$n_{r+1} = \min\left\{n > n_r : x_n \geq \frac{x_{n_r}}{\xi}\right\}$$

(n_{r+1} exists for the same reason as above) and let

$$y_{r+1} = x_{n_{r+1}}.$$

By this recursive construction we obtain a subsequence $(y_n)_{n\in\mathbb{N}^*}$ of $(x_n)_{n\in\mathbb{N}^*}$

with the property that

$$y_{n+1} \geq \frac{y_n}{\xi}, \qquad n = 1, 2, \ldots; \tag{11}$$

now set

$$a_n = y_n \xi \quad \text{and} \quad b_n = y_n.$$

It is easy to see that the sequence of intervals $\mathcal{I} = \big((a_n, b_n]\big)_{n \in \mathbb{N}^*}$ belongs to $\mathfrak{C}_{>1}$:

(i) $a_n < b_n \leq a_{n+1}$ since this means $y_n \xi < y_n \leq y_{n+1} \xi$, which is true by (11);

(ii)
$$\sum_{n=1}^{\infty} \left(\frac{b_n}{a_n} - 1\right)^2 = \sum_{n=1}^{\infty} \left(\frac{1}{\xi} - 1\right)^2 = \infty;$$

(iii)
$$\limsup_{n \to \infty} \frac{b_n}{a_n} = \frac{1}{\xi} > 1.$$

Since ultimately

$$\frac{F(b_n) - F(a_n)}{b_n - a_n} > R$$

by (10), we deduce that $R \in \mathfrak{R}_{>1}$, hence $R \leq \sup \mathfrak{R}_{>1} = b_{>1}(\Lambda)$. □

Now we are concerned with the reverse inequality.

Proposition 11. *We have*

$$b_{>1}(\Lambda) \leq \overline{p}(\Lambda).$$

P r o o f. Let $R \in \mathfrak{R}_{>1}$ and let $\mathcal{I} = \big((a_n, b_n]\big)_{n \in \mathbb{N}^*} \in \mathfrak{C}_{>1}$ with $\frac{\Lambda(b_n) - \Lambda(a_n)}{b_n - a_n} \geq R$. Denote

$$\limsup_{n \geq 1} \frac{b_n}{a_n} = L \geq 1.$$

Fix $\epsilon > 0$; we have ultimately

$$a_n > \frac{b_n}{L + \epsilon},$$

hence

$$\begin{aligned} R \leq \frac{F(b_n) - F(a_n)}{b_n - a_n} &\leq \frac{F(b_n) - F(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \cdot \frac{b_n(1 - \frac{1}{L+\epsilon})}{b_n - a_n} \\ &= \frac{F(b_n) - F(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \cdot \frac{\frac{b_n}{a_n}(1 - \frac{1}{L+\epsilon})}{\frac{b_n}{a_n} - 1} \\ &\leq \frac{F(b_n) - F(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \cdot \frac{L + \epsilon - 1}{\frac{b_n}{a_n} - 1}. \end{aligned}$$

We deduce

$$\frac{F(b_n)-F(\frac{b_n}{L+\epsilon})}{b_n-\frac{b_n}{L+\epsilon}} \geq R\cdot\frac{\frac{b_n}{a_n}-1}{L+\epsilon-1}$$

and, passing to the limsup in x,

$$\begin{aligned}\limsup_{x\to\infty}\frac{F(x)-F(\frac{x}{L+\epsilon})}{x-\frac{x}{L+\epsilon}} &\geq \limsup_{x\to\infty}\frac{F(b_n)-F(\frac{b_n}{L+\epsilon})}{b_n-\frac{b_n}{L+\epsilon}}\\ &\geq R\cdot\limsup_{n\to\infty}\frac{\frac{b_n}{a_n}-1}{L+\epsilon-1}\\ &= R\cdot\frac{L-1}{L+\epsilon-1}.\end{aligned}$$

Thus, observing that $\frac{1}{L+\epsilon}<1$ and by Proposition 8 (see relation (9)), we get

$$\begin{aligned}\overline{p}(\Lambda)=\sup_{\xi<1}\limsup_{x\to\infty}\frac{F(x)-F(x\xi)}{x-x\xi} &\geq \limsup_{x\to\infty}\frac{F(x)-F(\frac{x}{L+\epsilon})}{x-\frac{x}{L+\epsilon}}\\ &\geq R\cdot\frac{L-1}{L+\epsilon-1},\end{aligned}$$

for every $\epsilon>0$. Now pass to the limit as $\epsilon\to 0$ to obtain that

$$\overline{p}(\Lambda)\geq R, \qquad \forall\, R\in\mathfrak{R}_{>1},$$

and optimizing

$$\overline{p}(\Lambda)\geq\sup\mathfrak{R}_{>1}=b_{>1}(\Lambda). \qquad \square$$

5. On the limit in the definition of the upper Pólya density

As we have seen at the beginning of Section 4, the inner limit in the definition of $\overline{p}(\Lambda)$ is calculated as $x\to\infty$, where x is a real variable. Actually, Pólya in [8] uses the symbol r (instead of x) without specifying where r varies, but there is no reason to suppose that he didn't have the real numbers in mind. Anyway, in this Section we prove the following result:

Theorem 12. *The limit*

$$\ell := \lim_{\eta\to 1^-}\limsup_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{F(n)-F(\eta n)}{(1-\eta)n}$$

exists and its value is $\overline{p}(\Lambda)$*.*

Theorem 12 is easy to prove if we have some further assumption on F, for instance,

(i) if the function $x \mapsto \frac{F(x)-F(\eta x)}{(1-\eta)x}$ is monotone (for instance if F is concave or convex), since in this case

$$\limsup_{\substack{n\to\infty \\ n\in\mathbb{N}^*}} \frac{F(n)-F(\eta n)}{(1-\eta)n} = \limsup_{x\to\infty} \frac{F(x)-F(\eta x)}{(1-\eta)x};$$

(ii) if $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}^*}} \frac{F(n+1)-F(n)}{n} = 0$. In fact, putting $n_x = \lfloor x \rfloor$, we have easily

$$\begin{aligned} F(x) - F(\eta x) &\le F(n_x+1) - F(\eta n_x) \\ &= \{F(n_x+1) - F(n_x)\} + F(n_x) - F(\eta n_x), \end{aligned}$$

which implies

$$\begin{aligned} \limsup_{x\to\infty} \frac{F(x)-F(\eta x)}{(1-\eta)x} &\le \limsup_{x\to\infty} \frac{F(n_x)-F(\eta n_x)}{(1-\eta)n_x} \\ &\le \limsup_{\substack{n\to\infty \\ n\in\mathbb{N}^*}} \frac{F(n)-F(\eta n)}{(1-\eta)n}. \end{aligned}$$

Anyway, in general, the proof of Theorem 12 is rather intricated and needs some preparation. In particular we need to construct a particular covering of the interval $(\xi x, x]$, i.e., a finite family of intervals $\{(a_i, b_i]\}$ with right endpoints b_i belonging to $\mathbb{N}$ and such that

$$(\xi x, x] \subseteq \bigcup_i (a_i, b_i].$$

5.1. Construction of an η-covering of $(\xi x, x]$.

For $x > 0$ and $\eta \in (0,1)$ let

$$\phi(x) = \lceil x \rceil \quad \text{and} \quad \psi_\eta(x) = \eta x, \quad f_\eta(x) = (\phi \circ \psi)(x) = \lceil \eta x \rceil.$$

We denote by f_η^m the function obtained by composing f_η with itself m times, i.e.,

$$f_\eta^0(x) = x; \qquad f_\eta^{m+1}(x) = (f_\eta \circ f_\eta^m)(x).$$

Similarly for ψ_η^m.

For the sake of convenience, in the sequel we eliminate the suffix η and write f, ψ in place of f_η, ψ_η and so on.

Lemma 13. *We have the following facts:*

(i) $t \le \phi(t)$, *for every* $t > 0$;

(ii) $\psi^m(t) \le f^m(t)$ *for every integer* $m \ge 0$;

(iii) $(f^m \circ \phi)(x) \le \eta^m x + \sum_{k=0}^m \eta^k$ *for every integer* $m \ge 0$.

Proof. (i) is evident.

We prove (ii) by induction: the case $m = 0$ is obvious; the case $m = 1$ follows from (i) with $\psi(t)$ in place of t. For the case $m+1$ we have,

$$\psi^{m+1}(t) = \psi\big(\psi^m(t)\big) \le \psi\big(f^m(t)\big) \le f\big(f^m(t)\big) = f^{m+1}(t)$$

where, besides the inductive assumption, we have used the case $m = 1$ and the fact that ψ is increasing.

Now we prove (iii), again by induction. The case $m = 0$ is obvious (recall that $f^0(x) = x$) and reads as $\phi(x) \le x + 1$. The inductive step uses the inductive assumption and the fact that f is nondecreasing:

$$\begin{aligned} f^{m+1}\big(\phi(x)\big) &= f\Big(f^m\big(\phi(x)\big)\Big) \le f\left(\eta^m x + \sum_{k=0}^m \eta^k\right) \\ &= \left\lceil \eta\left(\eta^m x + \sum_{k=0}^m \eta^k\right)\right\rceil \le \eta\left(\eta^m x + \sum_{k=0}^m \eta^k\right) + 1 \\ &= \eta^{m+1} x + \sum_{k=0}^{m+1} \eta^k. \end{aligned}$$

□

Remark 14. By Lemma 13 (iii).

$$\eta^m x \le (f^m \circ \phi)(x) \le \eta^m x + \sum_{k=0}^m \eta^k$$

and, by putting $S_\eta = \sum_{k=0}^\infty \eta^k$, it provides the bound

$$0 \le \sup_{m,x}\big((f^m \circ \phi)(x) - \eta^m x\big) \le S_\eta.$$

For fixed x and for every integer $i \ge 0$ put $b_i^{(\eta)} = (f^i \circ \phi)(x)$ and

$$a_i^{(\eta)} = \eta b_i^{(\eta)} = \psi\big(b_i^{(\eta)}\big) = \big(\psi \circ f^i \circ \phi\big)(x);$$

here and in the sequel we drop the symbol η an write a_i and b_i.

Notice that b_i is an integer and $b_0=\phi(x)=\lceil x\rceil$; notice also that, for every $i\geq 1$,

$$\begin{aligned} b_i &= (f^i\circ\phi)(x) = f\big((f^{i-1}\circ\phi)(x)\big) \\ &= (\phi\circ\psi)\big((f^{i-1}\circ\phi)(x)\big) \\ &= \phi\big((\psi\circ f^{i-1}\circ\phi)(x)\big) \\ &= \phi(a_{i-1}) = \lceil a_{i-1}\rceil. \end{aligned} \tag{12}$$

Last, denote

$$r(t) = \frac{1-\eta^{t+1}}{\eta^{t-1}(1-\eta)^2}, \qquad t\in[1,+\infty);$$

the function r is increasing, as one can check easily by writing it in the form

$$r(t) = \frac{1}{(1-\eta)^2}\big(\eta e^{t\log\frac{1}{\eta}} - \eta^2\big).$$

Lemma 15. *Let $x\geq r(q)$ for some integer $q\geq 1$. Then, for every $i=1,\dots,q$, the following inequalities hold*

$$\eta^{i+1}x \;\leq\; a_i \;\leq\; \eta^i x \;\leq\; a_{i-1} \;\leq\; b_i \;\leq\; \eta^{i-1}x \;\leq\; b_{i-1}.$$

Proof.

(a) That $a_{i-1}\leq b_i$ follows from (ii) of Lemma 13, since, by the above definitions and putting $t=(f^{i-1}\circ\phi)(x)$,

$$a_{i-1} = \psi(t) \leq f(t) = b_i.$$

(b) Now we prove that $\eta^{i-1}x\leq b_{i-1}$, i.e., $\psi^{i-1}(x)\leq f^{i-1}\circ\phi(x)$, which follows from

$$\psi^{i-1}(x) \underbrace{\leq}_{\text{Lemma 13, (ii)}} f^{i-1}(x) \underbrace{\leq}_{\text{Lemma 13, (i)}} f^{i-1}\circ\phi(x),$$

(since f^m is non-decreasing for every m).

(c) We prove that $b_i\leq\eta^{i-1}x$, which means

$$(f^i\circ\phi)(x)\leq\eta^{i-1}x.$$

From Lemma 13 (iii) we know that

$$f^i\big(\phi(x)\big)\leq\eta^i x+\sum_{k=0}^{i}\eta^k$$

and the inequality $\eta^i x+\sum_{k=0}^{i}\eta^k\leq\eta^{i-1}x$ is equivalent to $x\geq r\,(i)$;

thus the claim follows from $x \geq r(q) \geq r(i)$, recalling that r is increasing.

(d) From the preceding points (b) and (c) we have

$$b_i \leq \eta^{i-1}x \leq b_{i-1}$$

and multiplying by η we get

$$a_i = \eta b_i \leq \eta^i x \leq \eta b_{i-1} = a_{i-1}.$$

(e) In order to show that $\eta^{i+1}x \leq a_i$ it suffices to multiply by η each side of the inequality (already proved) $\eta^i x \leq b_i$.

The proof is complete. □

Lemma 16. *Let $\xi \in (0,1)$ and $\eta \in (\xi, 1)$ be fixed; assume that $x \geq r(q)$ for some integer $q \geq \frac{\log \xi}{\log \eta}$. Then the set $\{i \geq 0 : a_i > \xi x\}$ is a finite interval of integers $\{0, 1, \dots, d-1\}$ for some integer d with $1 \leq d \leq q$.*

Proof. From Lemma 15 we know that the sequence $\{a_i, i = 0, \dots, q\}$ is non-increasing. Moreover, $a_0 = \eta\lceil x\rceil > \xi x$, while $a_q \leq \eta^q x < \xi x$, since $q \geq \frac{\log \xi}{\log \eta}$ by the assumption. This proves the statement. □

Let $\xi \in (0,1)$ and $\eta \in (\xi,1)$ be fixed and assume that $x \geq r(q)$ for some integer $q \geq \frac{\log \xi}{\log \eta}$. From now on, by the symbol $d(= d^{(\eta)})$ we denote the first integer such that $a_d \leq \xi x$, i.e., d verifies

$$a_d \leq \xi x < a_{d-1},$$

further, we recall that $b_0 = \lceil x\rceil \geq x$. Thus,

$$(\xi x, x] \subseteq \bigcup_{i=0}^{d} (a_i, b_i].$$

Motivated by these remarks, we can give the following

Definition 17. By the *η-covering of* $(\xi x, x]$ we mean the family of intervals

$$\left\{ \left(a_i^{(\eta)}, b_i^{(\eta)}\right], \, i = 0, \dots, d^{(\eta)} \right\},$$

constructed as shown above.

In the following Sections η will be fixed; thus we shall still adopt the convention of eliminating the symbol η when considering the η-covering of $(\xi x, x]$.

5.2. Some properties of the η-covering of $(\xi x, x]$.

Lemma 18. *Let $\xi \in (0,1)$ and $\eta \in (\xi, 1)$ be fixed; assume that $x \geq \frac{1-\eta^2\xi}{\xi(1-\eta)^2}$. Let*

$$\{(a_i, b_i], i = 0, 1, \dots, d\}$$

be the η-covering of $(\xi x, x]$. Then $d \leq \left\lceil \frac{\log \xi}{\log \eta} \right\rceil$.

P r o o f. In the proof we denote $\left\lceil \frac{\log \xi}{\log \eta} \right\rceil =: q$ for simplicity. Notice that

$$x \geq \frac{1-\eta^2\xi}{\xi(1-\eta)^2} = r\left(\frac{\log \xi}{\log \eta} + 1\right) > r(q).$$

Hence Lemma 15 is in force for the integers $1, \dots, q$. Applying this lemma with $i = q$ and by the definition of the ceiling function, we find

$$a_q \leq \eta^q x \leq \xi x.$$

By the definition of d, this relation says that $d \leq q$. □

Remark 19. Actually, we can prove even more, precisely that

$$d \in \left\{ \left\lfloor \frac{\log \xi}{\log \eta} \right\rfloor - 1, \ \left\lfloor \frac{\log \xi}{\log \eta} \right\rfloor \right\}.$$

The proof of this fact is rather complicated, and is postponed in the last Section 7. Here Lemma 18 will be sufficient for our scopes.

Lemma 20. *Let $\xi \in (0,1)$ and $\eta \in (\xi, 1)$ be fixed. Denote*

$$M = M(\xi, \eta) := \frac{1-\eta^2\xi}{\xi(1-\eta)^2} \vee \frac{\frac{\log \xi}{\log \eta} + 2}{\eta^2\xi(1-\eta)}.$$

Let $x \geq M$ and let $\{(a_i, b_i], i = 0, 1, \dots, d\}$ be the η-covering of $(\xi x, x]$). Then

$$\sum_{i=0}^{d} (b_i - a_i) \leq (1 - \xi\eta^3)x.$$

P r o o f. Once more by Lemma 15, for every i the two intervals $(a_{i-1}, b_{i-1}]$ and $(a_i, b_i]$ overlap on the interval $(a_{i-1}, b_i]$, the length of which is

$$b_i - a_{i-1} = \lceil a_{i-1} \rceil - a_{i-1} \leq 1,$$

due to (12). Hence (recall that $b_0 = \lceil x \rceil$)

$$\sum_{i=0}^{d} (b_i - a_i) \leq b_0 - a_d + d = \lceil x \rceil - a_d + d \leq x + 1 - a_d + d \leq x + 1 - \eta^{d+1}x + d,$$

where we have used the left-hand inequality in Lemma 15.

Continuing and using Lemma 18, we find

$$x + 1 - \eta^{d+1}x + d = (1 - \eta^{d+1})x + d + 1 \le (1 - \eta^2\xi)x + d + 1 \le$$
$$(1 - \eta^2\xi)x + \frac{\log \xi}{\log \eta} + 2 \le (1 - \eta^3\xi)x,$$

where the first and second inequalities come from $d \le \frac{\log \xi}{\log \eta} + 1$ (by Lemma 18), and the last one holds since $x \ge M$. □

5.3. Use of the η-covering of $(\xi x, x]$.

Now we are in a position to prove Theorem 12.

First notice the obvious relations

$$\liminf_{\eta \to 1^-} \left(\limsup_n \frac{F(n) - F(\eta n)}{(1-\eta)n} \right) \le \limsup_{\eta \to 1^-} \left(\limsup_n \frac{F(n) - F(\eta n)}{(1-\eta)n} \right)$$
$$\le \sup_{\eta \in (0,1)} \left(\limsup_n \frac{F(n) - F(\eta n)}{(1-\eta)n} \right) \le \sup_{\xi \in (0,1)} \left(\limsup_x \frac{F(x) - F(\xi x)}{(1-\xi)x} \right);$$

hence, in order to prove Theorem 12, it suffices to show that

Proposition 21. *We have*

$$\liminf_{\eta \to 1^-} \left(\limsup_{n \to \infty} \frac{F(n) - F(\eta n)}{(1-\eta)n} \right) \ge \sup_{\xi \in (0,1)} \left(\limsup_{x \to \infty} \frac{F(x) - F(\xi x)}{(1-\xi)x} \right).$$

Proof. We shall use the following Lemma, the proof of which is postponed at the end.

Lemma 22. *Let $\xi \in (0,1)$ and $\epsilon \in (0,1)$ be fixed. Then there exists $\delta = \delta(\xi, \epsilon)$ with the following property: for every $\eta \in (\delta, 1)$ there exists $M = M(\xi, \epsilon, \eta)$ such that for every $x > M$ there exists an integer $n = n(\xi, \epsilon, \eta, x)$*

$$\frac{F(n) - F(\eta n)}{(1-\eta)n} \ge \frac{F(x) - F(\xi x)}{(1-\xi)x}(1-\epsilon).$$

Now, in order to prove Proposition 21, observe that in Lemma 22 the integer n depends on x, so here we denote it by n_x. Thus, passing to the limsup as $x \to \infty$ in the relation of Lemma 10 we obtain that, for any fixed ξ and ϵ, there exists $\delta(\xi, \epsilon)$ such that, for every $\eta \in (\delta(\xi, \epsilon), 1)$,

$$\limsup_{x \to \infty} \frac{F(n_x) - F(\eta n_x)}{(1-\eta)n_x} \ge \limsup_{x \to \infty} \frac{F(x) - F(\xi x)}{(1-\xi)x}(1-\epsilon).$$

Consequently we have also

$$\limsup_{n \to \infty} \frac{F(n) - F(\eta n)}{(1-\eta)n} \ge \limsup_{x \to \infty} \frac{F(x) - F(\xi x)}{(1-\xi)x}(1-\epsilon).$$

Denote provisorily

$$A(\xi) = \limsup_{x\to\infty} \frac{F(x) - F(\xi x)}{(1-\xi)x}; \quad B(\eta) = \limsup_{n\to\infty} \frac{F(n) - F(\eta n)}{(1-\eta)n}.$$

Then we have that, for any fixed ξ and ϵ, there exists $\delta(\xi, \epsilon)$ such that, for every $\eta \in (\delta(\xi, \epsilon), 1)$,

$$B(\eta) \geq A(\xi)(1-\epsilon);$$

hence, for every ξ and ϵ,

$$\liminf_{\eta\to 1^-} B(\eta) \geq A(\xi)(1-\epsilon),$$

and now the statement follows by the arbitrariness of ξ and ϵ. This concludes the proof of Proposition 21. □

It remains to give the proof of Lemma 22. To this extent, we need the following

Lemma 23. *Let $\xi \in (0,1)$ and $\eta \in (\xi, 1)$ be fixed. Then there exists $M = M(\xi, \eta) > 0$ such that, for every $x > M$, there exists an integer $n = n(\xi, x, \eta)$ that verifies the inequality*

$$\frac{F(n) - F(\eta n)}{(1-\eta)n} \geq \frac{F(x) - F(\xi x)}{(1-\xi\eta^3)x}.$$

Proof of Lemma 23. Let M be as in Lemma 20. If $x > M$, the η-covering of the interval $(\xi x, x]$, $\{(a_i, b_i], i = 0, \ldots, d\}$ is such that

$$\sum_i (b_i - a_i) \leq (1-\xi\eta^3)x.$$

By the argument used in the proof of Proposition 5, there exists $j \in \{0, \ldots, d\}$ such that

$$\frac{F(b_j) - F(a_j)}{b_j - a_j} \geq \frac{F(x) - F(\xi x)}{\sum_i (b_i - a_i)}.$$

By construction of an η-covering, b_j is an integer and $a_j = \eta b_j$; thus, if we call $b_j = n$, we have obtained

$$\frac{F(n) - F(\eta n)}{(1-\eta)n} \geq \frac{F(x) - F(\xi x)}{\sum_i (b_i - a_i)} \geq \frac{F(x) - F(\xi x)}{(1-\xi\eta^3)x}.$$ □

Proof of Lemma 22. Since

$$\lim_{\eta\to 1^-}\frac{1-\xi}{1-\xi\eta^3}=1,$$

there exists $\overline{\delta}(\xi,\epsilon)$ such that, for every $\eta\in(\overline{\delta},1)$,

$$1-\xi\eta^3<\frac{1-\xi}{1-\epsilon}.$$

Now take $\delta(\xi,\epsilon)=\overline{\delta}(\xi,\epsilon)\vee\xi$ and apply Lemma 23 to conclude. □

6. On the number d

This Section is devoted to the proof of the announced result concerning d (see Remark 19). Precisely,

Proposition 24.

(i) *Assume that* $\frac{\log\xi}{\log\eta}$ *is not an integer and*

$$x\geq\frac{1-\eta^2\xi}{\xi(1-\eta)^2}\vee\frac{\left\lfloor\frac{\log\xi}{\log\eta}\right\rfloor+2}{\xi-\eta^{\left\lfloor\frac{\log\xi}{\log\eta}\right\rfloor+1}}\vee\frac{1}{\eta^{\left\lfloor\frac{\log\xi}{\log\eta}\right\rfloor}-\xi}. \tag{13}$$

Then

$$d=\left\lfloor\frac{\log\xi}{\log\eta}\right\rfloor.$$

(ii) *If* $\frac{\log\xi}{\log\eta}$ *is an integer and*

$$x\geq\frac{1-\eta^2\xi}{\xi(1-\eta)^2}\vee\frac{\frac{\log\xi}{\log\eta}+2}{\xi(1-\eta)},$$

then

$$d\in\left\{\frac{\log\xi}{\log\eta}-1,\ \frac{\log\xi}{\log\eta}\right\}.$$

Proof. In the proof we denote $\left\lfloor\frac{\log\xi}{\log\eta}\right\rfloor=:p$ for simplicity. Notice also that Lemma 15 is applicable for $i=1,\dots p+1$, since, in both (i) and (ii),

$$x\geq\frac{1-\eta^2\xi}{\xi(1-\eta)^2}=r\left(\frac{\log\xi}{\log\eta}+1\right)\geq r(p+1).$$

Last, in case (i), $\eta^{p+1}<\xi<\eta^p$, hence the denominators $\xi-\eta^{p+1}$ and $\eta^p-\xi$ appearing in (13) are strictly positive.

The trivial observation that $\eta^m x \le b_m$ and Lemma 13 (iii) yield for every $m \ge 0$,

$$\eta^m x \le b_m \le \eta^m x + m + 1,$$

and by (12), for every $m \ge 0$

$$b_m - 1 < a_{m-1} \le b_m.$$

Putting $m = p$ and $m = p+1$ we obtain in turn

$$\eta^p x \le b_p \le \eta^p x + p + 1, \qquad b_p - 1 < a_{p-1} \le b_p \tag{14}$$

and

$$\eta^{p+1} x \le b_{p+1} \le \eta^{p+1} x + p + 2, \qquad b_{p+1} - 1 < a_p \le b_{p+1}. \tag{15}$$

Since $x \ge \frac{p+2}{\xi - \eta^{p+1}}$, we have, by (15) and Lemma 15,

$$a_p \le b_{p+1} \le \eta^{p+1} x + p + 2 \le \xi x, \tag{16}$$

which says that $d \le p$.

Now we distinguish the two cases (i) and (ii).

(i) If $\frac{\log \xi}{\log \eta}$ is not an integer, by (14),

$$a_{p-1} > b_p - 1 \ge \eta^p x - 1 > \xi x,$$

then, by (16),

$$a_p \le \xi x < a_{p-1}, \tag{17}$$

implying that $d = p$.

(ii) If $\frac{\log \xi}{\log \eta}$ is an integer, then $\xi x = \eta^p x$ and, by Lemma 15, $\eta^p x \le a_{p-1}$. If $\xi x (= \eta^p x) < a_{p-1}$, we have the same inequalities as before in (17) and $d = p$ again. Otherwise, $\eta^p x (= \xi x) = a_{p-1}$. We prove that in this case $d = p - 1$, which means that

$$a_{p-1} \le \xi x < a_{p-2}. \tag{18}$$

The proof of (18) follows from

Lemma 25. *Assume that for some $i \ge 0$ we have $\eta^{i+1} x = a_i$. Then*

$$\eta = \frac{b_1}{b_0}, \qquad x = b_0, \qquad a_j = \frac{b_1^{j+1}}{b_0^j}, \qquad j = 0, \dots, i.$$

Proof of Lemma 25. First, recall the definition of every a_j, i.e.,

$$a_j = \eta b_j. \tag{19}$$

Now, from $a_i = \eta^{i+1} x$ we deduce by (19) that $b_i = \eta^i x$. Since $\eta^i x \le a_{i-1} \le b_i$ (by Lemma 15), we obtain that $a_{i-1} = b_i$, whence $\eta b_i = \eta a_{i-1}$ and (by (19)) $a_i = \eta a_{i-1}$. Repeating this argument we get that $a_j = \eta a_{j-1}$ for every

$j = 1, \dots, i$; from $a_1 = \eta a_0$ and (19) (applied to a_0 and a_1) we deduce that $b_1 = \eta b_0$, hence $\eta = \frac{b_1}{b_0}$, which implies that

$$a_0 = \eta b_0 = \frac{b_1}{b_0} \cdot b_0 = b_1, \quad a_1 = \eta a_0 = \frac{b_1}{b_0} \cdot b_1 = \frac{b_1^2}{b_0}, \dots, a_j = \frac{b_1^{j+1}}{b_0^j}, \quad j = 2, \dots, i;$$

further

$$x = \frac{a_i}{\eta^{i+1}} = \frac{b_1^{i+1}}{b_0^i} \cdot \frac{b_0^{i+1}}{b_1^{i+1}} = b_0. \qquad \square$$

We go back to the proof of Proposition 24; the left inequality in (18) holds obviously (it is nothing but the equality $a_{p-1} = \eta^p x = \xi x$). Concerning the right one, by the same equality and Lemma 25, it is equivalent to $\frac{b_1^p}{b_0^{p-1}} < \frac{b_1^{p-1}}{b_0^{p-2}}$ and, after simplification, to $\frac{b_1}{b_0} < 1$, which is true since it is nothing but $\eta < 1$ (by Lemma 25 again). $\square$

7. Appendix: Proof of Lemma 7

P r o o f. First it is easily seen by a simple recursive argument that, for every $a \geq 0$ and for every $n \in \mathbb{N}^*$,

$$g(na) \leq g(a). \tag{20}$$

Let $y > 0$ be fixed. For every $x \in (0, y)$, let $n(x) = \lfloor \frac{y}{x} \rfloor - 1$, put

$$z(x) = y - n(x) \cdot x \tag{21}$$

and observe that, by the inequalities $a - 1 < \lfloor a \rfloor \leq a$, we have

$$x \leq z(x) \leq 2x. \tag{22}$$

Since $y = n(x) \cdot x + z(x)$, we get from (8) and (20) that

$$g(y) \leq \frac{\phi(n(x) \cdot x)}{\phi(y)} g(x) + \left(1 - \frac{\phi(n(x) \cdot x)}{\phi(y)}\right) g\big(z(x)\big). \tag{23}$$

Now we distinguish the two cases: (i) $\sup_{x>0} g(x) \in \mathbb{R}$; (ii) $\sup_{x>0} g(x) = +\infty$.

(i) It follows from (23) that

$$g(y) \leq \frac{\phi(n(x) \cdot x)}{\phi(y)} g(x) + \left(1 - \frac{\phi(n(x) \cdot x)}{\phi(y)}\right) \left(\sup_{x>0} g(x)\right).$$

Let $x \to 0^+$; we have from (22) that $z(x) \to 0$, hence $n(x) \cdot x \to y$ by (21); thus, by the continuity of ϕ,

$$\lim_{x\to 0^+} \frac{\phi(n(x)\cdot x)}{\phi(y)} = 1$$

and

$$\begin{aligned} g(y) &\le \Big(\liminf_{x\to 0^+} g(x)\Big) \lim_{x\to 0^+} \frac{\phi(n(x)\cdot x)}{\phi(y)} \\ &\quad + \Big(\sup_{x>0} g(x)\Big) \lim_{x\to 0^+} \Big(1 - \frac{\phi(n(x)\cdot x)}{\phi(y)}\Big) \\ &= \liminf_{x\to 0^+} g(x). \end{aligned}$$

Now we pass to the supremum in y and get

$$\sup_{y>0} g(y) \le \liminf_{x\to 0^+} g(x) \le \limsup_{x\to 0^+} g(x) \le \sup_{y>0} g(y),$$

and we are done.

(ii) We get from (23)

$$g(y) - \Big(1 - \frac{\phi(n(x)\cdot x)}{\phi(y)}\Big) g\big(z(x)\big) \le \frac{\phi(n(x)\cdot x)}{\phi(y)} g(x).$$

Assume that $\liminf_{x\to 0^+} g(x) < +\infty$.

Then, passing to the $\liminf_{x\to 0^+}$, the above relation gives

$$g(y) \le \liminf_{x\to 0^+} g(x) < +\infty,$$

hence the absurdum by passing to the sup in y. □

Acknowledgements. The authors are grateful to the CIRM of Marseille for the hospitality offered to them in 2019 under the Program: *Research in pairs* `https://conferences.cirm-math.fr/2019-calendar.html`, when this research was at the beginning.

The authors wish to warmly thank an anonymous referee, who helped them in obtaining a more reader's friendly presentation of the results.

REFERENCES

[1] BEURLING, A.—MALLIAVIN, P.: *On the closure of characters and the zeros of entire functions*, Acta Math. **118** (1967), 79–93.

[2] FEKETE, M.: *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **17** (1923), 228–249.

[3] GIULIANO, R.—GREKOS, G.: *On the connection between the Beurling-Malliavin density and the asymptotic density*, Unif. Distrib. Theory **19** (2024), no. 1, 43–66.

[4] KOOSIS, P.: *The Logarithmic Integral I.* Cambridge Studies in Advanced Mathematics, Vol. 12. Cambridge University Press, Cambridge, 1988.

[5] KOOSIS, P.: *The Logarithmic Integral II.* Cambridge Studies in Advanced Mathematics, Vol. 21. Cambridge University Press, Cambridge. 1992.

[6] POLTORATSKI, A.: *Spectral gaps for sets and measures*, Acta Math. **208**, (2012), no. 1, 151–209.

[7] POLTORATSKI, A.: *Toeplitz methods in completeness and spectral problems.* In: Proc. of the International Congress of Mathematicians—Rio de Janeiro 2018, Vol. III, Invited lectures, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018. pp. 1771–1796.

[8] PÓLYA, G.: Untersuchungen über Lücken und Singularitäten von Potenzreihen, Math. Z. **29** (1929), no. 1, 549–640.

Received March 30, 2024
Accepted July 10, 2024

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