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uniform distribution theory

DOI: 10.2478/UDT-2024-0003
Unif. Distrib. Theory **19** (2024), no.1, 15–32

ON THE DISTRIBUTION MODULO ONE OF THE a-POINTS OF THE kth DERIVATIVE OF AN L-FUNCTION IN THE SELBERG CLASS

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ABSTRACT. Let F be an L-function from the Selberg class, $F^{(k)}$ be the kth derivative of F and a be an arbitrary fixed complex number. The solutions of $F^{(k)}(s) = a$ are called a-points. In this paper, we present a new zone without a-points of $F^{(k)}$ and we show that if F has a polynomial Euler product and satisfies the analogue of the Lindelöf hypothesis, then the a-points of $F^{(k)}$ are uniformly distributed modulo one.

Communicated by Radhakrishnan Nair

1. Introduction

Let $F(s)$ be a function from the Selberg class and a be a complex number. The zeros of $F(s) - a$, which are denoted by $\rho_a = \beta_a + i\gamma_a$, are called a-points of $F(s)$. It is well-known that there is an a-point near any trivial zero ρ of $F(s)$ provided that $|\rho|$ is sufficiently large and that there are at most finitely many further a-points in the region $Re(s) < 0$ (see Steuding book [10]). The a-points with $\beta_a \leq 0$ are said to be trivial. All other a-points lie in a strip $0 < \sigma < A$, where A depends on a, and are called the nontrivial a-points. Steuding [10, Chapter 7.2] found a Riemann von-Mangoldt formula type for $N_F^a(T)$ which denotes

2020 Mathematics Subject Classification: 11M06, 11M26, 11M36.
Keywords: Riemann zeta function, Selberg class, Riemann hypothesis, a-points.

the number of nontrivial a-points (according to multiplicities) with imaginary parts $0 < \gamma_a < T$. He showed exactly if $a \neq 1$[1] that

$$N_F^a(T) = \frac{d_F T}{2\pi} \log T + \frac{T}{2\pi} \log\left(\lambda Q^2\right) - \frac{d_F T}{2\pi} + O\left(\log T\right), \tag{2}$$

where d_F (is the degree of F), λ and Q are defined below. Hence, we observe that these asymptotics are essentially independent of a, that is,

$$N_F^a(T) \sim N_F^0(T) \sim \frac{d_F T}{2\pi} \log T.$$

In this paper, while extending some previous results due to Steuding [9] for the Riemann zeta function ζ and Jakhlouti, Mazhouda and Steuding [2] for L-functions in the Selberg class $\mathcal{S}$ having a polynomial Euler product representation, we consider the kth derivative of L-function in the Selberg class $\mathcal{S}$ having a polynomial Euler product representation. To do so, we use recent results of Chaudhary et al. [1] on the distribution of the a-points of the kth derivative of L-function in the Selberg class.

The Selberg class $\mathcal{S}$ was introduced by A. Selberg [7], which consists of the Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \qquad \Re(s) > 1,$$

satisfying the following properties:

- *Ramanujan hypothesis*: $a(n) = O(n^{\epsilon})$.
- *Euler product*: For s with sufficiently large real part, we have

  $$F(s) = \prod_p \exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right),$$

  where $b(p^k) = O(p^{k\theta})$, $\theta < \frac{1}{2}$.
- *Analytic continuation*: For a non-negative integer m the entire function defined by $(s-1)^m F(s)$ is of finite order. The smallest such number is denoted by m_F and called the polar order of F.

[1]When $a = 1$, according to Steuding's work [10, Chapter 7.2], we can easily prove the following result

$$N_F^1(T) = \frac{d_F T}{2\pi} \log T + \frac{T}{2\pi} \log\left(\lambda Q^2\right) - \frac{d_F T}{2\pi} - \frac{T}{2\pi} \log \ell + O\left(\log T\right), \tag{1}$$

where $\ell \in \mathbb{N}_{\geq 2}$ is the least number such that the Dirichlet series coefficient a_ℓ for $F(s)$ does not vanish.

- *Functional equation*: For $1 \leq j \leq r$, there exist positive real numbers (Q_F, λ_j) and complex numbers ($\mu_j,\ \omega$) with $Re(\mu_j) \geq 0$ and $|\omega| = 1$ such that

$$\phi_F(s) = \omega\overline{\phi_F(1-\overline{s})} = F(s)Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$

The degree of $F \in \mathcal{S}$ is defined by $d_F = 2\sum_{j=1}^{r} \lambda_j$. One of the most important conjectures about the Selberg class is the generalized Riemann hypothesis which states: for all $F \in \mathcal{S}$, the nontrivial (non-real) zeros of F are located on the critical line $\Re(s) = \frac{1}{2}$. The logarithmic derivative of F has a Dirichlet series expansion as given below

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n)n^{-s}, \qquad (\Re s > 1),$$

where $\Lambda_F(n) = b(n)\log n$ is the generalized von Mangoldt function (supported on the prime powers). The Euler product for $F \in \mathcal{S}$ can be written as

$$F(s) = \prod_p F_p(s), \quad \text{where} \quad F_p(s) = 1 + \sum_{m=1}^{\infty} \frac{a_F(p^m)}{p^{ms}}.$$

It was conjectured in [3] that the p-factors F_p are of polynomial type, and there exist a positive integer ν_F and complex numbers $\alpha_i(p)$ such that

$$F_p(s) = \prod_{i=1}^{\nu_F} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}, \qquad |\alpha_i(p)| \leq 1.$$

The subsemigroup of the function $F \in \mathcal{S}$ with polynomial Euler product is denoted by $\mathcal{S}^{\text{poly}}$, and it is conjectured that $\mathcal{S}^{\text{poly}} = \mathcal{S}$.

From now on, and until the end of the present paper, we consider $F \in \mathcal{S}^{\text{poly}}$ with $\nu_F = d$, where d is as small as possible and there exists at least one prime number p_0 such that $\prod_{j=1}^{d} \alpha_i(p_0) \neq 0$; d is called the *Euler degree* of F. Kaczorowski and Perelli [3] conjectured that $d = d_F$ for every $F \in \mathcal{S}^{\text{poly}}$. This conjecture holds for all classical L-functions in $\mathcal{S}^{\text{poly}}$ known to us. Then, the Euler product of F has the form

$$F(s) = \prod_p F_p(s) = \prod_p \prod_{i=1}^{d} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1};$$

we will say that F has a polynomial Euler product representation.

Let k be a positive integer, a be a complex number and $\rho_a^{(k)} = \beta_a^{(k)} + i\gamma_a^{(k)}$ denote an a-point of the kth derivative $F^{(k)}(s)$ of $F(s)$. Similar to the a-points of $F(s)$, Chaudhary et al. [1] studied the distribution of the a-points of $F^{(k)}(s)$, and they proved that there is an a-point of $F^{(k)}(s)$ near any trivial zero ρ of $F(s)$ provided that $|\rho|$ is sufficiently large. Those a-points can be regarded as trivial a-points of $F^{(k)}(s)$. All other a-points lie in a strip $E_2 < \sigma < E_1$, where E_1 and E_2 depend on a and k, and are called the nontrivial a-points. In Lemma 2.2 below, we improve on this result and prove that apart the trivial a-points, there are only finitely many other a-points in the half-plane $\sigma \leq C$ for any $C < 0$. Moreover, the authors in [1] investigated the number of nontrivial a-points of $F^{(k)}(s)$ with imaginary parts $0 < \gamma_a^{(k)} < T$ and proved

$$N^a_{F^{(k)}}(T) = \frac{d_F T}{2\pi}\log T + \frac{T}{2\pi}\log\left(\lambda Q^2\right) - \frac{d_F T}{2\pi} - c_a\frac{T}{2\pi}\log \ell + O\left(\log T\right), \quad (3)$$

where $c_a = 1$ if $a = 0$ and 0 otherwise, and $\ell \in \mathbb{N}$ is the least number such that the Dirichlet series coefficient a_ℓ for $F^{(k)}(s)$ does not vanish (see [8] for more details about the case $a = 0$).

Our main results are stated in the following theorems.

Theorem 1.1. *Let $F(s) \in \mathcal{S}^{\text{poly}}$ be a polynomial Euler product, a be an arbitrary complex number and $x \neq 1$ be a positive real number. Then, as $T \longrightarrow \infty$, we have*

$$\sum_{\rho_a^{(k)}\ :\ 0<\gamma_a^{(k)}<T} x^{\rho_a^{(k)}} = \left\{\alpha_{F^{(k)}}(x) - x\overline{\Lambda_{F^{(k)}}(1/x)}\right\}\frac{T}{2\pi} + O\left(T^{\frac{1}{2}+\epsilon}\right), \quad (4)$$

where $\rho_a^{(k)} = \beta_a^{(k)} + i\gamma_a^{(k)}$ denotes a nontrivial a-point of the kth derivative $F^{(k)}(s)$ of $F(s)$. Here the quantities $\Lambda_{F^{(k)}}$ and $\alpha_{F^{(k)}}$ are the coefficients of the Dirichlet series defined by

$$\frac{F^{(k+1)}(s)}{F^{(k)}(s)} = -\sum_{n\geq 1}\frac{\Lambda_{F^{(k)}}(n)}{n^s} \qquad \text{and} \qquad \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} = \sum_{n\geq 1}\frac{\alpha_{F^{(k)}}(n)}{n^s}, \quad (5)$$

respectively, if $x = \frac{1}{n}$ or $x = n$ for some integer $n \geq 2$, and zero otherwise.

As a consequence of Theorem 1.1, we deduce the following.

Theorem 1.2. *Let $F(s) \in \mathcal{S}^{\text{poly}}$ be a polynomial Euler product satisfying the analogue of the Lindelöf hypothesis, i.e., $F\left(\frac{1}{2}+it\right) \ll t^\epsilon$ for any positive ϵ, as $|t| \to \infty$. Then, for any complex number a and any positive real α, the sequence of multiples $\alpha\gamma_a^{(k)}$ of the ordinates of the nontrivial a-points of the kth derivative $F^{(k)}(s)$ of $F(s)$ is uniformly distributed modulo one.*

Remark 1.

1. Using Onozuka's results [6], one can prove the previous result unconditionally in the case of the the kth derivative of the Riemann zeta function.
2. In this paper, we will only discuss the cases $k \geq 1$. See [2] for the case $k = 0$.

2. Preliminary lemmas and equations

The purpose of this section is to give some results and lemmas which will be useful for the proofs of our main theorems.

In view of our investigations the functional equation is of special interest. We rewrite the functional equation as

$$F(s) = \Delta_F(s)\overline{F(1-\overline{s})}, \tag{6}$$

where

$$\Delta_F(s) = \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j})}{\Gamma(\lambda_j s+\mu_j)}. \tag{7}$$

By Stirling's formula, we have

$$\Delta_F(\sigma+it) = \left(\lambda Q^2 t^{d_F}\right)^{\frac{1}{2}-\sigma-it} \exp\left(itd_F + \frac{i\pi(\mu-d_F)}{4}\right)\{\omega + O(1/t)\} \tag{8}$$

and

$$-\frac{\Delta_F'}{\Delta_F}(\sigma+it) = \log\left(\lambda Q^2 t^{d_F}\right) + O\left(\frac{1}{t}\right), \tag{9}$$

where

$$\mu = 2\sum_{j=1}^{r}(1-2\mu_j) \quad \text{and} \quad \lambda = \prod_{j=1}^{r}\lambda_j^{2\lambda_j}.$$

Further, we have

$$\mu_F(\sigma) = \limsup_{t\to\pm\infty} \frac{\log|F(\sigma+it)|}{\log|t|} = \begin{cases} 0 & \text{for} \quad \sigma > 1, \\ \left(\frac{1}{2}-\sigma\right)d_F & \text{for} \quad \sigma < 0 \end{cases} \tag{10}$$

and

$$\mu_F(\sigma) \le \frac{1}{2}d_F(1-\sigma) \quad \text{for} \quad 0 \le \sigma \le 1. \tag{11}$$

Also

$$F(\sigma+it) \ll_\epsilon t^{\mu_F(\sigma)+\epsilon}. \tag{12}$$

For more details, kindly see [5, Lemma 2.1]. Moreover, by Cauchy's integral formula we have

$$F^{(k)}(s) = \frac{k!}{2\pi i}\int_{\mathbf{C}} \frac{F(w)}{(w-s)^{k+1}}\mathrm{d}s, \tag{13}$$

where $\mathbf{C}$ is any arbitrarily small circle centered at s. By using the last bound of $F(s)$, it follows that

$$F^{(k)}(s) \ll_\epsilon t^{\mu_F(\sigma)+\epsilon}. \tag{14}$$

Let k be a positive integer, a be a complex number and $F(s)$ be an L-function from the Selberg class. In the following lemma, we present an a-point free region for the kth derivative $F^{(k)}(s)$ of $F(s)$.

Lemma 2.1. *Let k be a positive integer, a be a complex number and $F(s)$ be an L-function from the Selberg class. For any real number $C < 0$, there exists a constant $T_{k,C} > 0$ such that there are no a-points of $F^{(k)}(s)$ in*

$$\{s \in \mathbb{C};\ \sigma \le C,\ |t| \ge T_{k,C}\}.$$

Proof. From [1, Lemma 2], we have

$$\begin{aligned}
\overline{F(1-\overline{s})}^{(k)} &= \frac{(d_F)^k}{\pi^r \omega} O^{2s-1} \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\Gamma(\lambda_j s + 1 - \lambda_j - \overline{\mu_j}) \\
&\quad \times \prod_{j=1}^{r} \sin \pi(-\lambda_j s + \lambda_j + \overline{\mu_j}) \log^k(s) F(s) \left(1 + O\left(\frac{1}{|\log(s)|}\right)\right) \\
&= (d_F)^k \overline{\Delta_F(1-\overline{s})} \log^k(s) F(s) \left(1 + O\left(\frac{1}{|\log(s)|}\right)\right) \qquad (15)
\end{aligned}$$

holds in the region $\{s \in \mathbb{C};\ \sigma > c,\ |t| \ge \tau_k\}$ for $c > 1$ and sufficiently large τ_k. Using equation (8), we get for fixed $\sigma \ge 1 - C$ and $|t| \ge \tau_k$

$$\left|F^{(k)}(1-s)\right| \asymp |t|^{d_F(\sigma - 1/2)} |\log(|t|)|^k.$$

Hence, there exists a constant $T_{k,C} > 0$ such that $|F^{(k)}(s)| > |a|$ holds for all $E_2(a) \le \sigma \le C$ and $|t| \ge T_{k,C}$. □

From Lemma 2.1, we easily deduce the following lemma.

Lemma 2.2. *Let k be a non-negative integer. For any real number $C < 0$, except the trivial a-points, there are only finitely many a-points of $F^{(k)}(s)$ in*

$$\{s \in \mathbb{C};\ \sigma \le C\}.$$

Lemma 2.3. *Let k be a non-negative integer, $F(s) \in \mathcal{S}^{\text{poly}}$ be a polynomial Euler product and $\epsilon > 0$ be sufficiently small. Then, for sufficiently large t*

$$F^{(k)}(-\epsilon + it) \gg t^{\frac{d_F}{2}} (\log t)^{k - \nu_F}. \qquad (16)$$

Proof. Using equations (15) and (8), we get

$$|F^{(k)}(\sigma + it)| \asymp t^{d_F(\frac{1}{2} - \sigma)} (\log t)^k |F(1 - \sigma + it)|$$

as $t \longrightarrow \infty$. Now, using [2, Lemma 1] we complete the proof. □

Lemma 2.4. *Let k be a positive integer, a be a complex number and $F(s)$ be L-function from the Selberg class. Assuming the truth of the Lindelöf hypothesis for $F(s)$, then, we have for any $\epsilon > 0$*

$$\sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>\frac12}} \left(\beta_a^{(k)} - \frac{1}{2}\right) \ll \epsilon T \log T. \tag{17}$$

P r o o f. For $a \neq 0$, we let

$$G_a(s) = \frac{a - F^{(k)}(s)}{a} = 1 - \frac{F^{(k)}(s)}{a}.$$

Obviously, the zeros of $G_a(s)$ correspond exactly to the a-points of $F^{(k)}(s)$. Then, for sufficiently large C and by Littlewood's lemma, we have

$$\begin{aligned} 2\pi & \sum_{\substack{\rho_0^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>\frac12}} \left(\beta_a^{(k)} - \frac{1}{2}\right) \\ = & \int_T^{2T} \log\left|G_a\left(\frac{1}{2}+it\right)\right| \mathrm{d}t - \int_T^{2T} \log|G_a(C+it)|\,\mathrm{d}t - \int_{\frac12}^{C} \arg\big(G_a(\sigma+iT)\big)\,\mathrm{d}\sigma \\ & + \int_{\frac12}^{C} \arg\big(G_a(\sigma+2iT)\big)\,\mathrm{d}\sigma \\ = & K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Let us start by estimating K_3 and K_4. To do so, we define $g(s)$ by

$$g(s) = \frac{G_a(s+iT) + \overline{G_a(\overline{s}+iT)}}{2}.$$

Obviously, we have $g(\sigma) = \Re G_a(\sigma + iT)$. This yields

$$\big|\arg\big(G_a(\sigma+iT)\big)\big| \le \left(1 + n_{C+iT}\left(C + \frac{1}{2}\right)\right)\pi$$

for $\frac12 < \sigma < C$, where $n_{C+iT}(C+\frac12)$ denotes the number of zeros of $g(s)$ in the disc with center $C+iT$ and radius $C+\frac12$. Hence by Jensen's theorem, we have

$$\big|\arg\big(G_a(\sigma+iT)\big)\big| \ll \int_0^{C+1} \frac{n_{C+iT}(r)}{r}\,\mathrm{d}r \ll \int_0^{2\pi} \log\big|g\big(C+iT+(C+1)e^{i\theta}\big)\big|\,\mathrm{d}\theta$$

for $\frac12 < \sigma < C$. Using equation (14), we obtain

$$\big|\arg\big(G_a(\sigma+iT)\big)\big| \ll \log T.$$

Thus $K_3 \ll \log T$ and $K_4 \ll \log T$.

Next we consider K_2. Since C is sufficiently large, we have

$$G_a(C+it) = 1 - \frac{F^{(k)}(C+it)}{a} = 1 - \frac{(-1)^k}{a}\sum_{n\geq 2}\frac{a(n)\log^k n}{n^{C+it}}$$

and the absolute value of the series is less than 1. Therefore, we find by the Taylor expansion of the logarithm

$$\begin{aligned}\log|G_a(C+it)| &= \Re\sum_{m=1}^{+\infty}\frac{1}{ma^m}\sum_{n_1=2}^{+\infty}\cdots \\ &\cdots\sum_{n_m=2}^{+\infty}\frac{a(n_1)\cdots a(n_m)\log^k(n_1)\cdots\log^k(n_m)}{(n_1\cdots n_m)^{C+it}}.\end{aligned}$$

This gives

$$\begin{aligned}K_2 &= \Re\sum_{m=1}^{+\infty}\frac{1}{ma^m}\sum_{n_1=2}^{+\infty}\cdots \\ &\cdots\sum_{n_m=2}^{+\infty}\frac{a(n_1)\cdots a(n_m)\log^k(n_1)\cdots\log^k(n_m)}{(n_1\cdots n_m)^{C}} \\ &\times\int_T^{2T}\frac{1}{(n_1\cdots n_m)^{it}}\mathrm{d}t \\ &\ll 1.\end{aligned}$$

Finally, we have to estimate the value of K_1. We have

$$\begin{aligned}K_1 &= \int_T^{2T}\log\left|G_a\left(\frac{1}{2}+it\right)\right|\mathrm{d}t \\ &= \int_T^{2T}\log\left|a - F^{(k)}\left(\frac{1}{2}+it\right)\right|\mathrm{d}t - T\log|a| \\ &\leq \int_T^{2T}\log\left|F^{(k)}\left(\frac{1}{2}+it\right)\right|\mathrm{d}t + T\log 2 - T\log|a| \\ &\leq \frac{1}{v}\int_T^{2T}\log\left|F^{(k)}\left(\frac{1}{2}+it\right)\right|^v\mathrm{d}t + T\log 2 - T\log|a|,\end{aligned}$$

for any positive v. By Jensen's inequality, we have

$$K_1 \leq \frac{T}{v}\log\left(\frac{1}{T}\int_T^{2T}\left|F^{(k)}\left(\frac{1}{2}+it\right)\right|^v\mathrm{d}t\right) + T\log 2 - T\log|a|.$$

For positive p, q with $\frac{1}{p}+\frac{1}{q}=1$, by Hölder's inequality, we have

$$\begin{aligned}\int_T^{2T}\left|F^{(k)}\left(\frac{1}{2}+it\right)\right|^v \mathrm{d}t &= \int_T^{2T}\left|\frac{F^{(k)}}{F}\left(\frac{1}{2}+it\right)\right|^v\left|F\left(\frac{1}{2}+it\right)\right|^v \mathrm{d}t\\ &\leq \left(\int_T^{2T}\left|\frac{F^{(k)}}{F}\left(\frac{1}{2}+it\right)\right|^{pv} \mathrm{d}t\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_T^{2T}\left|F\left(\frac{1}{2}+it\right)\right|^{qv} \mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

We put $v=\frac{2}{4k+1}$, $p=\frac{4k+1}{4k}$ and $q=4k+1$, we obtain

$$\begin{aligned}\int_T^{2T}\left|F^{(k)}\left(\frac{1}{2}+it\right)\right|^v \mathrm{d}t &\leq \left(\int_T^{2T}\left|\frac{F^{(k)}}{F}\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2k}} \mathrm{d}t\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_T^{2T}\left|F\left(\frac{1}{2}+it\right)\right|^{2} \mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Now, using Lemma 5 in [1] and equation (3), by the same argument as in [4, Claim], we have

$$\int_T^{2T}\left|\frac{F^{(k)}}{F}\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2k}} \mathrm{d}t \ll T\log T.$$

An immediate consequence of Lindelöf's hypothesis is

$$\int_T^{2T}\left|F\left(\frac{1}{2}+it\right)\right|^{2} \mathrm{d}t \ll T^{1+\epsilon},$$

for any positive ϵ. Hence we have

$$\int_T^{2T}\left|F^{(k)}\left(\frac{1}{2}+it\right)\right|^{v} \mathrm{d}t \ll T^{1+\epsilon}.$$

Thus

$$K_1 \ll \epsilon T\log T.$$

Collecting all estimates, the assertions of Lemma 2.4 follows.
In the case $a=0$, we consider the function

$$G_0(s) = \frac{\ell^s}{a_\ell}F^{(k)}(s),$$

where $\ell\in\mathbb{N}$ is the least number such that the Dirichlet series coefficient a_ℓ for$F^{(k)}(s)$ does not vanish. Then, similarly as above we complete the proof. $\square$

Remark 2. By the same argument as above and [10, Corollary 6.11], we can proof unconditionally that for any $b > \max\{\frac{1}{2}, 1 - \frac{1}{d_F}\}$,

$$\sum_{\substack{\rho_a^{(k)}:\ T < \gamma_a^{(k)} \le 2T \\ \beta_a^{(k)} > b}} \left(\beta_a^{(k)} - b\right) \ll T \log\log T. \tag{18}$$

Lemma 2.5. *Let k be a positive integer, $a \neq 0$ be a complex number and $F(s)$ be an L-function from the Selberg class. Then, for sufficiently large negative b,*

$$\begin{aligned} 2\pi \sum_{\rho_a^{(k)}: T < \gamma_a^{(k)} \le 2T} \left(\beta_a^{(k)} - b\right) &= \left(\frac{1}{2} - b\right)\left(d_F T \log\left(\frac{4T}{e}\right) + T\log\left(\lambda Q^2\right)\right) \\ &\quad + 2kT\log\log(2T) - kT\log\log T \\ &\quad + kT\log(d_F) \\ &\quad - T\log|a| + O\left(\frac{T}{\log T}\right). \end{aligned} \tag{19}$$

P r o o f.

Since b is sufficiently large negative, it follows from Lemmas 2.1 and 2.2 that

$$2\pi \sum_{\rho_a^{(k)}: T < \gamma_a^{(k)} \le 2T} \left(\beta_a^{(k)} - b\right) = 2\pi \sum_{\substack{\rho_a^{(k)}:\ T < \gamma_a^{(k)} \le 2T \\ \beta_a^{(k)} > b}} \left(\beta_a^{(k)} - b\right).$$

The same argument as above yields

$$\begin{aligned} 2\pi \sum_{\rho_a^{(k)}: T < \gamma_a^{(k)} \le 2T} \left(\beta_a^{(k)} - b\right) &= \int_T^{2T} \log\left|F^{(k)}(b+it) - a\right| \mathrm{d}t \\ &\quad - T\log|a| + O(\log T). \end{aligned}$$

By(15), the integrand can be expressed as

$$\begin{aligned} \log\left|F^{(k)}(b+it) - a\right| &= \log\left|F^{(k)}(b+it)\right| + \log\left|1 - \frac{a}{F^{(k)}(b+it)}\right| \\ &= k\log(d_F) + \log|\Delta_F(b+it)| \\ &\quad + k\log|\log(1-b-it)| + \log|F(1-b+it)| \\ &\quad + O\left(\frac{1}{F^{(k)}(b+it)}\right) + O\left(\frac{1}{\log t}\right). \end{aligned}$$

From [10, page 142], we have

$$\int_T^{2T} \log|\Delta_F(b+it)| + \log|F(1-b+it)| \mathrm{d}t$$
$$= \left(\frac{1}{2}-b\right) d_F T \log\left(\frac{4T}{e}\right) + \left(\frac{1}{2}-b\right) T\log\left(\lambda Q^2\right) + O\left(\log T\right).$$

By [6, page 73], we get

$$\int_T^{2T} k\log|\log(1-b-it)|\,\mathrm{d}t$$
$$= 2kT\log\log(2T) - kT\log\log T + O\left(\frac{T}{\log T}\right).$$

By equation (15) and Stirling's formula, we have $F^{(k)}(v+it) \gg |t|^{\frac{3}{2}}$, hence

$$\int_T^{2T} O\left(\frac{1}{F^{(k)}\left(b+it\right)}\right) \mathrm{d}t = O(1).$$

Collecting all the estimates, the proof is complete. □

Remark 3. By similar reasoning, we obtain for the case $a = 0$,

$$2\pi \sum_{\rho_0^{(k)}:T<\gamma_0^{(k)}\le 2T} \left(\beta_0^{(k)} - b\right) = \left(\frac{1}{2}-b\right)\left(d_F T\log\left(\frac{4T}{e}\right) + T\log\left(\lambda Q^2\right)\right)$$
$$+ 2kT\log\log(2T) - kT\log\log T$$
$$+ kT\log(d_F) + O\left(T\right). \tag{20}$$

Let us now define the following quantities

$$N_+^a(\sigma,T) = \sharp\left\{\rho_a^{(k)}, T<\gamma_a^{(k)} \le 2T, \beta_a^{(k)} > \sigma\right\}, \tag{21}$$

and

$$N_-^a(\sigma,T) = \sharp\left\{\rho_a^{(k)}, T<\gamma_a^{(k)} \le 2T, \beta_a^{(k)} < \sigma\right\}. \tag{22}$$

Lemma 2.6. *Let k be a positive integer, a be a complex number and $F(s)$ be an L-function from the Selberg class. Assuming the truth of the Lindelöf hypothesis, then, we have for any $\delta > 0$*

$$N_-^a\left(\frac{1}{2}-\delta, T\right) + N_+^a\left(\frac{1}{2}+\delta, T\right) \ll \delta T\log T. \tag{23}$$

Proof. We have

$$N_+^a\left(\frac{1}{2}+\delta,T\right) = \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>\frac{1}{2}+\delta}} 1 \le \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>\frac{1}{2}+\delta}} \left(\frac{\beta_a^{(k)}-\frac{1}{2}}{\delta}\right)$$
$$\le \frac{1}{\delta} \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>\frac{1}{2}+\delta}} \left(\beta_a^{(k)}-\frac{1}{2}\right).$$

From Lemma 2.4, we obtain

$$N_+^a\left(\frac{1}{2}+\delta,T\right) \ll \frac{\epsilon}{\delta} T\log T,$$

for any positive ϵ. For sufficiently large negative b, we have

$$\sum_{\rho_a^{(k)}:T<\gamma_a^{(k)}\le 2T} \left(\beta_a^{(k)}-b\right) = \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>b}} \left(\beta_a^{(k)}-b\right) = \left(\frac{1}{2}-b\right) \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}<\frac{1}{2}-\delta}} 1$$
$$+ \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}<\frac{1}{2}-\delta}} \left(\beta_a^{(k)}-\frac{1}{2}\right)$$
$$+ \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}\ge\frac{1}{2}-\delta}} \left(\beta_a^{(k)}-b\right).$$

Since

$$\sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}\ge\frac{1}{2}-\delta}} \left(\beta_a^{(k)}-b\right)$$
$$= \left(\frac{1}{2}-b\right) \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}\ge\frac{1}{2}-\delta}} 1 + \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}\ge\frac{1}{2}-\delta}} \left(\beta_a^{(k)}-\frac{1}{2}\right)$$
$$\le \left(\frac{1}{2}-b\right) \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}\ge\frac{1}{2}-\delta}} 1 + \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\le 2T\\ \beta_a^{(k)}>\frac{1}{2}}} \left(\beta_a^{(k)}-\frac{1}{2}\right),$$

we obtain

$$\begin{aligned}
&\sum_{\rho_a^{(k)}:T<\gamma_a^{(k)}\leq 2T}\left(\beta_a^{(k)}-b\right)\\
&\leq \left(\frac{1}{2}-b\right)\sum_{\rho_a^{(k)}:T<\gamma_a^{(k)}\leq 2T} 1 \quad + \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\leq 2T \\ \beta_a^{(k)}<\frac{1}{2}-\delta}}\left(\beta_a^{(k)}-\frac{1}{2}\right)\\
&\quad + \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\leq 2T \\ \beta_a^{(k)}>\frac{1}{2}}}\left(\beta_a^{(k)}-\frac{1}{2}\right)\\
&\leq \left(\frac{1}{2}-b\right)\left(N^a_{F^{(k)}}(2T)-N^a_{F^{(k)}}(T)\right)-\delta N^a_-\left(\frac{1}{2}-\delta,T\right)\\
&\quad + \sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\leq 2T \\ \beta_a^{(k)}>\frac{1}{2}}}\left(\beta_a^{(k)}-\frac{1}{2}\right).
\end{aligned}$$

This leads to

$$\begin{aligned}
\delta N^a_-\left(\frac{1}{2}-\delta,T\right) \quad \leq \quad & \left(\frac{1}{2}-b\right)\left(N^a_{F^{(k)}}(2T)-N^a_{F^{(k)}}(T)\right)\\
& -\sum_{\rho_a^{(k)}:T<\gamma_a^{(k)}\leq 2T}\left(\beta_a^{(k)}-b\right)\\
& +\sum_{\substack{\rho_a^{(k)}:\ T<\gamma_a^{(k)}\leq 2T \\ \beta_a^{(k)}>\frac{1}{2}}}\left(\beta_a^{(k)}-\frac{1}{2}\right).
\end{aligned}$$

Now, by (3) and Lemma 2.5 we get

$$\delta N^a_-\left(\frac{1}{2}-\delta,T\right) \quad \ll \quad \epsilon T\log T.$$

Hence

$$N^a_-\left(\frac{1}{2}-\delta,T\right) \quad \ll \quad \frac{\epsilon}{\delta}T\log T,$$

for any positive ϵ. Putting $\epsilon=\delta^2$ we obtain the assertion of the lemma. □

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1.

Let k be a positive integer, a be a complex number and $F(s) \in \mathcal{S}^{\text{poly}}$ be a polynomial Euler product. Then, from [1, Lemma 1] there exists a constant E_1 such that $F^{(k)} - a \neq 0$ for $Re(s) \geq E_1$ and by [1, Lemma 6] the function $\frac{F^{(k+1)}(s)}{F^{(k)}-a}$ has an absolutely convergent Dirichlet series expansion in the half-plane $Re(s) \geq E_1$, namely[2]

$$\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} = \sum_{n\geq 1} \frac{\alpha_{F^{(k)}}(n)}{n^s}. \tag{25}$$

Now, let $b = 1 + \frac{1}{\log T}$. Then, from Lemma 2.2 above, only finitely many trivial a-points lie to the left of the vertical line $\Re(s) = 1 - b$. Hence, by the residue theorem we have

$$\sum_{\rho_a^{(k)}\,:\,0<\gamma_a^{(k)}<T} x^{\rho_a^{(k)}} = \frac{1}{2\pi i}\int_{\mathbf{R}} x^s \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}\,\mathrm{d}s + O(1), \tag{26}$$

where the integration is taken over a rectangular contour in counter clockwise direction denoted by $\mathbf{R}$ with vertices $A+i, A+iT, 1-b+iT, 1-b+i$ with some constants $A \geq E_1$ such that $F^{(k)}(s)$ has no a-point on the lines $t = T$ and $\sigma = 1 - b$. Then, we have

$$\begin{aligned}
\sum_{\rho_a^{(k)}:0<\gamma_a^{(k)}<T} x^{\rho_a^{(k)}} &= \frac{1}{2\pi i}\int_{\mathbf{R}} x^s \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}\,\mathrm{d}s + O(1)\\
&= \frac{1}{2\pi i}\left\{\int_{1-b+i}^{A+i} + \int_{A+i}^{A+iT} + \int_{A+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i}\right\}\\
&\quad \times\ x^s \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}\,\mathrm{d}s + O(1)\\
&:= I_1 + I_2 + I_3 + I_4 + O(1).
\end{aligned}$$

The estimation of each integral I_1, I_2 and I_3 goes to the same way as J_1, J_2 and J_3 in the proof of Theorem 2 in [1], then we have $I_1 = O(1)$, $I_3 = O(\log T)$ and $I_2 = \frac{T}{2\pi}\alpha_{F^{(k)}}(x) + O(1)$, where $\alpha_{F^{(k)}}(x) = \alpha_{F^{(k)}}(n)$ if $x = n$ and 0 otherwise.

[2] When $a = 0$, one has

$$\frac{F^{(k+1)}(s)}{F^{(k)}(s)} = -\sum_{n\geq 1} \frac{\Lambda_{F^{(k)}}(n)}{n^s}, \tag{24}$$

where $\Lambda_{F^{(k)}}(n)$ denotes the generalized von Mangoldt function.

Next, we have to estimate the value of I_4. It follows from Lemma 2.3 that $F^{(k)}(\sigma+it)$ has absolute value larger than $2|a|$ for sufficiently large $t \geq t_0$. Then, we have

$$\begin{aligned} I_4 &= -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT} \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} x^s \,\mathrm{d}s + O(1) \\ &= -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} \times \frac{1}{1-\frac{a}{F^{(k)}(s)}} x^s \,\mathrm{d}s + O(1) \\ &= -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} \left(1+\sum_{n\geq 1}\left(\frac{a}{F^{(k)}(s)}\right)^n\right) x^s \,\mathrm{d}s + O(1) \\ &= -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} x^s \,\mathrm{d}s \\ &\quad -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} \sum_{n\geq 1}\left(\frac{a}{F^{(k)}(s)}\right)^n x^s \,\mathrm{d}s + O(1) \\ &= J_1 + J_2 + O(1). \end{aligned}$$

From the functional equations (6) and (9), it is easy to see that for any positive integer k

$$F^{(k)}(s) = (-1)^k \Delta_F(s) \overline{F}^{(k)}(1-s)\big(1+O(\log s)\big). \tag{27}$$

Then, we obtain for J_1

$$\begin{aligned} J_1 &= -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT} \left(\frac{\Delta_F'(s)}{\Delta_F(s)} - \frac{\overline{F}^{(k+1)}}{\overline{F}^{(k+1)}}(1-s)\right) x^s \,\mathrm{d}s + O(1) \\ &= \frac{x^{1-b}}{2\pi}\int_{t_0}^{T} x^{it}\left\{\log\left(\lambda Q^2 t^{d_F}\right) + O\left(\frac{1}{t}\right)\right\}\mathrm{d}t \\ &\quad -\frac{x^{1-b}}{2\pi}\sum_{n\geq 2}\frac{\overline{\Lambda_{F^{(k)}}(n)}}{n^b}\int_{t_0}^{T}(xn)^{it}\mathrm{d}t + O(1) \\ &= -x\overline{\Lambda_{F^{(k)}}(1/x)} + O(\log T), \end{aligned}$$

where $\Lambda_{F^{(k)}}(1/x) = \Lambda_{F^{(k)}}(n)$ if $x = 1/n$ and 0 otherwise.
Now, using Lemma 2.3 and that

$$\frac{F^{(k+1)}(v+it)}{F^{(k)}(v+it)} \ll \log t,$$

we get for J_2

$$J_2 = -\frac{1}{2\pi i}\int_{1-b+it_0}^{1-b+iT}\sum_{n\geq 1} a^n \frac{F^{(k+1)}(s)}{F^{(k)}(s)}\frac{1}{\left(F^{(k)}(s)\right)^n}x^s\,\mathrm{d}s$$

$$\ll \int_{t_0}^{T}\sum_{n\geq 1}\frac{(\log t)^{\nu_F+1}}{(t^{\frac{d_F}{2}}(\log t)^k)^n}\,\mathrm{d}t \ll (\log T)^{\nu_F}T^{\frac{1}{2}}.$$

Finally, Theorem 1.1 follows from the estimates of I_1, I_2, I_3 and I_4.

4. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2.

Let us write

$$\sum_{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T} e(m\alpha\gamma_a^{(k)}) = \sum_{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T} x^{i\gamma_a^{(k)}}$$

with $x = \exp(2\pi m\alpha)$. Then, we have

$$x^{\frac{1}{2}}\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} e(m\alpha\gamma_a^{(k)}) = \sum_{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T} x^{\beta_a^{(k)}+i\gamma_a^{(k)}} + \sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} x^{\frac{1}{2}+i\gamma_a^{(k)}} - x^{\beta_a^{(k)}+i\gamma_a^{(k)}},$$

which leads to

$$x^{\frac{1}{2}}\left|\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} e(m\alpha\gamma_a^{(k)})\right|$$

$$\leq \left|\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} x^{\beta_a^{(k)}+i\gamma_a^{(k)}}\right| + \sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T}\left|x^{\frac{1}{2}+i\gamma_a^{(k)}} - x^{\beta_a^{(k)}+i\gamma_a^{(k)}}\right|$$

$$\leq \left|\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} x^{\beta_a^{(k)}+i\gamma_a^{(k)}}\right| + |\log x|\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T}\left|\beta_a^{(k)} - \frac{1}{2}\right|\max\left\{x^{\beta_a^{(k)}}, x^{\frac{1}{2}}\right\}.$$

Using Theorem 1.1, we obtain

$$\left|\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} e(m\alpha\gamma_a^{(k)})\right| \ll \sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T}\left|\beta_a^{(k)} - \frac{1}{2}\right| + O(T).$$

Since

$$\begin{aligned}
&\sum_{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T} \left|\beta_a^{(k)} - \frac{1}{2}\right| \\
&= \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ \beta_a^{(k)} < \frac{1}{2}-\epsilon}} \left|\beta_a^{(k)} - \frac{1}{2}\right| + \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ |\beta_a^{(k)} \frac{1}{2}| \le \epsilon}} \left|\beta_a^{(k)} - \frac{1}{2}\right| \\
&\quad + \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ \beta_a^{(k)} > \frac{1}{2}+\epsilon}} \left|\beta_a^{(k)} - \frac{1}{2}\right| \\
&\le \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ \beta_a^{(k)} < \frac{1}{2}-\epsilon}} \left|\beta_a^{(k)} - \frac{1}{2}\right| + \epsilon N^a_{F^{(k)}}(T) + \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ \beta_a^{(k)} > \frac{1}{2}+\epsilon}} \left|\beta_a^{(k)} - \frac{1}{2}\right|,
\end{aligned}$$

we obtain from Lemma 2.6

$$\sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} \left|\beta_a^{(k)} - \frac{1}{2}\right| \ll \epsilon N^a_{F^{(k)}}(T)$$

for any $\epsilon > 0$. Therefore

$$\frac{1}{N^a_{F^{(k)}}(T)} \sum_{\rho_a^{(k)}:\,0<\gamma_a^{(k)}<T} e(m\alpha\gamma_a^{(k)}) \ll \epsilon$$

as $T \to \infty$. Consequently, by Weyl's criterion the sequence $(\alpha\gamma_a^{(k)})$ is uniformly distributed modulo one.

Acknowledgement. The authors thank the anonymous referee for his careful reading of the manuscript and constructive suggestions which improved the paper.

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Received November 9, 2023
Accepted April 13, 2024

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