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COPULAS

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ABSTRACT. Two-dimensional distribution function $g(x, y)$ defined in $[0, 1]^2$ is called copula, if $g(x, 1) = x$ and $g(1, y) = y$ for every x, y. Similarly, s-dimensional copula is a distribution function $g(x_1, x_2, \dots, x_s)$ such that every k-dimensional face function

$$g(1, \dots, 1, x_{i_1}, 1, \dots, 1, x_{i_2}, 1, \dots, 1, x_{i_k}, 1, \dots, 1)$$

is equal to $x_{i_1} x_{i_2} \dots x_{i_k}$ for some but fixed k. In this paper we summarize and extend all known parts of copulas.

In this paper we use the following abbreviations:

$\{x\}$	—	*fractional part of* x;
$\{x\}$	—	x mod 1;
$[x]$	—	*integer part of* x;
u.d.	—	*uniform distribution*;
d.f.	—	*distribution function*;
a.d.f.	—	*asymptotic distribution function*;
u.d.p.	—	*uniform distribution preserving*;
step d.f.	—	*step distribution function*;
a.e.	—	*almost everywhere*;
$\#X$	—	*cardinality of the set* X.

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Contents

1. Introduction

In this expository paper we presents some old results of copulas given in the books [18], [35], [36], [3], [15] and some new results. We also study the limit points of

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n),$$

where x_n and y_n are uniformly distributed (abbreviating u.d.) in $[0, 1)$. It can be studied by using distribution functions (abbreviating d.f.s) $g(x, y)$ of (x_n, y_n). These d.f.s satisfy $g(x, 1) = x$, $g(1, y) = y$ and are called copulas. As we shall see in Theorem 1, p. 151, that, each copula meets the following inequalities (3)

$$\max(x + y - 1, 0) \le g(x, y) \le \min(x, y)$$

then, assuming $\mathrm{d}_x\,\mathrm{d}_y F(x, y) > 0$, for an arbitrary u.d. x_n and y_n, we have in

Theorem 8, p. 185, inequalities (57) and (58)

$$\int_0^1 F(x, 1-x)\,\mathrm{d}x \le \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) \le \int_0^1 F(x, x)\,\mathrm{d}x.$$

Here the left boundary is attained in the sequence $(x_n, 1-x_n)$ and the right in (x_n, x_n) where x_n is u.d.

Let (x_n, y_n), $n=1,2,\dots$, be two-dimensional sequence in the unit square $[0,1)^2$. Define a step d.f. $F_N(x,y)$ of $(x_1,y_1),\dots,(x_N,y_N)$ by

$$F_N(x,y) = \frac{1}{N}\#\{n \le N; (x_n,y_n) \in [0,x)\times[0,y)\}$$

and put $F_N(1,1)=1$ and Δ_x and Δ_y are small real numbers. We have:

(i) $F_N(x+\Delta_x, y+\Delta_y) + F_N(x,y) - F_N(x+\Delta_x, y) - F_N(x, y+\Delta_y)$
$= \frac{1}{N}\#\{n \le N; (x_n,y_n) \in [x, x+\Delta_x)\times[y, y+\Delta_y)\} \ge 0;$
for $[x, x+\Delta_x)\times[y, y+\Delta_y) \subset [0,1)^2$.

(ii) $F_N(0,0) = F_N(x,0) = F_N(0,y) = 0$ for $x, y \in [0,1]$.

Since a two-dimensional d.f. $g(x,y)$ is a limit of $F_N(x,y)$ we have the following definition:

Definition 1 (Distribution function)**.** A two-dimensional function $g : [0,1]^2 \to [0,1]$ is a distribution function (abbreviating d.f.) if

(i) $\mathrm{d}g(x,y) = g(x+\mathrm{d}x, y+\mathrm{d}y) + g(x,y) - g(x+\mathrm{d}x, y) - g(x, y+\mathrm{d}y) \ge 0$,

(ii) $g(0,0) = g(x,0) = g(0,y) = 0$ for $x, y \in [0,1]$.

(iii) $g(0,0) = 0$.

From (i) and (ii) directly follows that $g(x,y)$ is non-decreasing for every variable x and y.

Definition 2 (Step distribution function)**.** For a sequence $x_1,\dots,x_N \bmod 1$ we define the step distribution function $F_N(x)$ for $x \in [0,1)$ by

$$F_N(x) = \frac{1}{N}\#\{n \le N; x_n \bmod 1 \in [0,x)\},$$

while $F_N(1)=1$. It can be also writen as

$$F_N(x) = \frac{A([0,x); N; x_n \bmod 1)}{N}.$$

Definition 3 (Distribution functions of a given sequence)**.** A d.f. g is called a distribution function of the sequence x_n mod 1 if an increasing sequence of positive integers $N_1, N_2, \dots$ exists such that the equality

$$g(x) = \lim_{k\to\infty} \frac{A([0,x); N_k; x_n \bmod 1)}{N_k} \left(= \lim_{k\to\infty} F_{N_k}(x)\right)$$

holds at every point x, $0 \le x \le 1$, of the continuity of $g(x)$ and thus a.e. on $[0,1]$.

Definition 4 (The set of all d.f.s. of a sequence)**.** The set of all distribution functions of a sequence $x_n \bmod 1$ will be denoted by $G(x_n \bmod 1)$. We shall identify the notion of the distribution of a sequence x_n mod 1 with the set $G(x_n$ mod 1), i.e. the distribution of x_n mod 1 is known if we know the set $G(x_n$ mod 1).

Definition 5 (Multidimensional d.f.s.)**.** An s-dimensional function $g : [0,1]^s \to [0,1]$ is *d.f.* if:

(i) $g(\mathbf{1}) = 1$,

(ii) $g(\mathbf{0}) = 0$, and also $g(\mathbf{x}) = 0$ for every $\mathbf{x}$ with a vanishing coordinate,

(iii) $g(\mathbf{x})$ is non-decreasing, see Section 4.

For multidimensional d.f.s. see p. 190.

2. Copulas

Definition 6 (Copula)**.** We denote by $G_{s,k}$ the set of all d.f.s $g(\mathbf{x})$ on $[0,1]^s$ for which all k-dimensional marginal, (i.e., face) d.f.s satisfy

$$g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_k},1,\dots,1) = x_{i_1}x_{i_2}\dots x_{i_k}.$$

For $k = 1$, these d.f.s are called *copulas*, which were introduced by M. Sklar (1959) [27]. Basic properties of copulas can be found in monographs R. B. Nelsen (1999) [18], N. Balakrishnan and Chin-Diew Lai (2009) [3], in Proccedings of EDS: P. Jaworski, F. Durante, W. Härdle and T. Rychlik (2009) [15], and in O. Strauch [35]. Let

$$\mathbf{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,s}), n = 1,2,\dots,$$

be an infinite s-dimensional sequence in the unit cube $[0,1)^s$. We assume that, for fixed $k < s$, all k-dimensional marginal sequences $(x_{n,i_1},\dots,x_{n,i_k})$ are u.d. Then $G(\mathbf{x}_n)$ have $G_{s,k}$ copulas.

Example 1. Let θ be an algebraic number of the degree s such that for its minimal polynomial $p(x) = \sum_{i=0}^{s} a_i x^i$ we have $a_i \neq 0$ for $i = 0, 1, \dots, s$. Define the sequence $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$, $n = 1, 2, \dots$, with coordinate sequences $x_{n,i} = n\theta^i \bmod 1$, $i = 1, 2, \dots, s$. Applying the well-known Kronecker theorem (cf. [DT, p. 15, Coroll. 1.20], [SP, p. 3-9, 3.4.1]) all marginal sequences with dimensions $k = 1, 2, \dots, s-1$ are u.d. but $\mathbf{x}_n$ is not. Then $G(\mathbf{x}_n)$ contains $G_{s,s-1}$ copulas.

2.1. $\mathbf{G}_{s,1}$

If $f_1, f_2, \dots, f_s$ are measure preserving transformations on $[0,1] \to [0,1]$, then the function

$$c_{f_1,f_2,\dots,f_s}(x_1, x_2, \dots, x_s) = \left|\left(f_1^{-1}([0,x_1]) \cap f_2^{-1}([0,x_2]) \cdots \cap f_s^{-1}([0,x_n])\right)\right| \quad (1)$$

is $G_{s,1}$ copula. Conversely, for every copula $c \in G_{s,1}$ there exists s measure preserving functions $f_1, f_2, \dots, f_s$ such that $c = c_{f_1,f_2,\dots,f_s}$.

A copula c is called Archimedean if it admits the representation

$$c(x_1, \dots, x_s) = \psi\big(\psi^{-1}(x_1) + \dots + \psi^{-1}(x_s)\big), \quad (2)$$

where the kth derivatives of generator $\psi(x)$ satisfy $(-1)^k \psi^{(k)}(x) \geq 0$ for all $x \geq 0$ and $k = 0, 1, \dots, s-2$ and $(-1)^{s-2}\psi^{(s-2)}(x)$ is non-increasing and convex.

Theorem 1 (Fréchet-Hoeffding bounds). *For any copula $g(x_1, \dots, x_s) \in G_{s,1}$ and any $(x_1, \dots, x_s) \in [0,1]^s$ the following bounds hold:*

$$\max\left(1 - s + \sum_{i=1}^{s} x_i, 0\right) \leq g(x_1, \dots, x_s) \leq \min(x_1, \dots, x_s). \quad (3)$$

Here $\min(x_1, \dots, x_s)$ is always a copula. The lower bound is a copula only in dimension $s = 2$. It is only point-wise sharp, in the sense that for fixed $(u_1, \dots, u_s)$ there exists a copula $g(x_1, \dots, x_s)$ such that $g(u_1, \dots, u_s) = \max(1 - s + \sum_{i=1}^{s} u_i, 0)$, see [16] and [8].

Theorem 2 (Sklar's theorem). *A multivariate d.f. $g(x_1, \dots, x_s)$ with marginals $g_i(x_i) = g(1, \dots, 1, x_i, 1, \dots, 1)$, $i = 1, 2, \dots, s$ can be written as*

$$g(x_1, \dots, x_s) = c\big(g_1(x_1), \dots, g_s(x_s)\big), \quad (4)$$

where $c(x_1, \dots, x_s)$ is a copula. This copula is defined unique, assuming that the marginals g_i are continuous.

This theorem provides the theoretical foundation for the application of copulas.

2.2. $\mathbf{G}_{2,1}$

As we denoted $G_{2,1}$ is the set of all two-dimensional d.f.s $g(x,y)$ defined on $[0,1]^2$ such that their marginal d.f.s satisfy $g(x,1)=x$ and $g(1,y)=y$. Telegraphically, to illustrate $G_{2,1}$ we give some basic copulas:

$g_1(x,y)=xy$,

$g_2(x,y)=\min(x,y)$,[1]

$g_3(x,y)=\max(x+y-1,0)$,

$g_\theta(x,y)=\big(\min(x,y)\big)^\theta(xy)^{1-\theta}$, where $\theta\in[0,1]$ (Cuadras-Augé family, cf. [18, p. 12, Ex. 2.5]),

$g_4(x,y)=\frac{xy}{x+y-xy}$ (see [18, p. 19, 2.3.4]),

$\tilde{g}(x,y)=x+y-1+g(1-x,1-y)$ for every $g(x,y)\in G_{2,1}$ (Survival copula, see [18, p. 28, 2.6.1]),

$g_5(x,y)=\min\big(ya(x),xb(y)\big)$, where $a(0)=b(0)=0$, $a(1)=b(1)=1$ and $a(x)/x$, $b(y)/y$ are both decreasing on $(0,1]$ (Marshall copula, cf. [18, p. 51, Exerc. 3.3]).

$g_6(x,y)=xy+3\alpha xy(1-x)(1-y)$, $-1/3\le\alpha\le1/3$, [9, p. 618].

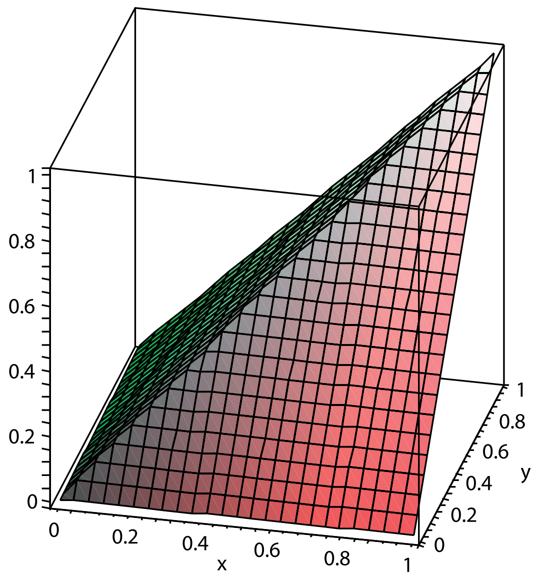


Fig.: Graph of $g(x,y)=\min(x,y)$.

[1] $\min(x,y)\le\frac{2xy}{x+y}$ for every $x>0,y>0$.

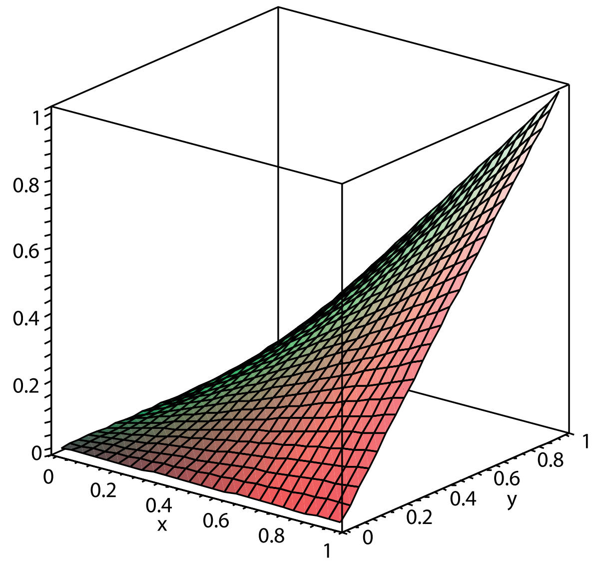


Fig.: Graph of $g(x, y) = xy$.

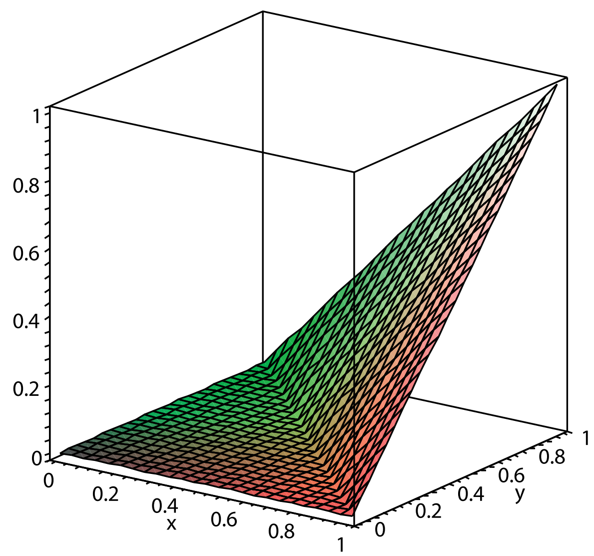


Fig.: Graph of $g(x, y) = \max(x + y - 1, 0)$.

2.3. Basic properties of $\mathbf{G}_{2,1}$

(I) $G_{2,1}$ is closed under pointwise limit and convex linear combinations.

(II) For every $g(x,y) \in G_{2,1}$ and every $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ we have

$|g(x_2,y_2) - g(x_1,y_1)| \le |x_2 - x_1| + |y_2 - y_1|$.
I.e. every copula is uniformly continuous, c.f. [19, p. 9]. Furthermore, the horizontal, vertical and diagonal sections of copula are all non-decreasing and uniformly continuous on $[0,1]$, see [19, p. 9, Cor. 2.26]. Also for every $y \in [0,1]$ the partial derivative $\frac{\partial g}{\partial x}$ exists for almost all x and $0 \le \frac{\partial g}{\partial x} \le 1$. Similarly, $0 \le \frac{\partial g}{\partial y} \le 1$.

(III) For every $g(x,y) \in G_{2,1}$

$g_3(x,y) = \max(x+y-1,0) \le g(x,y) \le \min(x,y) = g_2(x,y)$ (Fréchet--Hoeffding bounds [18, p. 9]).

(IV) M. Sklar (1959) proved that for every d.f. $g(x,y)$ on $[0,1]^2$ there exists $c(x,y) \in G_{2,1}$ such that $g(x,y) = c\big(g(x,1),g(1,y)\big)$ for every $(x,y) \in [0,1]^2$. If $g(x,1)$ and $g(1,y)$ are continuous, then $c(x,y)$ is defined unique (cf. [18, p. 15, Th. 2.3.3] and for multidimensional case, see Theorem 2, p. 151).

(V) For d.f. $g(x,y)$ denote the marginals $g_1(x) = g(x,1)$ and $g_2(y) = g(1,y)$ and by Sklar $g(x,y) = c\big(g_1(x),g_2(y)\big)$. Then for every continuous $F(x,y)$ we have

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x,y) = \int_0^1\int_0^1 F\big(g_1^{(-1)}(x),g_2^{(-1)}(y)\big)\,\mathrm{d}c(x,y), \tag{5}$$

see M. Hofer and M.R. Iacò [13].
Here are some new copulas:

(VI) $g_6(x,y) = \frac{1}{z_0}\min(xy,xz_0,yz_0)$ for fixed z_0, $0 < z_0 \le 1$.

(VII) $g_7(x,y) = \frac{1}{z_0u_0}\min(xyz_0,xyu_0,xz_0u_0,yz_0u_0)$ for fixed $z_0,u_0 \in (0,1]^2$.
In Subsection 4.1, Definition 8, p. 191, is a copula called

(VIII) Let $f : [0,1] \to [0,1]$ be a function and denote

$$\text{graph } f = \big\{\big(x,f(x)\big);x \in [0,1]\big\}.$$

Then

$$c_{\mathrm{graph}f}(x,y) = \big|\mathrm{Project}_x\big(\text{graph } f \cap [0,x) \times [0,y)\big)\big| \tag{6}$$

is a copula if and only if the function f preserves the Lebesgue measure (called *measure preserving function*, also see Section 4.2 p. 193).

In the books [3] and [15] copulas are studied in the position d.f.s of random variables. In the following we apply some parts of uniform distribution theory.

(IX) Let f_1 and f_2 be measure preserving functions on $[0,1] \to [0,1]$. Then

$$c_{f_1,f_2}(x,y) = \left|f_1^{-1}[0,x] \cap f_2^{-1}[0,y]\right| \tag{7}$$

is a copula. Furthermore, for every copula $c(x,y)$ there exist measure preserving f_1, f_2 such that $c(x,y) = c_{f_1,f_2}(x,y)$.

(X) If ϕ is a measure preserving function, then

$$c_{f_1,f_2}(x,y) = c_{f_1\circ\phi,f_2\circ\phi}(x,y). \tag{8}$$

(XI) Examples:
$c_{f,f}(x,y) = \min(x,y)$ for arbitrary measure preserving f.
$c_{\mathrm{graph} f}(x,y) = c_{f_1,f}(x,y)$ if $f_1(x) = x$.
$c_{f_1,f_2}(x,y) = \min(x+y+1,0)$ if $f_1(x) = x$ and $f_2(x) = 1-x$.

(XII) If (x_n, y_n), $n = 1,2,\dots$, is u.d. sequence in $[0,1)^2$ and f_1, f_2 are u.d.p. functions (see Sec. 4.2), then the sequence $\big(f_1(x_n), f_2(y_n)\big)$, $n = 1,2,\dots$, has a.d.f. the copula $c_{f_1,f_2}(x,y)$.

(XIII) In the following are the most popular ones:
(a) $\frac{1-t}{e^x-t}, t \in (0,1)$ Ali-Mikhail-Haq, precise $\frac{xy}{1-\theta(1-x)(1-y)}$;
(b) $(1+tx)^{-1/t}, t \in (0,\infty)$ Clayton, precise $[\max\{x^{-\theta}+y^{-\theta}-1;0\}]^{-1/\theta}$, $\theta \in [-1,\infty) - \{\theta\}$;
(c) $e^{-x^{1/t}}, t \in [1,\infty)$ Gumbel, precise $e^{[-((-\log x)^\theta + (-\log y)^\theta)^{1/\theta}]}$, $\theta \in [1,\infty)$;
(d) $1-(1-e^{-x})^{1/t}, t \in [1,\infty)$ Joe, precise $1-[(1-x)^\theta + (1-y)^\theta - (1-x)^\theta(1-y)^\theta]^{1/\theta}$, $\theta \in [1,\infty)$;
Put $W = \max(x+y-1,0)$, $\Pi = xy$, $M = \min(x,y)$.
(e) $\frac{\alpha^2(1-\alpha)}{2}W + (1-\alpha^2)\Pi + \frac{\alpha^2(1+\alpha)}{2}M, |\alpha| \le 1$.

(XIV) Product of copulas was introduced by W. F. Darsow—B. Nguyen–E. T. Olsen [9] and L. Overbeck—W. M. Schmidt [20]:
Given two copulas c_1 and c_2 we can define a product between term by

$$(c_1 * c_2)(x,y) = \int_0^1 \frac{\partial c_1(x,z)}{\partial z}\frac{\partial c_2(z,y)}{\partial z}\,\mathrm{d}z. \tag{9}$$

The product of two copulas is an another copula. This product is associative but not commutative. There is an identity element, so copulas with this product form a semigroup. The identity element is the copula $M(x,y) = \min(x,y)$ that is $M * c = c * M = c$. There is a null element for our semigroup and that is the independence copula $\Pi(x,y) = xy$.

It is called the independence copula because it is the copula for two independent random variables. It acts like a null elements because $\Pi * c = c * \Pi = \Pi$. Also follows [9]:
$(W * c)(x, y) = y - c(1 - x, y)$;
$(c * W)(x, y) = x - c(x, 1 - y)$.
Consider copulas A_n, B such that $A_n \to A$. Then $A_n * B \to A * B$ and $B * A_n \to B * A$.

(XV) F_1, F_2 given marginal of $F(x, y)$. Then
$W(x, y) \le F(x, y) \le M(x, y)$, where
$W(x, y) = \max(F_1(x) + F_2(y) - 1, 0)$,
$M(x, y) = \min\big(F_1(x), F_2(y)\big)$.

(XVI) A necessary and sufficient condition for the copula $c_{f,g}$ to be equal to Π is that f and g be independent as random variables (see [2]).

(XVII)
$$c(x, y) = \begin{cases} \min\big(x, h_1(y)\big) & \text{if } x \in [0, x_1], \\ \max\big(x + h_2(y) - x_2, h_1(y)\big) & \text{if } x \in [x_1, x_2], \\ \min(x - x_2 + h_2(y), y) & \text{if } x \in [x_2, 1] \end{cases}$$

is a copula if and only if:
(i) $h_1(y)$ and $h_2(y)$ are increasing;
(ii) $h_1(0) = 0, h_2(0) = 0$;
(iii) $h_1(1) = x_1, h_2(1) = x_2$;
(iv) $0 \le h_1(y) \le h_2(y) \le y$;
(v) $0 \le h_1'(y) \le h_2'(y) \le 1$,
see [6].

(XVIII) Let X be a non-empty, closed and connected set of copulas. Then there exists a two-dimensional sequence (x_n, y_n), $n = 1, 2, \dots$ in $[0, 1)^2$ such that $G\big((x_n, y_n)\big) = X$.

2.4. $\mathbf{G}_{3,2}$

Let $G_{3,2}$ be the set of all three-dimensional d.f.s $g(x, y, z)$ defined on $[0, 1]^3$ such that their two-dimensional marginals (or faces) d.f.s satisfy $g(x, y, 1) = xy$, $g(1, y, z) = yz$ and $g(x, 1, z) = xz$. For example, the $G_{3,2}$ contains

$g_1(x, y, z) = xyz$,

$g_2(x, y, z) = \min(xy, xz, yz)$,

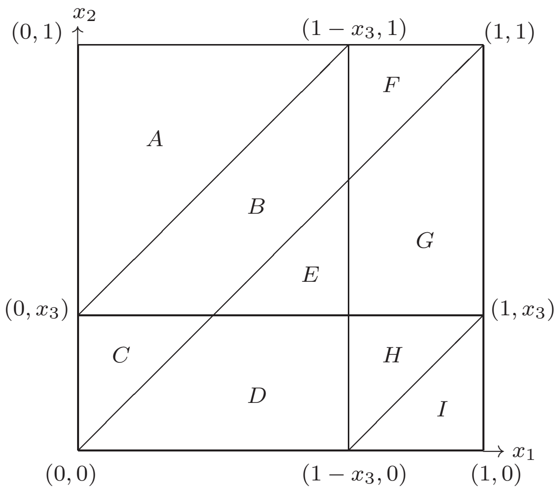
$g_3(x, y, z) = \frac{1}{u_0} \min(xyz, xyu_0, xzu_0, yzu_0)$, for fixed u_0, $0 < u_0 \le 1$,

$g_5(x, y, z) = xyz + \alpha x(1 - x)y(1 - y)z(1 - z)$, $|\alpha| < 1$, [9], p. 616, $G_{3,1}$.

Example 2. Let $g_4(x,y,z)$ be the a.d.f. of a three-dimensional sequence $(u_n, v_n, \{u_n - v_n\})$, where two-dimensional (u_n, v_n) is u.d. in $[0,1]^2$. Applying Weyl's criterion (cf. [10, p. 14], [36, p. 3–1]) we see that also $(u_n, \{u_n - v_n\})$ and $(v_n, \{u_n - v_n\})$ are u.d., thus the marginal d.f.s are

$$g_4(1,x_2,x_3) = x_2x_3, \quad g_4(x_1,1,x_3) = x_1x_3, \quad g_4(x_1,x_2,1) = x_1x_2.$$

The d.f. $g_4(x_1,x_2,x_3)$ has the following explicit form: Divide the unit square $[0,1]^2$ into regions $A, B, C, D, E, F, G, H, I$ as it is shown on the following Fig.



Then

$$g_4(x_1,x_2,x_3) = \begin{cases} x_1x_3 & \text{if } (x_1,x_2) \in A, \\ -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + x_1x_2 + x_2x_3 & \text{if } (x_1,x_2) \in B, \\ -\frac{1}{2}x_1^2 + x_1x_2 & \text{if } (x_1,x_2) \in C, \\ \frac{1}{2}x_2^2 & \text{if } (x_1,x_2) \in D, \\ -\frac{1}{2}x_3^2 + x_2x_3 & \text{if } (x_1,x_2) \in E, \\ -\frac{1}{2}x_2^2 + x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_3 + \frac{1}{2} & \text{if } (x_1,x_2) \in F, \\ \frac{1}{2}x_1^2 + x_1x_3 + x_2x_3 - x_1 - x_3 + \frac{1}{2} & \text{if } (x_1,x_2) \in G, \\ \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + x_1x_3 - x_1 - x_3 + \frac{1}{2} & \text{if } (x_1,x_2) \in H, \\ x_1x_2 + x_2x_3 - x_2 & \text{if } (x_1,x_2) \in I. \end{cases}$$

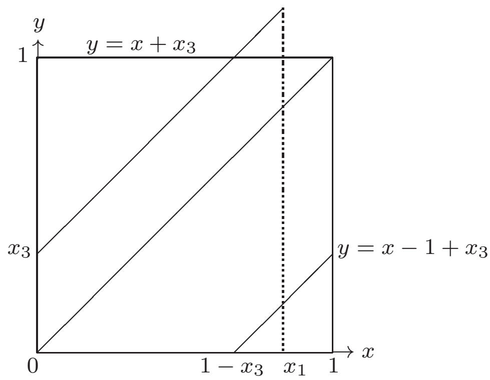
P r o o f. If (u_n, v_n) is u.d. in $[0,1]^2$, then $(u_n, \{v_n - u_n\})$ is also u.d. It follows from Weyl criterion

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\left(k_1 u_n + k_2(v_n - u_n)\right)} = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\left((k_1-k_2)u_n + k_2 v_n\right)}$$

and $(k_1, k_2) \neq (0,0) \iff (k_1 - k_2, k_2) \neq (0,0)$. Another proof follows directly from that

$$\{y - x\} < x_3 \iff \begin{cases} 0 \le y - x < x_3 & \text{if } y \ge x, \\ 0 \le 1 + y - x < x_3 & \text{if } y < x. \end{cases}$$

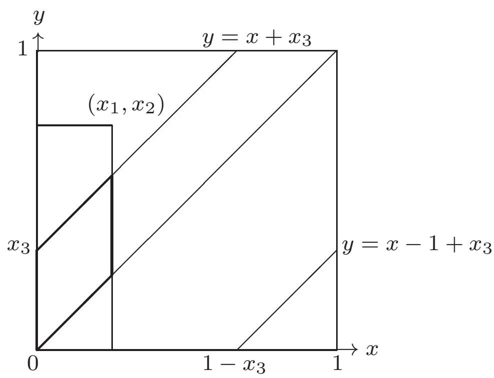
Thus the set $\{(x,y) \in [0,1]^2; \{y - x\} < x_3\}$ has the form



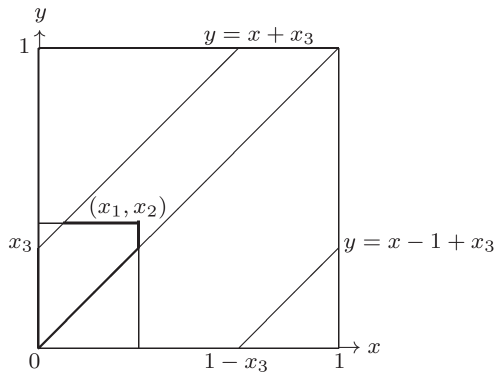
which gives $|\{(x,y) \in [0,1]^2; 0 \le x < x_1, \{y - x\} < x_3\}| = x_1 x_3$. Thus we have

$$g_4(x_1, x_2, x_3) = |\{(x,y,z) \in [0,1]^3; 0 \le x < x_1, 0 \le y < x_2, \{y - x\} < x_3\}|.$$

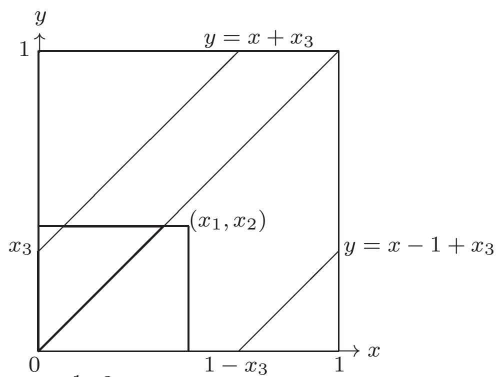
We investigate nine cases:



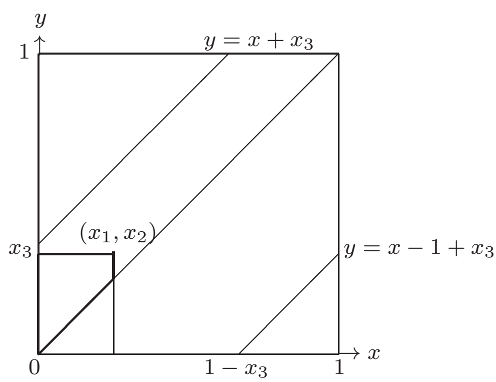
In this case $g_4(x_1, x_2, x_3) = x_1 x_3$, where $x_1 + x_3 \le x_2$.



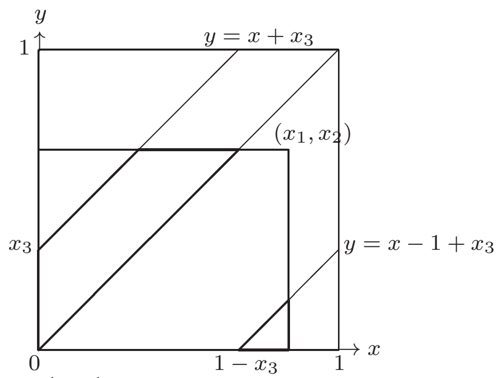
In this case $g_4(x_1, x_2, x_3) = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + x_1x_2 + x_2x_3$, where $x_1 \leq x_2 \leq x_1 + x_3$, $x_1 \leq 1 - x_3$ and $x_2 \geq x_3$.



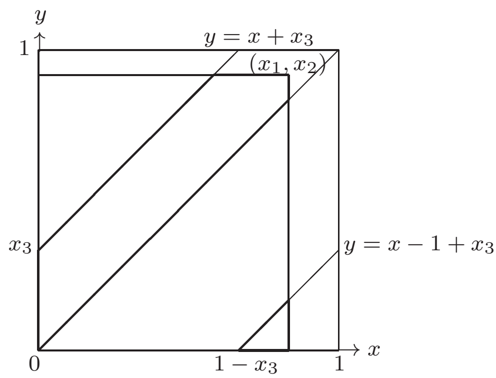
In this case $g_4(x_1, x_2, x_3) = -\frac{1}{2}x_3^2 + x_2x_3$, where $x_2 \leq x_1 \leq 1 - x_3$ and $x_3 \leq x_2$.



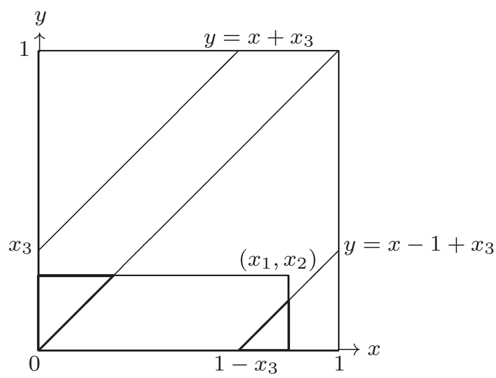
In this case $g_4(x_1, x_2, x_3) = -\frac{1}{2}x_1^2 + x_1x_2$, where $x_1 \leq x_2 \leq x_3$ and $x_1 \leq 1 - x_3$.



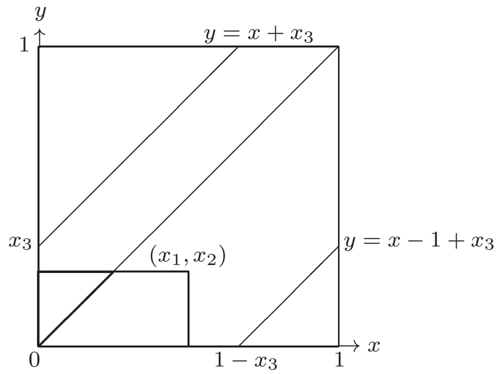
In this case $g_4(x_1, x_2, x_3) = \frac{1}{2} + \frac{1}{2}x_1^2 - x_1 - x_3 + x_2x_3 + x_1x_3$, where $x_3 \le x_2 \le x_1$ and $x_1 \ge 1 - x_3$.



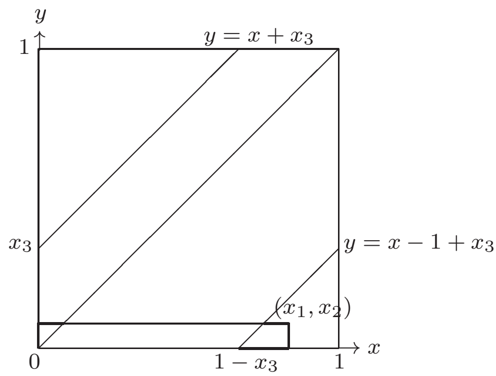
In this case $g_4(x_1, x_2, x_3) = \frac{1}{2} - x_1 - x_3 + x_1x_3 + x_1x_2 + x_2x_3 - \frac{1}{2}x_2^2$, where $x_1 \le x_2$, $x_1 \ge 1 - x_3$ and $x_2 \ge x_3$.



In this case $g_4(x_1, x_2, x_3) = \frac{1}{2} + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - x_1 - x_3 + x_1x_3$ where $x_2 \le x_3$ and $x_1 - x_2 \le 1 - x_3 \le x_1$.



In this case $g_4(x_1, x_2, x_3) = \frac{1}{2}x_2^2$ where $x_2 \le x_3$, $x_1 \le 1 - x_3$ and $x_2 \ge x_3$.



In this case $g_4(x_1, x_2, x_3) = x_1x_2 + x_2x_3 - x_2$ where $x_2 \le x_3$ and $x_2 \le x_1 - (1 - x_3)$. □

Furthermore, $G_{3,2}$ has the following properties:

For every $g(x, y, z) \in G_{3,2}$ and fixed z_0, $0 < z_0 \le 1$ we have $\frac{1}{z_0}g(x, y, z_0) \in G_{2,1}$. Vice versa, if $g_z(x, y)$, $z \in [0, 1]$ is a system of d.f.s in $G_{2,1}$ such that

(i) (i) $g_1(x, y) = xy$;

(ii) (ii) for every $z' \le z$, we have $z' \,\mathrm{d}_x\, \mathrm{d}_y\, g_{z'}(x, y) \le z \,\mathrm{d}_x\, \mathrm{d}_y\, g_z(x, y)$ on $[0, 1]^2$,

then $g(x, y, z) = zg_z(x, y) \in G_{3,2}$.

This directly follows from definition of $G_{2,1}$ and $G_{3,2}$ and that $g(x, y, z)$ is non-decreasing on $[0, 1]^3$, if every $(x_1, y_1, z_1), (x_2, y_2, z_2) \in [0, 1]^3$, $x_1 \le x_2$, $y_1 \le y_2$, $z_1 \le z_2$ satisfies

$$g(x_1, y_1, z_2) + g(x_2, y_2, z_2) - g(x_1, y_2, z_2) - g(x_2, y_1, z_2) \\ - \big(g(x_1, y_1, z_1) + g(x_2, y_2, z_1) - g(x_1, y_2, z_1) - g(x_2, y_1, z_1)\big) \ge 0.$$

2.5. Theorems over $\mathbf{G}_{2,1}$

See 1. Introduction.

Let (x_n, y_n), $n = 1, 2, \dots$, be a sequence in $[0, 1)^2$ such that both the coordinate sequences x_n, $n = 1, 2, \dots$, and y_n, $n = 1, 2, \dots$ are u.d. Then the set $G\big((x_n, y_n)\big)$ of all d.f. of (x_n, y_n), $n = 1, 2, \dots$ satisfies

(i) $G\big((x_n, y_n)\big) \subset G_{2,1}$,

(ii) $G\big((x_n, y_n)\big)$ is nonempty, closed and connected, and vice-versa

(iii) for every nonempty, closed and connected $H \subset G_{2,1}$, there exists a sequence $(x_n, y_n) \in [0, 1)^2$ such that $G\big((x_n, y_n)\big) = H$.

This is a two-dimensional version of Theorem 11, p. 191, with the metric

$$\rho(g_1, g_2) = \left(\int_0^1\int_0^1 \big(g_1(x, y) - g_2(x, y)\big)^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

Directly, from the theory of L^2 discrepancies [30] it follows:

Theorem 3. *Let (x_n, y_n), $n = 1, 2, \dots$, be a sequence in $[0, 1)^2$ such that both the coordinate sequences x_n, $n = 1, 2, \dots$, and y_n, $n = 1, 2, \dots$ are u.d. Then the sequence (x_n, y_n), $n = 1, 2, \dots$ is u.d. if and only if one of the following conditions is satisfied:*

(i) $\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^N F_0\big((x_m, y_m), (x_n, y_n)\big) = 0$;

(ii) $\lim_{N\to\infty} \frac{1}{N^4} \sum_{m,n,k,l=1}^N F_1\big((x_m, y_m), (x_n, y_n), (x_k, y_k), (x_l, y_l)\big) = 0$.

Here

$$F_0\big((x, y), (u, v)\big) = \left(\frac{1}{3}\right)^2 - \frac{1-x^2}{2}\frac{1-y^2}{2} - \frac{1-u^2}{2}\frac{1-v^2}{2}$$
$$+ \big(1 - \max(x, u)\big)\big(1 - \max(y, v)\big); \tag{10}$$
$$F_1\big((x, y), (u, v), (s, t), (z, w)\big) = \big(1 - \max(x, u)\big)\big(1 - \max(y, v)\big)$$
$$+ \big(1 - \max(x, s)\big)\big(1 - \max(v, w)\big) - 2\big(1 - \max(x, s)\big)\big(1 - \max(y, w)\big). \tag{11}$$

Applying [30, Th. 4] (cf. [36, p. 1–56, 1.10.9]) for searching $G\big((x_n, y_n)\big)$ the following theorem can be used.

Theorem 4. *Let (x_n, y_n), $n = 1, 2, \dots$, be a sequence in $[0, 1)^2$ for which both coordinate sequences x_n, $n = 1, 2, \dots$ and y_n, $n = 1, 2, \dots$ are u.d. Let $F(x, y, u, v)$ be a continuous function defined on $[0, 1]^4$ and assume that*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^N F(x_m, y_m, x_n, y_n) = 0.$$

Then every d.f. $g(x,y) \in G\big((x_n,y_n)\big)$ *is a copula which satisfies the following equation:*

$$\begin{aligned}
&\int_0^1\int_0^1 g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v F(1,1,u,v)\\
&\quad+\int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y,1,1)\\
&\quad-\int_0^1\int_0^1\int_0^1 g(u,v)y\,\mathrm{d}_y\,\mathrm{d}_u\,\mathrm{d}_v F(1,y,u,v)\\
&\quad-\int_0^1\int_0^1\int_0^1 g(u,v)x\,\mathrm{d}_x\,\mathrm{d}_u\,\mathrm{d}_v F(x,1,u,v)\\
&\quad-\int_0^1\int_0^1\int_0^1 g(x,y)v\,\mathrm{d}_v\,\mathrm{d}_x\,\mathrm{d}_y F(x,y,1,v)\\
&\quad-\int_0^1\int_0^1\int_0^1 g(x,y)u\,\mathrm{d}_u\,\mathrm{d}_x\,\mathrm{d}_y F(x,y,u,1)\\
&\quad+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y)g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v\,\mathrm{d}_x\,\mathrm{d}_y F(x,y,u,v)\\
&=-F(1,1,1,1)+\int_0^1 v\,\mathrm{d}_v F(1,1,1,v)+\int_0^1 u\,\mathrm{d}_u F(1,1,u,1)\\
&\quad+\int_0^1 x\,\mathrm{d}_x F(x,1,1,1)+\int_0^1 y\,\mathrm{d}_y F(1,y,1,1)\\
&\quad-\int_0^1\int_0^1 yv\,\mathrm{d}_y\,\mathrm{d}_v F(1,y,1,v)-\int_0^1\int_0^1 yu\,\mathrm{d}_y\,\mathrm{d}_u F(1,y,u,1)\\
&\quad-\int_0^1\int_0^1 xv\,\mathrm{d}_x\,\mathrm{d}_v F(x,1,1,v)-\int_0^1\int_0^1 xu\,\mathrm{d}_x\,\mathrm{d}_u F(x,1,u,1)=0.
\end{aligned}$$

P r o o f. By Helly theorem

$$\lim_{k\to\infty}\frac{1}{N_k^2}\sum_{m,n=1}^{N_k} F(x_m,y_m,x_n,y_n)=\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,u,v)\,\mathrm{d}g(x,y)\,\mathrm{d}g(u,v)=0$$

for $F_{N_k}(x,y)\to g(x,y)$. □

Next, for the 4-dimensional integral we use the following Theorem 5.

Theorem 5. *For every continuous $F(x,y,u,v)$ and any d.f. $g(x,y)$ we have*

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,u,v)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)\,\mathrm{d}_u\,\mathrm{d}_v g(u,v)\\
&= F(1,1,1,1) - \int_0^1 g(1,v)\,\mathrm{d}_v F(1,1,1,v) - \int_0^1 g(u,1)\,\mathrm{d}_u F(1,1,u,1)\\
&\quad + \int_0^1\int_0^1 g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v F(1,1,u,v)\\
&\quad - \int_0^1 g(1,y)\,\mathrm{d}_y F(1,y,1,1) + \int_0^1\int_0^1 g(1,y)g(1,v)\,\mathrm{d}_v\,\mathrm{d}_y F(1,y,1,v)\\
&\quad + \int_0^1\int_0^1 g(1,y)g(u,1)\,\mathrm{d}_u\,\mathrm{d}_y F(1,y,u,1)\\
&\quad - \int_0^1\int_0^1\int_0^1 g(1,y)g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v\,\mathrm{d}_y F(1,y,u,v)\\
&\quad - \int_0^1 g(x,1)\,\mathrm{d}_x F(x,1,1,1) + \int_0^1\int_0^1 g(x,1)g(1,v)\,\mathrm{d}_v\,\mathrm{d}_x F(x,1,1,v)\\
&\quad + \int_0^1\int_0^1 g(x,1)g(u,1)\,\mathrm{d}_u\,\mathrm{d}_x F(x,1,u,1)\\
&\quad - \int_0^1\int_0^1\int_0^1 g(x,1)g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v\,\mathrm{d}_x F(x,1,u,v)\\
&\quad + \int_0^1\int_0^1 g(x,y)\,\mathrm{d}_y\,\mathrm{d}_x F(x,y,1,1)\\
&\quad - \int_0^1\int_0^1\int_0^1 g(x,y)g(1,v)\,\mathrm{d}_v\,\mathrm{d}_y\,\mathrm{d}_x F(x,y,1,v)\\
&\quad - \int_0^1\int_0^1\int_0^1 g(x,y)g(u,1)\,\mathrm{d}_u\,\mathrm{d}_y\,\mathrm{d}_x F(x,y,u,1)\\
&\quad + \int_0^1\int_0^1\int_0^1\int_0^1 g(x,y)g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v\,\mathrm{d}_y\,\mathrm{d}_x F(x,y,u,v).
\end{aligned}$$

For copula $g(x,y)$ we can apply it to $g(x,1)=x$ and $g(1,y)=y$.

2.6. Examples in $\mathbf{G}_{2,1}$

As an application we give the following examples.

Example 3. Let p_n, $n = 1, 2, \dots$, be the increasing sequence of all primes. For the two-dimensional finite sequence

$$X_n = \left(\left(\frac{p_1}{p_n}, \frac{1}{n}\right), \left(\frac{p_2}{p_n}, \frac{2}{n}\right), \dots, \left(\frac{p_n}{p_n}, \frac{n}{n}\right)\right)$$

the coordinate sequences

$$\left(\frac{p_1}{p_n}, \frac{p_2}{p_n}, \dots, \frac{p_n}{p_n}\right), \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right)$$

are u.d., see [37] (cf. [36, p. 2–181, 2.19.16]). Using the well-known $p_n = n \log n + o(n \log n)$ we have that

$$\frac{1}{n} \sum_{i=1}^{n} \frac{p_i}{p_n} \frac{i}{n} \to \frac{1}{3}.$$

Putting $F(x, y, u, v) = xyuv - \frac{1}{9}$ and applying Theorem 4 we find that every d.f. $g(x, y) \in G(X_n)$ satisfies

$$\int_0^1 \int_0^1 g(x, y) \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{3},$$

thus the sequence X_n, $n = 1, 2, \dots$ is not u.d. in $[0, 1)^2$.

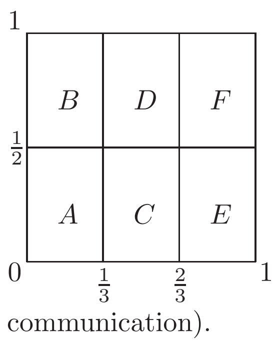
Example 4. For every u.d. sequence $x_n \in [0, 1)$ the two-dimensional sequence

$$(\{2x_n\}, \{3x_n\}), \quad n = 1, 2, \dots$$

has a.d.f. (see p. 183) that is copula

$$g(x, y) = \begin{cases} \min\left(\frac{x}{2}, \frac{y}{3}\right) & \text{if } (x, y) \in A, \\ \frac{x-1}{2} + \min\left(\frac{x+1}{2}, \frac{y+1}{3}\right) & \text{if } (x, y) \in B, \\ \frac{y-2}{3} + \min\left(\frac{x+1}{2}, \frac{y+2}{3}\right) & \text{if } (x, y) \in C, \\ \frac{x}{2} + \frac{y-1}{3} + \min\left(\frac{x}{2}, \frac{y}{3}\right) & \text{if } (x, y) \in D, \\ \frac{2y-1}{3} + \min\left(\frac{x}{2}, \frac{y+1}{3}\right) & \text{if } (x, y) \in E, \\ \frac{x-1}{2} + \frac{2y-2}{3} + \min\left(\frac{x+1}{2}, \frac{y+2}{3}\right) & \text{if } (x, y) \in F, \end{cases}$$

where

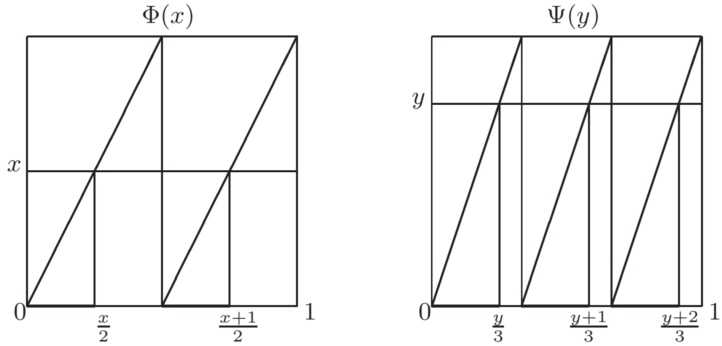


(Source J. Fialová, personal communication).

Putting $\Psi(x) = 2x \bmod 1$ and $\Psi(x) = 3x \bmod 1$ then

$$g(x,y) = \left|\Phi^{-1}\big([0,x)\big) \cap \Psi^{-1}\big([0,y)\big)\right|,$$

see the following Fig.



Example 5. By Theorem 5 in [22] for $x_n = n\alpha \bmod 1$, $n = 1, 2, \dots$, α irrational, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = 2\{\alpha\}(1 - \{\alpha\}). \tag{12}$$

P r o o f. Applying d.f. Because

$$\{(n+1)\alpha\} = \begin{cases} \{n\alpha\} + \{\alpha\}, & \text{if } \{n\alpha\} + \{\alpha\} < 1, \\ \{n\alpha\} + \{\alpha\} - 1, & \text{if } \{n\alpha\} + \{\alpha\} \geq 1, \end{cases} \tag{13}$$

then every point $(\{n\alpha\}, \{(n+1)\alpha\})$ lies on the line $Y = X + \{\alpha\} \bmod 1$. Using this and u.d. of $\{n\alpha\}$ we can compute a.d.f. $g(x,y)$ of the sequence $\big(\{n\alpha\}, \{(n+1)\alpha\}\big)$, $n = 1, 2, \dots$ by means the following Fig. and similarly to (27).

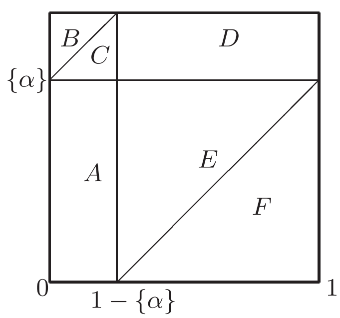


Fig.: Graph of the line $Y = x + \{\alpha\} \bmod 1$.

If $\{\alpha\} > \frac{1}{2}$, then the copula $g(x,y)$ has the form

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ x & \text{if } (x,y) \in B, \\ \{\alpha\} - y & \text{if } (x,y) \in C, \\ \{\alpha\} - y + x - (1 - \{\alpha\}) & \text{if } (x,y) \in D, \\ x - (1 - \{\alpha\}) & \text{if } (x,y) \in E, \\ y & \text{if } (x,y) \in F \end{cases} \tag{14}$$

and from it follows

$$g(x,x) = \begin{cases} 0 & \text{if } x \in [0, 1 - \{\alpha\}], \\ x - (1 - \{\alpha\}) & \text{if } x \in [1 - \{\alpha\}, \{\alpha\}], \\ x - (1 - \{\alpha\}) + x - \{\alpha\} & \text{if } x \in [\{\alpha\}, 1]. \end{cases} \tag{15}$$

Then (15) and (65) imply (12), since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\{n\alpha\} - \{(n+1)\alpha\}| = 1 - 2\int_0^1 g(x,x)\,\mathrm{d}x = 2\{\alpha\}(1 - \{\alpha\}).$$

Similarly for $\{\alpha\} < \frac{1}{2}$. □

Note that simplify (13) to

$$\{(n+1)\alpha\} = \begin{cases} \{n\alpha\} + \{\alpha\} & \text{if } \{n\alpha\} \in [0, 1 - \{\alpha\}), \\ \{n\alpha\} + \{\alpha\} - 1 & \text{if } \{n\alpha\} \in [1 - \{\alpha\}, 1) \end{cases}$$

we have

$$|\{(n+1)\alpha\} - \{n\alpha\}| = \begin{cases} \{\alpha\} & \text{if } \{n\alpha\} \in [0, 1 - \{\alpha\}), \\ 1 - \{\alpha\} & \text{if } \{n\alpha\} \in [1 - \{\alpha\}, 1) \end{cases} \tag{16}$$

and then for $F(x,y) = |x - y|$ we can use

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^{N} F\big(\{n\alpha\}, \{(n+1)\alpha\}\big) &\to \int_0^1 \int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\, g(x,y) \\ &= \int_0^{1-\{\alpha\}} F(x, x + \{\alpha\})\,\mathrm{d}x + \int_{1-\{\alpha\}}^1 F(x, x + \{\alpha\} - 1)\ \mathrm{d}x \end{aligned}$$

which gives (12), again.

More generally: Let

— $x_n \in [0,1)$ be u.d.,

— $y_n = f(x_n)$, where $f : [0,1) \to [0,1)$ be continuous,

— $F(x,y)$ be continuous,

— $g(x,y)$ be d.f. of (x_n, y_n).

Then, simultaneously

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) \to \int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\, g(x,y),$$
$$\frac{1}{N}\sum_{n=1}^{N} F\big(x_n, f(x_n)\big) \to \int_0^1 F\big(x, f(x)\big)\,\mathrm{d}x.$$

Example 6. Similar method can be used to prove S. Steinerberger [28] result that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}(x_n - y_n)^2 \le \frac{1}{3} \tag{17}$$

for every two u.d. sequences x_n and y_n in $[0,1)$.

Put $F(x,y) = (x-y)^2$. Then

— $F(1,1) = 0$;

— $\mathrm{d}_y F(1,y) = -2(1-y)\,\mathrm{d}y$;

— $\mathrm{d}_x F(x,1) = -2(1-x)\,\mathrm{d}x$;

— $\mathrm{d}_x\,\mathrm{d}_y F(x,y) = -2\,\mathrm{d}x\,\mathrm{d}y$.

Applying (59) then for every copula $g(x,y)$ we have

$$\int_0^1\int_0^1 (x-y)^2\ \mathrm{d}_x\,\mathrm{d}_y\, g(x,y) = \frac{2}{3} - 2\int_0^1\int_0^1 g(x,y)\ \mathrm{d}x\,\mathrm{d}y.$$

Now, using lower bound in (61) and computing

$$\int_0^1\int_0^1 \max(x+y-1, 0)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{6},$$

then we have (17) in the form

$$\int_0^1\int_0^1 (x-y)^2\,\mathrm{d}_x\,\mathrm{d}_y\, g(x,y) \le \frac{1}{3}$$

for an arbitrary copula $g(x,y)$.

Theorem 6. *For every copula $g(x,y)$ we have*

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) \le \frac{1}{2}.$$

P r o o f. Input $F(x,y) = |x-y|$ to

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 g(1,y)\,\mathrm{d}_y F(1,y) \\ - \int_0^1 g(x,1)\,\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y) \quad (18)$$

and bearing in mind

$$F(1,1) = 0, \qquad F(1,y) = 1-y, \qquad F(x,1) = 1-x,$$

$$\begin{aligned}\mathrm{d}_x\,\mathrm{d}_y|x-y| &= \mathrm{d}_x\,\mathrm{d}_y(y-x)\\ &= \big(y+\mathrm{d}y-(x+\mathrm{d}x)\big)+(y-x)\\ &\quad -\big(y-(x+\mathrm{d}x)\big)-(y+\mathrm{d}y-x) = 0\end{aligned}$$

for $y > x$, similarly for $y < x$ and for $y = x$, $\mathrm{d}x = \mathrm{d}y$

$$\begin{aligned}\mathrm{d}_x\,\mathrm{d}_y|x-y| &= |x+\mathrm{d}x-(x+\mathrm{d}x)|+|x-x|\\ &\quad -|(x+\mathrm{d}x)-x|-|x-(x+\mathrm{d}x)| = -2\,\mathrm{d}x\end{aligned}$$

we have

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) = 1 - 2\int_0^1 g(x,x)\,\mathrm{d}x. \quad (19)$$

Thus if $g_1(x,y) \le g_2(x,y)$ for $(x,y) \in [0,1]^2$, then

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g_2(x,y) \le \int_0^1\int_0^1 |x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g_1(x,y).$$

The lower bond $\max(x+y-1,0) \le g(x,y)$ and

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,\max(x+y-1,0) = \frac{1}{2}$$

implies theorem. □

Now, let again x_n and y_n, $n = 1, 2, \dots$, be two u.d. sequences in the unit interval $[0, 1)$. F. Pillichshammer and S. Steinerberger in [22] study the sequence $\frac{1}{N}\sum_{n=1}^{N} |x_n - y_n|$ and proved that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |x_n - y_n| \le \frac{1}{2}. \tag{20}$$

Putting $y_n = x_{n+1}$, $n = 1, 2, \dots$, they found a new necessary condition

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |x_{n+1} - x_n| \le \frac{1}{2} \tag{21}$$

for u.d. of the sequence x_n. They also found

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1} - x_n| = \frac{2(q-1)}{q^2}$$

for van der Corput sequence $x_n = \gamma_q(n)$ in the base q and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1} - x_n| = 2\{\alpha\}(1 - \{\alpha\})$$

for $x_n = n\alpha \bmod 1$, where α is irrational. Alternative proofs via d.f.s are in the following Example:

Example 7. For van der Corput sequence $x_n = \gamma_q(n)$, $n = 0, 1, \dots$, in the base q we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1} - x_n| = \frac{2(q-1)}{q^2}. \tag{22}$$

P r o o f. Every point $\big(\gamma_q(n), \gamma_q(n+1)\big)$, $n = 0, 1, 2, \dots$, lies on the line segments

$$Y = X + \frac{1}{q}, \qquad X \in \left[0, 1 - \frac{1}{q}\right], \tag{23}$$

$$Y = X - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}}, \qquad X \in \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right], \quad i = 1, 2, \dots \tag{24}$$

P r o o f. Express an n in the base q

$$n = n_k q^k + n_{k-1} q^{k-1} + \dots + n_1 q + n_0,$$

where $n_i < q$ and $n_k > 0$. We consider two following cases:

1^0. $n_0 < q - 1$,

2^0. $n_0 = q - 1$.

1^0. Let $n_0 < q-1$. Then:

$$n = n_k q^k + \dots + n_0,$$

$$n+1 = n_k q^k + \dots + n_0 + 1 \quad \text{and} \quad \gamma_q(n+1) - \gamma_q(n) = \frac{1}{q}.$$

In this case

$$\gamma_q(n) = \frac{n_0}{q} + \dots + \frac{n_k}{q^{k+1}} \le \frac{q-2}{q} + \frac{q-1}{q^2} + \dots = \frac{q-1}{q}.$$

Thus such $\big(\gamma_q(n), \gamma_q(n+1)\big)$ lies on the line-segment (23).

2^0. Let $n_0 = q-1$. Then:

$$n = n_k q^k + \dots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \dots + (q-1)$$

and

$$n_{i+1} < q-1, \quad \text{where} \quad i = 0, 1, 2, \dots$$

Then

$$n+1 = n_k q^k + \dots + (n_{i+1}+1) q^{i+1} + 0.q^i + 0.q^{i-1} + \dots + 0.$$

Thus

$$\gamma_q(n) = \frac{q-1}{q} + \dots + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+1) = \frac{n_{i+1}+1}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

and we have

$$\gamma_q(n+1) - \gamma_q(n) = \frac{1}{q^{i+2}} - \frac{q-1}{q}\left(1 + \frac{1}{q} + \dots + \frac{1}{q^i}\right) = \frac{1}{q^{i+2}} - 1 + \frac{1}{q^{i+1}},$$

and

$$1 - \frac{1}{q^{i+1}} = \frac{q-1}{q} + \dots + \frac{q-1}{q^{i+1}} \le \gamma_q(n)$$

and

$$\gamma_q(n) \le \frac{q-1}{q} + \dots + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \frac{q-1}{q^{i+3}} + \dots = 1 - \frac{1}{q^{i+2}}.$$

Thus such $\big(\gamma_q(n), \gamma_q(n+1)\big)$ lies on the segment (24).
Thus, for 1^0, the sequence $\big(\gamma_q(n), \gamma_q(n+1)\big)$ lies on the diagonal of the interval

$$I_X \times I_Y := \left[0, 1 - \frac{1}{q}\right] \times \left[\frac{1}{q}, 1\right] \tag{25}$$

and for 2^0, the sequence $\big(\gamma_q(n), \gamma_q(n+1)\big)$ lies on the diagonals of the intervals

$$I_X^{(i)} \times I_Y^{(i)} := \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], \quad i = 1, 2, \dots \tag{26}$$

These intervals are maximal with respect to inclusion. □

Adding the maps (23) and (24) we found so-called the von Neumann-Kakutani transformation $T:[0,1]\to[0,1]$, see the following Fig. Because $\gamma_q(n)$ is u.d., the sequence $\big(\gamma_q(n),\gamma_q(n+1)\big)$ has a.d.f. copula $g(x,y)$ of the form

$$\begin{aligned} g(x,y) &= |\mathrm{Project}_X\big(([0,x]\times[0,y])\cap \mathrm{graph}\ T\big)| \\ &= \min\big(\big|[0,x]\cap I_X\big|,\big|[0,y]\cap I_Y\big|\big) \\ &\quad+\sum_{i=1}^{\infty}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|\right), \end{aligned} \tag{27}$$

where $\mathrm{Project}_X$ is the projection of a two dimensional set to the X-axis.

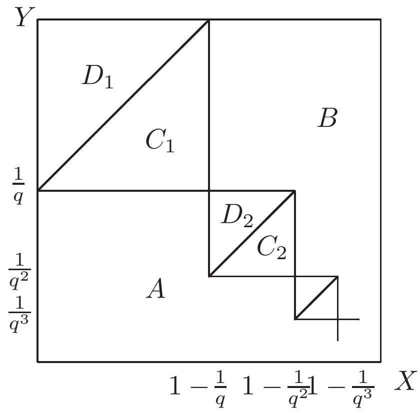


Fig.: Line segments containing $\big(\gamma_q(n),\gamma_q(n+1)\big), n=1,2,\dots$
The graph of the von Neumann-Kakutani transformation T.

The sum (27) implies

$$g(x,y)=\begin{cases}0 & \text{if } (x,y)\in A,\\ 1-(1-y)-(1-x)=x+y-1 & \text{if } (x,y)\in B,\\ y-\frac{1}{q^i} & \text{if } (x,y)\in C_i,\\ x-1+\frac{1}{q^{i-1}} & \text{if } (x,y)\in D_i,\end{cases} \tag{28}$$

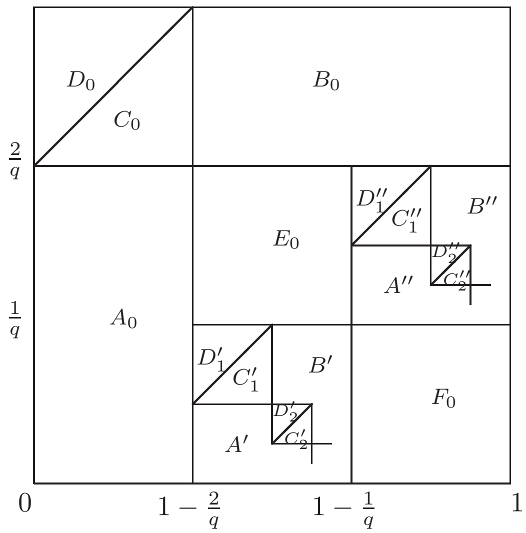
$i=1,2,\dots$ Thus

$$g(x,x)=\begin{cases}0, & \text{if } x\in\left[0,\frac1q\right],\\ x-\frac1q, & \text{if } x\in\left[\frac1q,1-\frac1q\right],\\ 2x-1, & \text{if } x\in\left[1-\frac1q,1\right]\end{cases} \tag{29}$$

and by (65)

$$\lim_{N\to\infty}\frac1N\sum_{n=0}^{N-1}|\gamma_q(n)-\gamma_q(n+1)|=1-2\int_0^1 g(x,x)\,\mathrm{d}x=\frac{2(q-1)}{q^2}. \qquad \square$$

Note that the dividing $[0,1]^2$ in the following Fig.



gives the following copula $g(x,y)$ of the sequence $\big(\gamma_q(n), \gamma_q(n+2)\big)$.

$$g(x,y) = \begin{cases} x & \text{if } (x,y) \in D_0, \\ y - \frac{2}{q} & \text{if } (x,y) \in C_0, \\ 0 & \text{if } (x,y) \in A_0, \\ y + x - 1 & \text{if } (x,y) \in B_0, \\ x - 1 + \frac{2}{q} & \text{if } (x,y) \in E_0, \\ y & \text{if } (x,y) \in F_0, \\ 0 & \text{if } (x,y) \in A', \\ x + y - 1 + \frac{1}{q} & \text{if } (x,y) \in B', \\ x - 1 + \frac{1}{q} + \frac{1}{q^i} & \text{if } (x,y) \in D_i', \\ y - \frac{1}{q^{i+1}} & \text{if } (x,y) \in C_i', \\ \frac{1}{q} & \text{if } (x,y) \in A'', \\ x + y - 1 & \text{if } (x,y) \in B'', \\ x - 1 + \frac{1}{q} + \frac{1}{q^i} & \text{if } (x,y) \in D_i'', \\ y - \frac{1}{q^{i+1}} & \text{if } (x,y) \in C_i''. \end{cases} \tag{30}$$

2.7. Theorems in $\mathbf{G}_{3,1}$

See [11], [34]. Every maximal 3-dimensional interval I containing points

$$\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)$$

will be written as

$$I = I_X \times I_Y \times I_Z,$$

where I_X, I_Y, I_Z are projections of I to the $X, Y, Z,$ axes, respectively. Moreover if

$$\gamma_q(n) \in I_X,$$

then

$$\gamma_q(n+1) \in I_Y \quad \text{and} \quad \gamma_q(n+2) \in I_Z.$$

From u.d. of $\gamma_q(n)$ follows that the lengths

$$|I_X| = |I_Y| = |I_Z|.$$

Combining intervals of equal lengths by following Fig. we find: every point

$$\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)$$

is contained in diagonals of the intervals

$$I = \left[0, 1 - \frac{2}{q}\right] \times \left[\frac{1}{q}, 1 - \frac{1}{q}\right] \times \left[\frac{2}{q}, 1\right], \tag{31}$$

$$I^{(i)} = \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right] \times \left[\frac{1}{q} + \frac{1}{q^{i+1}}, \frac{1}{q} + \frac{1}{q^i}\right],$$

$$i = 1, 2, \dots, \tag{32}$$

$$J^{(k)} = \left[1 - \frac{1}{q} - \frac{1}{q^k}, 1 - \frac{1}{q} - \frac{1}{q^{k+1}}\right] \times \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right] \times \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right],$$

$$k = 1, 2, \dots, \tag{33}$$

where $|I| = 0$ if $q = 2$. These intervals are maximal with respect to inclusion.

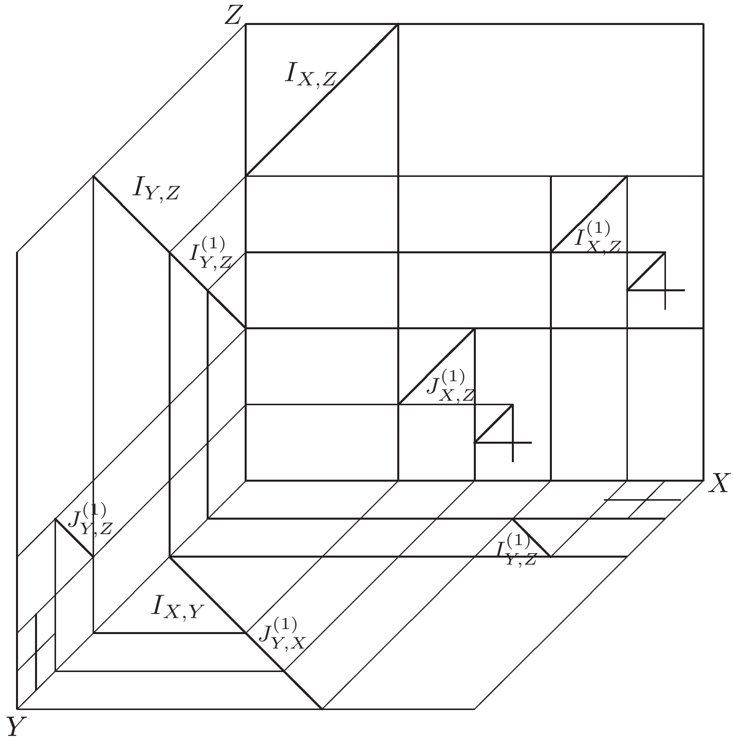


Fig.: Mapping between intervals with equal lengths.

Now, let T be the union of diagonals. Again, as in (27), the a.d.f. $g(x, y, z)$ has the form[2]

$$g(x, y, z) = |\text{Project}_X([0, x] \times [0, y] \times [0, z] \cap T)| \tag{34}$$

and it can be rewritten as

$$\begin{aligned} g(x, y, z) = \min(&|[0, x] \cap I_X|, |[0, y] \cap I_Y|, |[0, z] \cap I_Z|) \\ &+ \sum_{i=1}^{\infty} \min\left(\left|[0, x] \cap I_X^{(i)}\right|, \left|[0, y] \cap I_Y^{(i)}\right|, \left|[0, z] \cap I_Z^{(i)}\right|\right) \\ &+ \sum_{k=1}^{\infty} \min\left(\left|[0, x] \cap J_X^{(k)}\right|, \left|[0, y] \cap J_Y^{(k)}\right|, \left|[0, z] \cap J_Z^{(k)}\right|\right). \end{aligned} \tag{35}$$

[2]Since $g(x, y, z)$ is continuous, we use in the calculation closed intervals.

To calculate minimums in (45) we can use the following Fig.(here $q=3$):

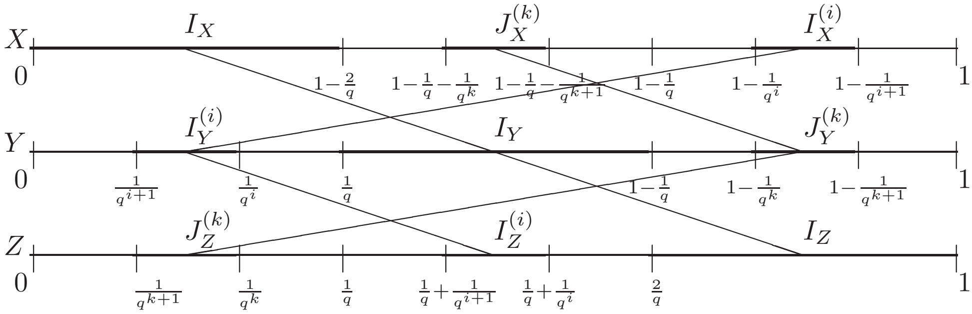


Fig.: Projections of intervals $I, I^{(i)}, J^{(k)}$ on axes X, Y, Z.

As an example of application of (45) and by Fig. we compute $g(x,x,x)$ for $q\ge 3$ without using the knowledge of $g(x,y,z)$,[3]

$$g(x,x,x)=\begin{cases}0 & \text{if } x\in\left[0,\frac{2}{q}\right],\\ x-\frac{2}{q} & \text{if } x\in\left[\frac{2}{q},1-\frac{1}{q}\right],\\ 3x-2 & \text{if } x\in\left[1-\frac{1}{q},1\right].\end{cases}\qquad(36)$$

P r o o f.

1. Let $x\in\left[0,\frac{1}{q}\right]$.
Then $|[0,x]\cap I_Z|=0$, $\left|[0,x]\cap I_Z^{(i)}\right|=0$, $\left|[0,x]\cap J_Y^{(k)}\right|=0$, consequently $g(x,x,x)=0$.

2. Let $x\in\left[\frac{1}{q},\frac{2}{q}\right]$.
Then $|[0,x]\cap I_Z|=0$, $\left|[0,x]\cap J_Y^{(k)}\right|=0$, $\left|[0,x]\cap I_X^{(i)}\right|=0$, consequently $g(x,x,x)=0$.

3. Let $x\in\left[\frac{2}{q},1-\frac{1}{q}\right]$.
Then $\left|[0,x]\cap I_X^{(i)}\right|=0$, $\left|[0,x]\cap J_Y^{(k)}\right|=0$, consequently
$g(x,x,x)=\min\left(1-\frac{2}{q},x-\frac{1}{q},x-\frac{2}{q}\right)=x-\frac{2}{q}$.

[3]For $q=3$ the middle member in (36) is omited.

4. Let $x \in \left[1-\frac{1}{q},1\right]$.

Specify $x \in I_X^{(k_1)}$, $x \in J_Y^{(k_1)}$. Then $\left|[0,x]\cap I_X^{(k)}\right| = 0$, $\left|[0,x]\cap J_Y^{(k)}\right| = 0$ for $k > k_1$. Thus (45) implies

$$\begin{aligned} g(x,x,x) &= \min\left(1-\frac{2}{q}, 1-\frac{1}{q}-\frac{1}{q}, x-\frac{2}{q}\right) \\ &\quad + \sum_{i=1}^{k_1} \min\left(\left|[0,x]\cap I_X^{(i)}\right|, \left|[0,y]\cap I_Y^{(i)}\right|, \left|[0,z]\cap I_Z^{(i)}\right|\right) \\ &\quad + \sum_{k=1}^{k_1} \min\left(\left|[0,x]\cap J_X^{(k)}\right|, \left|[0,y]\cap J_Y^{(k)}\right|, \left|[0,z]\cap J_Z^{(k)}\right|\right) \\ &= x-\frac{2}{q}+\sum_{i=1}^{k_1-1}\left(\frac{1}{q^i}-\frac{1}{q^{i+1}}\right)+x-1+\frac{1}{q^{k_1}} \\ &\quad + \sum_{k=1}^{k_1-1}\left(\frac{1}{q^k}-\frac{1}{q^{k+1}}\right)+x-1+\frac{1}{q^{k_1}} = 3x-2. \end{aligned}$$

For $q = 2$ we have

$$g(x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0,\frac{1}{2}\right], \\ x-\frac{1}{2} & \text{if } x \in \left[\frac{1}{2},\frac{3}{4}\right], \\ 3x-2 & \text{if } x \in \left[\frac{3}{4},1\right]. \end{cases} \tag{37}$$

□

2.8. $\mathbf{G}_{4,1}$

Von Neumann-Kakutani transformation: The map $T : [0,1] \to [0,1]$ for which $T\big(\gamma_q(n)\big) = \gamma_q(n+1)$ is called the von Neumann-Kakutani transformation. It is known that

$$T(x) = \begin{cases} x+\frac{1}{q} & \text{if } \quad x \in \left[0,1-\frac{1}{q}\right], \\ x-1+\frac{1}{q^i}+\frac{1}{q^{i+1}} & \text{if } \quad x \in \left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right], \quad i = 1,2,\dots \end{cases} \tag{38}$$

The third iteration

$$T^3(x)=\begin{cases} x+\frac{3}{q} & \text{if} \quad x\in\left[0,1-\frac{3}{q}\right], \\ x+\frac{2}{q}-1+\frac{1}{q^i}+\frac{1}{q^{i+1}} & \text{if} \quad x\in\left[1-\frac{2}{q}-\frac{1}{q^i},1-\frac{2}{q}-\frac{1}{q^{i+1}}\right]\cup, \\ & \qquad\left[1-\frac{1}{q}-\frac{1}{q^i},1-\frac{1}{q}-\frac{1}{q^{i+1}}\right]\cup, \\ & \qquad\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right], \qquad i=1,2,\dots \end{cases} \tag{39}$$

Copula of a.d.f.

$$\big(\gamma_q(n),\gamma_q(n{+}1),\gamma_q(n{+}2),\gamma_q(n{+}3)\big), n=0,1,2,\dots$$

In this part we find 4-dimensional maximal intervals in axes (X,Y,Z,U) containing the sequence $\big(\gamma_q(n),\gamma_q(n{+}1),\gamma_q(n{+}2),\gamma_q(n{+}3)\big)$, $n=0,1,2,\dots$ in diagonals. We started with 2-dimensional intervals in $(X,Y),(Y,Z),(Z,U)$ axis, respectively, containing $\big(\gamma_q(n),\gamma_q(n+1)\big)$, $n=0,1,2,\dots$, in diagonals. By (25), (26) they have length dimensions

$$\left[0,1-\frac{1}{q}\right]\times\left[\frac{1}{q},1\right]; \quad \left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right], \quad i=1,2,\dots;$$

Put

$$I,\ I^{(i)},\ i=1,2,\dots,\quad J^{(j)},\ j=1,2,\dots,\quad K^{(k)},\ k=1,2,\dots\in(X,U)$$

axis the maximal intervals containing $\big(\gamma_q(n),\gamma_q(n+3)\big)$. All these we plotted in the following Fig. Collecting intervals of equal length the maximal 4-dimensional intervals containing points

$$\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big), \quad n=0,1,2,\dots$$

in diagonals are (see [5])

$$I=\left[0,1-\frac{3}{q}\right]\times\left[\frac{1}{q},1-\frac{2}{q}\right]\times\left[\frac{2}{q},1-\frac{1}{q}\right]\times\left[\frac{3}{q},1\right], \tag{40}$$

$$I^{(i)}=\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right]$$
$$\times\left[\frac{2}{q}+\frac{1}{q^{i+1}},\frac{2}{q}+\frac{1}{q^i}\right], \qquad i=1,2,\dots, \tag{41}$$

$$J^{(j)}=\left[1-\frac{2}{q}-\frac{1}{q^j},1-\frac{2}{q}-\frac{1}{q^{j+1}}\right]\times\left[1-\frac{1}{q}-\frac{1}{q^j},1-\frac{1}{q}-\frac{1}{q^{j+1}}\right]$$
$$\times\left[1-\frac{1}{q^j},1-\frac{1}{q^{j+1}}\right]\times\left[\frac{1}{q^{j+1}},\frac{1}{q^j}\right], \qquad j=1,2,\dots, \tag{42}$$

$$K^{(k)} = \left[1 - \frac{1}{q} - \frac{1}{q^k}, 1 - \frac{1}{q} - \frac{1}{q^{k+1}}\right] \times \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right] \times \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right] \times \left[\frac{1}{q} + \frac{1}{q^{k+1}}, \frac{1}{q} + \frac{1}{q^k}\right], \qquad k = 1, 2, \ldots \tag{43}$$

Now, let D be a union of diagonals of (40), (41), (42) and (43). Then copula

$$g(x, y, z, u) = |\text{Project}_X([0, x] \times [0, y] \times [0, z] \times [0, u] \cap D)| \tag{44}$$

and it can be rewritten as

$$\begin{aligned} &g(x, y, z, u) \\ &= \min(|[0, x] \cap I_X|, |[0, y] \cap I_Y|, |[0, z] \cap I_Z|, |[0, u] \cap I_U|) \\ &\quad + \sum_{i=1}^{\infty} \min\left(\left|[0, x] \cap I_X^{(i)}\right|, \left|[0, y] \cap I_Y^{(i)}\right|, \left|[0, z] \cap I_Z^{(i)}\right|, \left|[0, u] \cap I_U^{(i)}\right|\right) \\ &\quad + \sum_{j=1}^{\infty} \min\left(\left|[0, x] \cap J_X^{(j)}\right|, \left|[0, y] \cap J_Y^{(j)}\right|, \left|[0, z] \cap J_Z^{(j)}\right|, \left|[0, u] \cap J_U^{(j)}\right|\right) \\ &\quad + \sum_{k=1}^{\infty} \min\left(\left|[0, x] \cap K_X^{(k)}\right|, \left|[0, y] \cap K_Y^{(k)}\right|, \left|[0, z] \cap K_Z^{(k)}\right|, \left|[0, u] \cap K_U^{(k)}\right|\right) \\ &= g_1(x, y, z, u) + g_2(x, y, z, u) + g_3(x, y, z, u) + g_4(x, y, z, u) \end{aligned} \tag{45}$$

respectively. Then copula

$$g(x, y, z, u) = |\text{Project}_X([0, x] \times [0, y] \times [0, z] \times [0, u] \cap D)| \tag{46}$$

and it can be rewritten as

$$\begin{aligned} &g(x, y, z, u) \\ &= \min\left(\left|[0, x] \cap I_X\right|, \left|[0, y] \cap I_Y\right|, \left|[0, z] \cap I_Z\right|, \left|[0, u] \cap I_U\right|\right) \\ &\quad + \sum_{i=1}^{\infty} \min\left(\left|[0, x] \cap I_X^{(i)}\right|, \left|[0, y] \cap I_Y^{(i)}\right|, \left|[0, z] \cap I_Z^{(i)}\right|, \left|[0, u] \cap I_U^{(i)}\right|\right) \\ &\quad + \sum_{j=1}^{\infty} \min\left(\left|[0, x] \cap J_X^{(j)}\right|, \left|[0, y] \cap J_Y^{(j)}\right|, \left|[0, z] \cap J_Z^{(j)}\right|, \left|[0, u] \cap J_U^{(j)}\right|\right) \\ &\quad + \sum_{k=1}^{\infty} \min\left(\left|[0, x] \cap K_X^{(k)}\right|, \left|[0, y] \cap K_Y^{(k)}\right|, \left|[0, z] \cap K_Z^{(k)}\right|, \left|[0, u] \cap K_U^{(k)}\right|\right) \\ &= g_1(x, y, z, u) + g_2(x, y, z, u) + g_3(x, y, z, u) + g_4(x, y, z, u) \end{aligned} \tag{47}$$

respectively. To calculate (47), as a guide, we use the following Fig. (here $q = 4$) for $x = y = z = u$.

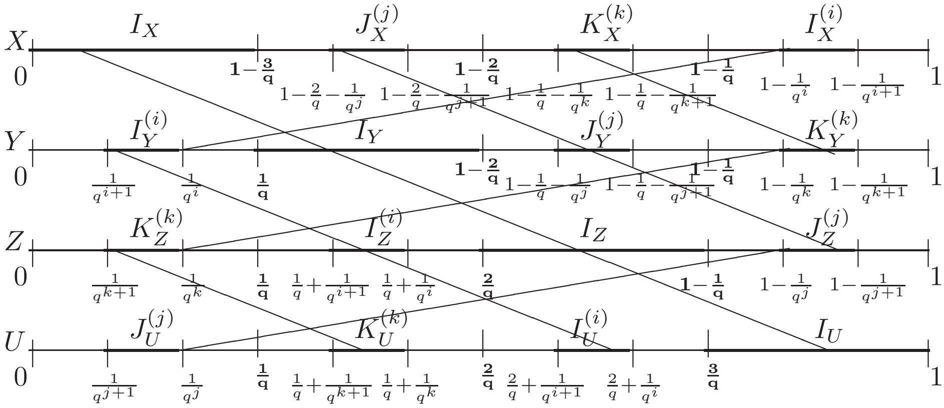


Fig.: Projections of intervals $I, I^{(i)}, J^{(j)}, K^{(k)}$ on axes X, Y, Z, U.

Assume that $q \geq 4$. Then by Fig.

$$g_1(x,x,x,x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{3}{q}\right], \\ x - \frac{3}{q}, & \text{if } x \in \left[\frac{3}{q}, 1\right], \end{cases} \tag{48}$$

and

$$g_2(x,x,x,x) = \begin{cases} 0, & \text{if } x \in \left[0, 1 - \frac{1}{q}\right], \\ x - \left(1 - \frac{1}{q}\right) & \text{if } x \in I_X^{(1)}, \\ x - \left(1 - \frac{1}{q^2}\right) + \left|I_X^{(1)}\right| & \text{if } x \in I_X^{(2)}, \dots \\ x - \left(1 - \frac{1}{q^i}\right) + \left|I_X^{(1)}\right| + \dots + \left|I_X^{(i-1)}\right| & \text{if } x \in I_X^{(i)}, \dots \end{cases}$$

Since

$$x - \left(1 - \frac{1}{q^i}\right) + \left|I_X^{(1)}\right| + \dots + \left|I_X^{(i-1)}\right| = x - 1 + \frac{1}{q}$$

we have

$$g_2(x,x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, 1 - \frac{1}{q}\right], \\ x - 1 + \frac{1}{q} & \text{if } x \in \left[1 - \frac{1}{q}, 1\right]. \end{cases} \tag{49}$$

As (49) the same holds for $g_3(x,x,x,x)$ and $g_4(x,x,x,x)$ and summing up this we have

$$g(x,x,x,x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{3}{q}\right], \\ x - \frac{3}{q}, & \text{if } x \in \left[\frac{3}{q}, 1 - \frac{1}{q}\right], \\ 4x - 3, & \text{if } x \in \left[1 - \frac{1}{q}, 1\right] \end{cases} \tag{50}$$

for $q \geq 4$.

2.9. Concluding remarks

Find an a.d.f. of the s-dimensional sequence

$$\big(\gamma_q(n), \dots, \gamma_q(n+s-1)\big), \qquad n = 0, 1, 2, \dots$$

Formal solution is given by Ch. Aisleitner and M. Hofer [1]: Let T denote von Neuman-Kakutani transformation. Define an s-dimensional curve

$$\{\gamma(t); t \in [0,1)\},$$

where

$$\gamma(t) = \big(t, T(t), T^2(t), \dots, T^{s-1}(t)\big).$$

Then the searched a.d.f. copula is

$$g(x_1, x_2, \dots, x_s) = |\{t \in [0,1]; \gamma(t) \in [0,x_1] \times [0,x_2] \times \dots \times [0,x_s]\}|\,.$$

2.10. U.d. of some copulas $\mathbf{G}_{3,2}$

Sequences (x_n, y_n, z_n) where (x_n, y_n), (x_n, z_n), and (y_n, z_n) are u.d.

Theorem 7. *Let (x_n, y_n, z_n), $n = 1, 2, \dots,$ be a sequence in $[0,1)^3$ such that both the marginal sequences*

$$(x_n, y_n), \quad n = 1, 2, \dots,$$

$$(x_n, z_n), \quad n = 1, 2, \dots$$

and

$$(y_n, z_n), \quad n = 1, 2, \dots$$

are u.d. in $[0,1)^2$. Then the sequence

$$(x_n, y_n, z_n), \quad n = 1, 2, \dots$$

is u.d. if and only if one of the following conditions is satisfied:

(i) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} F_0\big((x_m, y_m, z_m), (x_n, y_n, z_n)\big) = 0;$

(ii) $$\lim_{N\to\infty} \frac{1}{N^6}\sum_{m,n,k,l=1}^{N} F_1\Big((x_m, y_m, z_m), (x_n, y_n, z_n), (x_k, y_k, z_k), (x_{m'}, y_{m'}, z_{m'})(x_{n'}, y_{n'}, z_{n'})\Big) = 0;$$

(iii) $$\lim_{N\to\infty} \frac{1}{N^4}\sum_{m,n,k,l=1}^{N} F_1\Big((x_m, y_m, z_m), (x_n, y_n, z_n), (x_{m'}, y_{m'}, z_{m'}), (x_{n'}, y_{n'}, z_{n'})\Big) = 0.$$

Here

$$F_1\big((x_m,y_m,z_m),(x_n,y_n,z_n),(x_k,y_k,z_k),(x_{m'},y_{m'},z_{m'}),(x_{n'},y_{n'},z_{n'}),(x_{k'},y_{k'},z_{k'})\big)$$
$$=\Big(\big(1-\max(x_m,x_{m'})\big)\big(1-\max(y_m,y_{m'})\big)\big(1-\max(z_m,z_{m'})\big)$$
$$+\big(1-\max(x_m,x_{m'})\big)\big(1-\max(y_n,y_{n'})\big)\big(1-\max(z_k,z_{k'})\big)$$
$$-2\big(1-\max(x_m,x_{m'})\big)\big(1-\max(y_m,y_{n'})\big)\big(1-\max(z_m,z_{k'})\big)\Big);$$

$$F_1\big((x_m,y_m,z_m),(x_n,y_n,z_n),(x_{m'},y_{m'},z_{m'}),(x_{n'},y_{n'},z_{n'})\big)$$
$$=\Big(\big(1-\max(x_m,x_n)\big)\big(1-\max(y_m,y_n)\big)\big(1-\max(z_m,z_n)\big)$$
$$+\big(1-\max(x_m,x_{m'})\big)\big(1-\max(y_m,y_{m'})\big)\big(1-\max(z_n,z_{n'})\big)$$
$$-2\big(1-\max(x_m,x_{m'})\big)\big(1-\max(y_m,y_{m'})\big)\big(1-\max(z_m,z_{n'})\big)\Big);$$

$$F_0\big((x,y,z),(u,v,w)\big)=\big(\tfrac{1}{3}\big)^3-\tfrac{1-x^2}{2}\tfrac{1-y^2}{2}\tfrac{1-z^2}{2}-\tfrac{1-u^2}{2}\tfrac{1-v^2}{2}\tfrac{1-w^2}{2}$$
$$+\big(1-\max(x,u)\big)\big(1-\max(y,v)\big)\big(1-\max(z,w)\big);$$

P r o o f. Assume that the marginal sequences (x_n,y_n), (y_n,z_n) and (x_n,z_n) of (x_n,y_n,z_n) are u.d. Using the theory of L^2 discrepancy, every of the following zero-limits:

$$\text{(i)}\quad \lim_{N\to\infty}\int_0^1\int_0^1\int_0^1\big(F_N(x,y,z)-xyz\big)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0,$$

$$\text{(ii)}\quad \lim_{N\to\infty}\int_0^1\int_0^1\int_0^1\big(F_N(x,y,z)-F_N^{(1)}(x)F_N^{(2)}(y)F_N^{(3)}(z)\big)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0,$$

$$\text{(iii)}\quad \lim_{N\to\infty}\int_0^1\int_0^1\int_0^1\big(F_N(x,y,z)-F_N(x,y)F_N^{(3)}(z)\big)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0,$$

implies the u.d. of the sequence (x_n,y_n,z_n). Here (i) is the limit of the classical L^2 discrepancy characterizing u.d. of (x_n,y_n,z_n) and limits (ii) and (iii) are of L^2 discrepancies of statistical independence. The (ii) implies that every d.f. $g(x,y,z)\in G\big((x_n,y_n,z_n)\big)$ has the form

$$g(x,y,z)=g(x,1,1).g(1,y,1).g(1,1,z)(=x.y.z)$$

and (iii) implies

$$g(x,y,z)=g(x,y,1).g(1,1,z)(=xy.z).$$ □

3. Method of d.f.s for $\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$

Let $F(x, y)$ be a continuous function defined on $[0, 1]^2$ and $x_n, y_n, n = 1, 2, \dots,$ be two sequences in $[0, 1)$. In this section we study the limit points of the sequence of arithmetic means

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n), \quad N = 1, 2, \dots \tag{51}$$

To do this we use a theory of d.f.s. We also use this theory for computing integrals of the type

$$\int_0^1 f_1\big(\Phi(x)\big) f_2\big(\Psi(x)\big) dx \tag{52}$$

where f_1, f_2 are Riemann integrable functions and $\Phi, \Psi : [0, 1] \to [0, 1]$ are so called uniform distribution preserving (u.d.p.) maps (see, p. 193). This problem was introduced by S. Steinerberger [28, p. 127].

Recapitulation. Let x_n and y_n, $n = 1, 2, \dots,$ be an arbitrary two sequences in $[0, 1)$. Denote the step d.f.

$$F_N(x, y) = \frac{\#\{n \le N; (x_n, y_n) \in [0, x) \times [0, y)\}}{N},$$

and let $G\big((x_n, y_n)\big)$ be the set of all possible limits $F_{N_k}(x, y) \to g(x, y)$, which hold for all continuity points (x, y) of $g(x, y)$ (cf. Section 2.5). These $g(x, y)$ are called d.f.s of the sequence (x_n, y_n), $n = 1, 2, \dots$ If $G\big((x_n, y_n)\big)$ is singleton, i.e., $G\big((x_n, y_n)\big) = \{g(x, y)\}$ then $g(x, y)$ is called *asymptotic d.f.* (a.d.f.) of (x_n, y_n), $n = 1, 2, \dots$

By Riemann-Stieltjes integration we have

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1\int_0^1 F(x, y)\, \mathrm{d}_x\, \mathrm{d}_y F_N(x, y) \tag{53}$$

a) If $F_{N_k}(x, y) \to g(x, y)$ as $k \to \infty$ then by the second Helly theorem the equation (53) implies

$$\frac{1}{N_k}\sum_{n=1}^{N_k} F(x_n, y_n) \to \int_0^1\int_0^1 F(x, y)\, \mathrm{d}_x\, \mathrm{d}_y g(x, y) \quad \text{as } k \to \infty. \tag{54}$$

b) If $\frac{1}{N_k}\sum_{n=1}^{N_k} F(x_n, y_n) \to A$ as $k \to \infty$, then by the first Helly theorem there exists subsequence N'_k of N_k such that for some d.f. $g(x, y)$ of (x_n, y_n) we have $F_{N'_k}(x, y) \to g(x, y)$.

Summary, the set of limit points of (51) has the form

$$\left\{ \int_0^1 \int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y); g(x,y) \in G\big((x_n,y_n)\big) \right\}. \tag{55}$$

In (55) we can apply

$$\int_0^1 \int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 g(1,y)\,\mathrm{d}_y F(1,y) - \int_0^1 g(x,1)\,\mathrm{d}_x F(x,1) + \int_0^1 \int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y). \tag{56}$$

It is proved by integration by parts in Riemann-Stieltjes integration

$$\begin{aligned}
&\int_0^1 \int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y) \\
&= \left[\int_0^1 F(x,y)\,\mathrm{d}_y\, g(x,y) \right]_{x=0}^{x=1} - \int_0^1 \int_0^1 \,\mathrm{d}_y\, g(x,y)\,\mathrm{d}_x F(x,y) \\
&= \int_0^1 F(1,y)\,\mathrm{d}_y\, g(1,y) \ - \int_0^1 \int_0^1 \,\mathrm{d}_y\, g(x,y)\,\mathrm{d}_x F(x,y) \\
&= [F(1,y)g(1,y)]_{y=0}^{y=1} \ - \int_0^1 g(1,y)\,\mathrm{d}_y F(1,y) \\
&\quad - \left[\int_0^1 g(x,y)\,\mathrm{d}_x F(x,y) \right]_{y=0}^{y=1} + \int_0^1 \int_0^1 g(x,y)\,\mathrm{d}_y\,\mathrm{d}_x F(x,y).
\end{aligned}$$

or by induction in Theorem 10, p. 189.

Thus the problem is to find extreme values of $\int_0^1 \int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\, g(x,y)$, where $g(x,y)$ is a copula.

3.1. Boundaries of $\frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n)$

Let $x_n, y_n \in [0,1)$, $n=1,2,\dots$, both u.d. sequences. In this case two-dimensional sequence (x_n,y_n) need not be u.d. but every d.f. $g(x,y) \in G\big((x_n,y_n)\big)$ satisfies:

(i) $g(x,1) = x$ for $x \in [0,1]$ and
(ii) $g(1,y) = y$ for $y \in [0,1]$.

As we see in previous part the d.f. $g(x,y)$ satisfying (i) and (ii) is copula and its basic theory can be found in R. B. Nelsen [18] and in Section 8 of the book [35]. Applying this we find (see also (62) and (63), p. 186)

Theorem 8. *Let $F(x,y)$ be a continuous function defined on $[0,1]^2$. For differential of $F(x,y)$ assume that $\mathrm{d}_x\,\mathrm{d}_y F(x,y) > 0$ for every $(x,y) \in [0,1]^2$. Then for every two u.d. sequences $x_n, y_n \in [0,1)$, $n = 1,2,\dots$, we have*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) \le \int_0^1 F(x,x)\,\mathrm{d}x, \tag{57}$$

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) \ge \int_0^1 F(x,1-x)\,\mathrm{d}x. \tag{58}$$

Furthermore, for the sequence (x_n,y_n), $n = 1,2,\dots$, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) = \int_0^1 F(x,x)\,\mathrm{d}x$$

if and only if (x_n,y_n) has a.d.f. $g(x,y) = \min(x,y)$, and we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n,y_n) = \int_0^1 F(x,1-x)\,\mathrm{d}x$$

if and only if $g(x,y) = \max(x+y-1,0)$.

If $\mathrm{d}_x\,\mathrm{d}_y F(x,y) < 0$ the right hand sides of (57) and (58) are exchanged.

P r o o f. For a copula $g(x,y)$ the equation (56) has the form

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 y\,\mathrm{d}_y F(1,y) - \int_0^1 x\,\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y). \tag{59}$$

Thus the set (55) of limit points of (51) coincide with

$$\left\{\int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y)\right\} \tag{60}$$

shifting by $F(1,1) - \int_0^1 y\,\mathrm{d}_y F(1,y) - \int_0^1 x\,\mathrm{d}_x F(x,1)$. Then we use Fréchet--Hoeffding bounds [18, p. 9]

$$\max(x+y-1,0) \le g(x,y) \le \min(x,y) \tag{61}$$

which holds for every $(x,y) \in [0,1]^2$ and for every copula $g(x,y)$. The assumption $\mathrm{d}_x\,\mathrm{d}_y F(x,y) > 0$ implies

$$\int_0^1\int_0^1 \max(x+y-1,0)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y) \le \int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y)$$

and

$$\int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y) \le \int_0^1\int_0^1 \min(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y).$$

Since every copula is continuous, then the left inequality is attained if and only if $g(x,y)=\max(x+y-1,0)$ and the right if and only if $g(x,y)=\min(x,y)$.

Directly by definition of a.d.f., for every u.d. sequence $x_n\in[0,1)$, it can be proved that

a) the sequence (x_n,x_n), $n=1,2,\dots$, is not u.d., has the a.d.f.
$$g(x,y)=\min(x,y)\quad\text{and}$$

b) the sequence $(x_n,1-x_n)$, $n=1,2,\dots$, has the a.d.f.
$$g(x,y)=\max(x+y-1,0).$$

From it
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(x_n,x_n)=\int_0^1 F(x,x)\,\mathrm{d}x=\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\min(x,y),\tag{62}$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(x_n,1-x_n)=\int_0^1 F(x,1-x)\,\mathrm{d}x$$
$$=\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\max(x+y-1,0).\tag{63}$$
□

Once again in the following we prove the result (20) appeared in Pillichshammer and Steinerberger [22] repeating the proof of Theorem 6.

Example 8. Putting $F(x,y)=|x-y|$, we have $F(1,1)=0$, $F(1,x)=1-x$, $F(y,1)=1-y$, and computing, for $y>x$,
$$\mathrm{d}_x\,\mathrm{d}_y|y-x|=\big(y+\mathrm{d}y-(x+\mathrm{d}x)\big)=(y-x)-\big(y-(x+\mathrm{d}x)\big)-(y+\mathrm{d}y-x)=0,$$
and for $y=x$, $\mathrm{d}y=\mathrm{d}x$,
$$\mathrm{d}_x\,\mathrm{d}_y|y-x|=|x+\mathrm{d}x-(x+\mathrm{d}x)|+|x-x|-|(x+\mathrm{d}x)-x|-|x-(x+\mathrm{d}x)|=-2\,\mathrm{d}x$$
and then applying (59) we have
$$\int_0^1\int_0^1|x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)=\int_0^1 g(x,1)\,\mathrm{d}x+\int_0^1 g(1,y)\,\mathrm{d}y-2\int_0^1 g(x,x)\,\mathrm{d}x.\tag{64}$$
Thus for a copula $g(x,y)$, $g(x,1)=x$, $g(1,y)=y$ we have
$$\int_0^1\int_0^1|x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)=1-2\int_0^1 g(x,x)\,\mathrm{d}x.\tag{65}$$

Finally, the lower bound in (61) for copulas $g(x,y)$ give F. Pillichshammer and S. Steinerberger result [22] in the form
$$\int_0^1\int_0^1|x-y|\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)\le\int_0^1\int_0^1|x-y|\,\mathrm{d}_x\,\mathrm{d}_y\max(x+y-1,0)=\frac{1}{2}.\tag{66}$$

3.2. D.f. $\mathbf{g}(x, y)$ with given marginal $\mathbf{g}(1, y)$ and $\mathbf{g}(x, 1)$

In the following theorem the sequences x_n and y_n are not u.d., but with assigned a.d.f. $g_1(x)$ and $g_2(x)$, respectively.

Theorem 9. *Let $x_n \in [0, 1)$ be a sequence with an a.d.f. $g_1(x)$ and $y_n \in [0, 1)$ with an a.d.f. $g_2(x)$. Assume that $F(x, y)$ is a continuous function such that $\mathrm{d}_x\, \mathrm{d}_y F(x, y) \geq 0$ for every $(x, y) \in [0, 1]^2$. Then we have*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) \leq \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(x)\big)\, \mathrm{d}x, \tag{67}$$

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) \geq \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(1-x)\big)\, \mathrm{d}x. \tag{68}$$

Furthermore, for the sequence (x_n, y_n), $n = 1, 2, \dots,$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(x)\big)\, \mathrm{d}x \tag{69}$$

if (x_n, y_n) has the a.d.f. $g(x, y) = \min\big(g_1(x), g_2(y)\big)$ and we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(1-x)\big)\, \mathrm{d}x \tag{70}$$

if $g(x, y) = \max(g_1(x) + g_2(y) - 1, 0)$.

If $\mathrm{d}_x\, \mathrm{d}_y F(x, y) \leq 0$ the right hand sides of (67) and (68) are exchanged.

Proof. Let $g(x, y)$ be a d.f. of the sequence (x_n, y_n), i.e., there exists a sequence $N_k \to \infty$ such that $F_{N_k}(x, y) \to g(x, y)$ and

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F(x_n, y_n) \to \int_0^1 \int_0^1 F(x, y)\, \mathrm{d}_x\, \mathrm{d}_y\, g(x, y).$$

Furthermore we have $g(x, 1) = g_1(x)$ and $g(1, y) = g_2(y)$. Then by the Sklar's theorem for any such d.f. $g(x, y)$ there exists a copula $c(x, y)$ such that

$$g(x, y) = c\big(g_1(x), g_2(y)\big)$$

for every $(x, y) \in [0, 1]^2$. Applying the Fréchet-Hoeffding bounds (61) we find

$$\max\big(g_1(x) + g_2(y) - 1, 0\big) \leq g(x, y) \leq \min\big(g_1(x), g_2(y)\big). \tag{71}$$

Now, apply (59) we find (67) and (68). Let z_n, $n = 1, 2, \dots$, be u.d. sequence in $[0, 1)$. Then the sequences

(i) $\big(g_1^{-1}(z_n), g_2^{-1}(z_n)\big)$ has the a.d.f. $g(x, y) = \min\big(g_1(x), g_2(y)\big)$, and

(ii) $\big(g_1^{-1}(z_n), g_2^{-1}(1 - z_n)\big)$ has the a.d.f. $g(x, y) = \max\big(g_1(x) + g_2(y) - 1, 0\big)$.

By Helly theorem, for sequences (i) and (ii) we have (69) and (70)

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F\big(g_1^{-1}(z_n), g_2^{-1}(z_n)\big) \to \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(x)\big)\,\mathrm{d}x = \int_0^1 \int_0^1 F(x, y)\,\mathrm{d}_x\,\mathrm{d}_y \min\big(g_1(x), g_2(y)\big),$$

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F\big(g_1^{-1}(z_n), g_2^{-1}(1 - z_n)\big) \to \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(1 - x)\big)\,\mathrm{d}x = \int_0^1 \int_0^1 F(x, y)\,\mathrm{d}_x\,\mathrm{d}_y \max\big(g_1(x)+g_2(y)-1, 0\big).$$

P r o o f o f (i) a n d (ii): Assume that $g_1(x)$ and $g_2(x)$ are strictly increasing.

For (i): we note that $g_1^{-1}(z_n) < x \Leftrightarrow z_n < g_1(x)$ and $g_2^{-1}(z_n) < y \Leftrightarrow z_n < g_2(y)$ and thus

$$\big(g_1^{-1}(z_n), g_2^{-1}(z_n)\big) \in [0, x) \times [0, y) \Leftrightarrow z_n \in \big[0, \min\big(g_1(x), g_2(y)\big)\big).$$

For (ii): we note that $g_1^{-1}(z_n) < x \Leftrightarrow z_n < g_1(x)$ and $g_2^{-1}(1 - z_n) < y \Leftrightarrow 1 - z_n < g_2(y)$, $1 - z_n < g_2(y) \Leftrightarrow 1 - g_2(y) < z_n$ and thus

$$\big(g_1^{-1}(z_n), g_2^{-1}(1 - z_n)\big) \in [0, x) \times [0, y) \Leftrightarrow z_n \in \big(1 - g_2(y), g_1(x)\big).$$

On the contrary, if a d.f. $g(x)$ is constant on the interval $I = (\alpha, \beta)$ with value c and to the left of α and to the right of β d.f. $g(x)$ increases simultaneously, then we put in (i) and (ii)

$$g^{-1}(c) = \beta \tag{72}$$

because in this case the $g_1^{-1}(z_n) < x \Leftrightarrow z_n < g_1(x)$ also holds for $z_n = c$.

Finally, the uniqueness of extremal d.f. $g(x, y)$ follows from the existence of common point (x, y) of continuity for any two d.f.s $g(x, y)$. □

Completing the above proof we prove (59) in the following discrete form (73) assuming $F_N(x, y) \to g(x, y)$.

Theorem 10. *Let $F(x,y)$ be a continuous function defined on $[0,1]^2$. Then for arbitrary N-terms sequence*

$$(x_1,y_1),\dots,(x_N,y_N) \quad \textit{in } [0,1)^2 \textit{ with the step d.f. } F_N(x,y,)$$

we have

$$\begin{aligned} N\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F_N(x,y) = NF(1,1) &- N\int_0^1 F_N(1,y)\,\mathrm{d}_y F(1,y) \\ &- N\int_0^1 F_N(x,1)\,\mathrm{d}_x F(x,1) \\ &+ N\int_0^1\int_0^1 F_N(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y). \end{aligned} \tag{73}$$

P r o o f. We employ induction. Assume that for N the equation (73) holds and add in $(x_1,y_1),\dots,(x_N,y_N)$ a new point (x_{N+1},y_{N+1}) and exchange $F_N(x,y)$ by $F_{N+1}(x,y)$. We have

$$(N+1)F(1,1) = NF(1,1) + F(1,1)$$

and

$$\begin{aligned} (N+1)\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F_{N+1}(x,y) &= N\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F_N(x,y) \\ &\quad + F(x_{N+1},y_{N+1}), \\ -(N+1)\int_0^1 F_{N+1}(x,1)\,\mathrm{d}_x F(x,1) &= -N\int_0^1 F_N(x,1)\,\mathrm{d}_x F(x,1) \\ &\quad - 1.\big(F(1,1) - F(x_{N+1},1)\big), \\ -(N+1)\int_0^1 F_{N+1}(1,y)\,\mathrm{d}_y F(1,y) &= -N\int_0^1 F_N(1,y)\,\mathrm{d}_y F(1,y) \\ &\quad - 1.\big(F(1,1) - F(1,y_{N+1})\big), \\ (N+1)\int_0^1\int_0^1 F_{N+1}(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y) &= N\int_0^1\int_0^1 F_N(x,y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y) \\ &\quad + \big(F(1,1) + F(x_{N+1},y_{N+1}) \\ &\quad - F(x_{N+1},1) - F(1,y_{N+1})\big). \end{aligned}$$

Summing up

$$\begin{aligned} &F(1,1) - 1.\big(F(1,1) - F(x_{N+1},1)\big) - 1.\big(F(1,1) - F(1,y_{N+1})\big) \\ &+ \big(F(1,1) + F(x_{N+1},y_{N+1}) - F(x_{N+1},1) - F(1,y_{N+1})\big) = F(x_{N+1},y_{N+1}) \end{aligned}$$

we have that (73) valid also for $N+1$. □

4. The multidimensional d.f.s

Definition 7. An s-dimensional function $g : [0,1]^s \to [0,1]$ is d.f. if

(i) $g(\mathbf{1}) = 1$,

(ii) $g(\mathbf{0}) = 0$, and also $g(\mathbf{x}) = 0$ for every $\mathbf{x}$ with a vanishing coordinate,

(iii) $g(\mathbf{x})$ is non–decreasing, i.e.,

$$\Delta(g, J) = \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} g(x_{\varepsilon_1}^{(1)}, \dots, x_{\varepsilon_s}^{(s)}) \ge 0 \tag{74}$$

for every interval

$$J = \left[x_1^{(1)}, x_2^{(1)}\right] \times \left[x_1^{(2)}, x_2^{(2)}\right] \times \cdots \times \left[x_1^{(s)}, x_2^{(s)}\right] \subset [0,1]^s.$$

(iv) For every $l = 1, 2, \dots, s-1$ the *l–dimensional marginal d.f.*

$$g(1, \dots, 1, x_{i_1}, 1, \dots, 1, x_{i_2}, 1, \dots, 1, x_{i_l}, 1 \dots, 1)$$

of g in variables

$$(x_{i_1}, x_{i_2}, \dots, x_{i_l}) \in (0,1)^l, \text{ is d.f.}$$

Let

$$\mathbf{x} = (x_1, \dots, x_s), \qquad x_i \in [0,1), \qquad i = 1, \dots, s$$

and

$$\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}) \in \mathbb{R}^s, \qquad n = 1, 2, \dots$$

- Define the *s–dimensional step d.f.* $F_N(\mathbf{x})$ of the sequence $\mathbf{x_n}$ as

  (i) $F_N(\mathbf{x}) = \frac{1}{N}\#\{n \le N; \{x_{n,1}\} \in [0, x_1), \dots, \{x_{n,s}\} \in [0, x_s)\}$,

  (ii) $F_N(\mathbf{x}) = 0$ for every $\mathbf{x}$ having a vanishing coordinate,

  (iii) $F_N(\mathbf{1}) = \mathbf{1}$,

  (iv) $F_N(1, \dots, 1, x_{i_1}, 1, \dots, 1, x_{i_2}, 1, \dots, 1, x_{i_l}, 1 \dots, 1) = F_N(x_{i_1}, x_{i_2}, \dots, x_{i_l})$ for every restricted l–dimensional face sequence $(x_{n,i_1}, x_{n,i_2}, \dots, x_{n,i_l})$ of $\mathbf{x}_n$ for $l = 1, 2, \dots, s$.

Then

- If $f : [0,1]^s \to \mathbb{R}$ is continuous, again

$$\frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n \bmod 1) = \int_{[0,1]^s} f(\mathbf{x}) \, \mathrm{d}F_N(\mathbf{x}).$$

The s–dimensional d.f. $g(\mathbf{x})$ is a d.f. of the sequence $\mathbf{x}_n \bmod 1$ if

(i) $g(\mathbf{x}) = \lim_{k\to\infty} F_{N_k}(\mathbf{x})$ for all continuity points $\mathbf{x} \in (0,1)^s$ of g (the so-called *weak limit*) and,

(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1) = \lim_{k\to\infty} F_{N_k}(x_{i_1},x_{i_2},\dots,x_{i_l})$ weakly over $(0,1)^l$ and every l–dimensional marginal sequence of $\mathbf{x}_n$ for $l = 1,2,\dots,s-1$, and for a suitable sequence of indices $N_1 < N_2 < \dots$

- The Second Helly theorem shows that the weak limit[4] $F_{N_k}(\mathbf{x}) \to g(\mathbf{x})$ implies
$$\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}F_{N_k}(\mathbf{x}) \to \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}g(\mathbf{x})$$
for every continuous $f : [0,1]^s \to \mathbb{R}$.
- $G(\mathbf{x}_n \bmod 1)$ is the set of all d.f.s of $\mathbf{x}_n \bmod 1$.

Theorem 11. $G(\mathbf{x}_n \bmod 1)$ *is again a non–empty, closed and connected set, and either it is a singleton or it has infinitely many elements.*

Proof can be found in R. Winkler (1997) (cf. [41, p. 1–9]). Note that the connection is in the weak topology on which is metrizable by the metric
$$d(g_1,g_2) = \left(\int_{\mathbf{0}}^{\mathbf{1}} \big(g_1(\mathbf{x}) - g_2(\mathbf{x})\big)^2\,\mathrm{d}\mathbf{x}\right)^{1/2}.$$

4.1. Shuffles of M

Definition 8. Let I_i, $i = 1,2,\dots,n$ be a decomposition of the unit interval $[0,1]$, let π be a permutation of $(1,2,\dots,n)$, and let $T : [0,1] \to [0,1]$ be an one-to-one map whose graph T is formed by diagonals or anti-diagonals of squares $I_i \times I_{\pi(i)}$, $i = 1,2,\dots,n$. Then the copula $C(x,y)$ defined by
$$C(x,y) = \Big|\mathrm{Project}_x\Big(\big([0,x)\times[0,y)\big)\cap T\Big)\Big| \tag{75}$$
is called the shuffle of M.

Note that if x_n, $n = 1,2,\dots$, is an u.d. sequence, then two-dimensional sequence $\big(x_n, T(x_n)\big)$ has a.d.f $C(x,y)$ and thus for every continuous $F(x,y)$ we have
$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}C(x,y) = \int_0^1 F\big(x,T(x)\big)\,\mathrm{d}x. \tag{76}$$
For example (28), (30), (14) are shuffles of M.

[4] it means that (i) and (ii) above are fulfilled

M. Hofer and M. R. Iacò [13] proved:

Theorem 12.

— *Let* $(a_{i,j})$, $i,j=1,2,\dots,n$ *be a real-valued* $n\times n$ *matrix.*

— *Let* $I_{i,j}=\left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right]$, $i,j=1,2,\dots,n$ *and*

— *let the piecewise constant function* $F(x,y)$ *be defined as*

$$F(x,y)=a_{i,j}\ \text{if}\ (x,y)\in I_{i,j}, i,j=1,2,\dots,n.$$

Then

$$\max_{g(x,y)\text{-copula}}\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)=\frac{1}{n}\sum_{i=1}^n a_{i,\pi^*(i)}. \tag{77}$$

Here $\pi^*(i)$ *maximizes* $\sum_{i=1}^n a_{i,\pi(i)}$, *where* π *is a permutation of* $(1,2,\dots,n)$. *The maximum in* (77) *is attained at* $g(x,y)=C(x,y)$, *where* $C(x,y)$ *is the shuffle of* M *whose graph* T *is formed by diagonals or anti-diagonals in*

$$I_{i,\pi^*(i)},\qquad i=1,2,\dots,n.$$

Proof.

a) Let $C_k(x,y)$, $k=1,2,\dots,n!=N$ be all copulas defined by shuffles of M and $t_k\ge 0$, $\sum_{k=1}^N t_k=1$. Then $C(x,y)=\sum_{k=1}^N t_kC_k(x,y)$ is a copula and satisfies

$$\int_{[0,1]^2} f(x,y)\,\mathrm{d}C(x,y)\le\frac{1}{n}\sum_{i=1}^n a_{i,\pi^*(i)}.$$

b) To every copula $C(x,y)$ we add the matrix

$$C(i,j)=n\int_{I_{i,j}}1.\,\mathrm{d}C(x,y).$$

Then $C(i,j)$ is a doubly-stochastic and $C_k(i,j)$ is a permutation matrix. By Birkhoff theorem (see [7]) the set of doubly-stochastic matrices is identical with the convex hull of the set of permutation matrices. Thus there exists

$$t_k\ge 0,\quad \sum_{k=1}^N t_k=1$$

such that

$$C(i,j)=\sum_{k=1}^N t_kC_k(i,j)\quad\text{for every }\ i,j,$$

$$\frac{1}{n}a_{i,j}C(i,j)=\sum_{k=1}^N t_k\frac{1}{n}a_{i,j}C_k(i,j),$$

and thus

$$\int_{[0,1]^2} f(x,y)\,\mathrm{d}C(x,y) = \sum_{k=1}^{N} t_k \int_{[0,1]^2} f(x,y)\,\mathrm{d}C_k(x,y)$$
$$\leq \frac{1}{n}\sum_{i=1}^{n} a_{i,\pi^*(i)}. \qquad \square$$

Applying Theorem 12 M. Hofer and M. R. Iacò (2014) [13] approximate extremes of $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x,y)$ which respect to copulas $g(x,y)$ by the following.

Theorem 13. *For a continuous $F(x,y)$ on $[0,1]^2$ define piecewise constant functions $F_1(x,y)$, $F_2(x,y)$ as*

$$F_1(x,y) = \min_{(u,v)\in I_{i,j}} F(u,v) \text{ if } (x,y)\in I_{i,j}, i,j = 1,2,\dots,n,$$
$$F_2(x,y) = \max_{(u,v)\in I_{i,j}} F(u,v) \text{ if } (x,y)\in I_{i,j}, i,j = 1,2,\dots,n,$$

where

$$I_{i,j} = \left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right].$$

Let $C_0(x,y), C_1(x,y), C_2(x,y)$ be copulas such that:

— *$C_1(x,y)$ maximize $\int_0^1\int_0^1 F_1(x,y)\,\mathrm{d}g(x,y)$,*

— *$C_2(x,y)$ maximize $\int_0^1\int_0^1 F_2(x,y)\,\mathrm{d}g(x,y)$ and*

— *$C_0(x,y)$ maximize $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x,y)$*

over all copulas $g(x,y)$. Then

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}C_0(x,y)$$
$$= \lim_{n\to\infty}\int_0^1\int_0^1 F_1(x,y)\,\mathrm{d}C_1(x,y)$$
$$= \lim_{n\to\infty}\int_0^1\int_0^1 F_2(x,y)\,\mathrm{d}C_2(x,y).$$

4.2. Uniform distribution preserving map (u.d.p.)

In p. 7, (IX) we have constructed copulas by using measure preserving functions. In the following to construct copulas we use simply measure preserving function that is Jordan measure preserving functions.

The map $u : [0,1] \to [0,1]$ is called *uniform distribution preserving* (abbreviated u.d.p.) if for any u.d. sequence x_n, $n = 1, 2, \dots$, in $[0,1]$ the sequence $u(x_n)$ is also u.d. By following (vii) u.d.p. transformations are Jordan measure preserving and then we can also construct the copula (7), p. 155,

$$c_{f_1,f_2}(x,y) = |f_1^{-1}[0,x] \cap f_2^{-1}[0,y]| \quad \text{for u.d.p. } f_1, f_2.$$

In this u.d.p. theory we register the following properties:

A Riemann integrable function $u : [0,1] \to [0,1]$ is a u.d.p. transformation if and only if one of the following conditions is satisfied:

(i) $\int_0^1 h(x)\,\mathrm{d}x = \int_0^1 h\big(u(x)\big)\,\mathrm{d}x$ for every continuous $h : [0,1] \to \mathbb{R}$.

(ii) $\int_0^1 \big(u(x)\big)^k\,\mathrm{d}x = \frac{1}{k+1}$ for every $k = 1, 2, \dots$

(iii) $\int_0^1 e^{2\pi i k u(x)}\,\mathrm{d}x = 0$ for every $k = \pm 1, \pm 2, \dots$

(iv) There exists an increasing sequence of positive integers N_k such that $x_{N_k}, k = 1, 2, \dots$ is u.d sequence. Such sequence $x_n, n = 1, 2, \dots$ is called *almost u.d. sequence.*

(v) There exists an almost u.d. sequence x_n in $[0,1)$ such that the sequence $u(x_n) - x_n$ converges to a finite limit.

(vi) There exists at least one $x \in [0,1]$ of which orbit $x, u(x), u\big(u(x)\big), \dots$ is almost u.d.

(vii) u is measurable in the Jordan sense and $|u^{-1}(I)| = |I|$ for every subinterval $I \subset [0,1]$.

(viii) $\int_0^1 u(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$,
$\int_0^1 \big(u(x)\big)^2\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$,
$\int_0^1 \int_0^1 |u(x) - u(y)|\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 |x - y|\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{3}$.

From the other properties of u.d.p. transformations let us mention:

(ix) Let u_1, u_2 be u.d.p. transformations and α a real number. Then $u_1\big(u_2(x)\big)$, $1 - u_1(x)$ and $u_1(x) + \alpha \bmod 1$ are again u.d.p. transformations.

(x) Let u_n be a sequence of u.d.p. transformations uniformly converging to u. Then u is u.d.p.

(xi) Let $u : [0,1] \to [0,1]$ be piecewise differentiable. Then u is u.d.p. if and only if $\sum_{x \in u^{-1}(y)} \frac{1}{|u'(x)|} = 1$ for all but a finite number of points $y \in [0,1]$.

(xii) A piecewise linear transformation $u : [0,1] \to [0,1]$ is u.d.p. if and only if

$$|J_j| = |I_{j,1}| + \cdots + |I_{j,n_j}| \quad \text{for every} \quad J_j = (y_{j-1}, y_j),$$

where

$$0 = y_0 < y_1 < \cdots < y_m = 1$$

is the sequence of coordinates of the ends of line segment components of the graph of u and $u^{-1}(J_j) = I_{j,1} \cup \cdots \cup I_{j,n_j}$.

(xiii) U.d.p. function $f : [0,1] \to [0,1]$ is equal $f(x) = x$ or $f(x) = 1 - x$, if some of the following properties hold:

— f is monotone;

— f has a derivative in every point of the interval $(0,1)$;

— f has a Darboux property;

— f is continuous and either $f(x) \le x$ for every $x \in [0,1]$, or $f(x) \ge 1-x$ for every $x \in [0,1]$.

(xiv) $u(x)$ is u.d.p. if and only if

$$\int_0^1 \int_0^1 F\big(u(x), u(y)\big)\, \mathrm{d}x\, \mathrm{d}y = 0,$$

where

$$F(x,y) = (1/2)(|x - u(y)| + |y - u(x)| - |x - y| - |u(x) - u(y)|).$$

(xv) If $u_1(x), \dots, u_s(x)$ are u.d.p. transformations and $f(x_1, \dots, x_s)$ is continuous, then again

$$\int_0^1 f\big(u_1(x_1), \dots, u_s(x_s)\big)\, \mathrm{d}x_1 \dots \mathrm{d}x_s = \int_0^1 f(x_1, \dots, x_s)\, \mathrm{d}x_1 \dots \mathrm{d}x_s.$$

The problem to find all continuous u.d.p. is formulated in Ja.-I. Rivkid (1973) [24]. The results (i)-(vii), (ix)-(xiii) are proved in Š. Porubský, T. Šalát and O. Strauch (1988) [23]. The criterion (viii) and (xiii) are given in O. Strauch (1999, p. 116, 67) [33]. R. F. Tichy and R. Winkler (1991) [40] gave a generalization for compact metric spaces. Some related results can be found in: M. Paštéka (1987) [21], Y. Sun (1993) [38], and (1995) [39], P. Schatte (1993) [25], S. H. Molnár (1994) [17] and J. Schmeling and R. Winkler (1995) [26].

4.3. Multidimensional u.d.p. map

$\Phi : [0,1]^s \to [0,1]^s$ is called *uniform distribution preserving* (u.d.p.) map if for every uniformly distributed (u.d.) sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, the image $\Phi(\mathbf{x}_n)$ is again u.d. For one-dimensional case basic properties of u.d.p. maps can be found in [32] and [36, 2.5.1].

For example, if $\Phi(x)$, $\Psi(x)$ are u.d.p. transformation and α a real number, then $\Psi\big(\Phi(x)\big)$, $1 - \Phi(x)$, $\Phi(x) + \alpha \bmod 1$ are also u.d.p

For multi-dimensional case we have only known the following u.d.p.:

(i) $\Phi(\mathbf{x}) = \mathbf{x} \oplus \boldsymbol{\sigma}$;

(ii) $\Phi(\mathbf{x}) = \big(\Phi_1(x_1), \dots, \Phi_s(x_s)\big)$, where $\Phi_n(x)$ are one-dimensional u.d.p. maps, especially

(iii) $\Phi(\mathbf{x}) = \mathbf{b}^{\boldsymbol{\alpha}}\mathbf{x} \bmod 1 = (b_1^{\alpha_1}x_1, \dots, b_s^{\alpha_s}x_s) \bmod 1$;

(iv) $\Phi(\mathbf{x}) = \mathbf{x} + \boldsymbol{\sigma} \bmod 1 = (x_1 + \sigma_1, \dots, x_s + \sigma_s) \bmod 1$;

(v) $\Phi(\mathbf{x}) = (A\mathbf{x})^T \bmod 1$, where A is an $s \times s$ non-singular integer matrix, cf. S. Steinerberger [28, Th.2];

(vi) $\Phi(\mathbf{x}) = \pi(\mathbf{x})$, where $\pi(\mathbf{x}) = (x_{\pi(1)}, \dots, x_{\pi(n)})$ is a permutation.

(vii) The function $u(x, y) = y$.

We have the following main criterion.

Theorem 14. *A map $\Phi(\mathbf{x})$ is u.d.p. if and only if for every continuous $f : [0,1]^s \to \mathbb{R}$ we have*

$$\int_{[0,1]^s} f\big(\Phi(\mathbf{x})\big)\,\mathrm{d}\mathbf{x} = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{78}$$

In [4] is used a map $\Psi(\mathbf{x}, \boldsymbol{\sigma}) : [0,1]^{2s} \to [0,1]^s$ which is u.d.p. which respect to $\mathbf{x}$ and $\boldsymbol{\sigma}$, simultaneously. We know such maps only in the form $\Phi(\mathbf{x} \oplus \boldsymbol{\sigma})$ and $\Phi(\mathbf{x} + \boldsymbol{\sigma} \bmod 1)$, where $\Phi : [0,1]^s \to [0,1]^s$ is an arbitrary u.d.p. map. Let us remember the following definitions:

- $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ is a b-adic representation of $x \in [0,1)$, and
- $\sigma = \frac{\sigma_0}{b} + \frac{\sigma_1}{b^2} + \dots$, then
- $x \oplus \sigma = \frac{x_0 + \sigma_0 \pmod b}{b} + \frac{x_1 + \sigma_1 \pmod b}{b^2} + \dots$,
- $\mathbf{x} \oplus \boldsymbol{\sigma} = (x_1 \oplus \sigma_1, x_2 \oplus \sigma_2, \dots, x_s \oplus \sigma_s)$.
- $\{\mathbf{x} + \boldsymbol{\sigma}\} = (\{x_1 + \sigma_1\}, \{x_2 + \sigma_2\}, \dots, \{x_s + \sigma_s\})$,
- $\boldsymbol{\sigma}_i$, $i = 0, 1, \dots$, is a u.d. sequence in $[0,1)^s$,

- $g_{m,n}(\mathbf{x},\mathbf{y})$ is the asymptotic distribution function (a.d.f.) of the sequence
$$(\mathbf{x}_m \oplus \boldsymbol{\sigma}_i, \mathbf{x}_n \oplus \boldsymbol{\sigma}_i), \quad i = 0, 1, 2, \dots,$$
- also the same notation $g_{m,n}(\mathbf{x},\mathbf{y})$ is used for a.d.f. of the sequence
$$(\{\mathbf{x}_m + \boldsymbol{\sigma}_i\}, \{\mathbf{x}_n + \boldsymbol{\sigma}_i\}), \quad i = 0, 1, 2, \dots$$
- We distinguish $g_{m,n}(\mathbf{x},\mathbf{y})$ depending on whether
$$\Psi(\mathbf{x}, \boldsymbol{\sigma}) = \Phi(\mathbf{x} \oplus \boldsymbol{\sigma})$$
or
$$\Psi(\mathbf{x}, \boldsymbol{\sigma}) = \Phi(\{\mathbf{x} + \boldsymbol{\sigma}\}).$$
- Note that
$$g_{m,n}(\mathbf{x}, \mathbf{1}) = \mathbf{x} \quad \text{and} \quad g_{m,n}(\mathbf{1}, \mathbf{y}) = \mathbf{y},$$
and thus $g_{m,n}(\mathbf{x},\mathbf{y})$ is a copula, see Definition 6, p. 150.

Between u.d.p. maps and copulas we have

Theorem 15. *Let $f_i : [0,1] \to [0,1]$, $i = 1, 2, \dots s$ are u.d.p. maps, then*
$$\begin{aligned} &g(y_1, y_2, \dots, y_s) \\ &= \left| f_1^{-1}\big([0,y_1)\big) \cap f_2^{-1}\big([0,y_2)\big) \cap \dots \cap f_s^{-1}\big([0,y_s)\big) \right|. \end{aligned} \tag{79}$$
is s-dimensional copula in $G_{s,1}$.

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