

A NOTE ON WELL DISTRIBUTED SEQUENCES

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ABSTRACT. We prove an easy statement about inhomogeneous approximation for non-singular vectors in metric theory of Diophantine approximation.

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1. Well distributed sequences

Let

$$\boldsymbol{\xi}_k = (\xi_{1,k}, \dots, \xi_{n,k}), \quad k = 1, 2, 3, \dots \tag{1}$$

be an infinite sequence of points in the unit cube $[0, 1)^n$. For

$$\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in [0, 1)^n$$

we consider boxes of the form

$$B_{C,q}(\boldsymbol{\eta}) = \left[\eta_1, \eta_1 + \frac{C}{q^{1/n}} \right) \times \dots \times \left[\eta_n, \eta_n + \frac{C}{q^{1/n}} \right) \subset [0, 1)^n. \tag{2}$$

We define sequence (1) to be *well distributed* if there exists a positive constant C and infinitely many positive integers q such that for any box of the form (2) there exists a positive integer $k \leq q$ with

$$\boldsymbol{\xi}_k \in B_{C,q}(\boldsymbol{\eta}). \tag{3}$$

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Chebyshev showed that for any irrational α there exists infinitely many positive integers q such that for any $\eta \in [0, 1)$ there exists positive integer $k \leq q$ such that

$$\|k\alpha - \eta\| \leq \frac{3}{q}.$$

This means that the sequence

$$\{k\alpha\}, \quad k = 1, 2, 3, \dots \tag{4}$$

is well distributed. Generalising a result by Chebyshev, Khintchine [8] proved that the Kronecker sequence

$$\boldsymbol{\xi}_k = (\{k\alpha_1\}, \dots, \{k\alpha_n\}), \quad k = 1, 2, 3, \dots; \quad \alpha_1, \dots, \alpha_n \in \mathbb{R} \tag{5}$$

is well distributed if and only if vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ is non-singular. For the sake of completeness in the next section, we formulate this result as it is given in Cassels' book [2]. Certain discussion about different formulations of this result one can find in [7]. Here we would like to add that Khintchine's result is valid not only for the Kronecker sequence but for general systems of n linear forms in m variables.

2. Khintchine's criterium

Consider irrationality measure function

$$\psi_{\boldsymbol{\alpha}}(t) = \min_{q \in \mathbb{Z}_+ : q \leq t} \max_{1 \leq j \leq n} \|q\alpha_j\|, \quad \|\cdot\| = \min_{a \in \mathbb{Z}} |\cdot - a|.$$

By Dirichlet theorem

$$\psi_{\boldsymbol{\alpha}}(t) \leq \frac{1}{t^{1/n}} \quad \forall t \geq 1.$$

Vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ is called *non-singular* if

$$\limsup_{t \rightarrow \infty} t^{1/n} \psi_{\boldsymbol{\alpha}}(t) > 0.$$

Theorem XIII from [2] (to be more precise formula (4) from the proof of Theorem XII which is based on Theorem VI) together with transference theorem for singular vectors (Theorem XII) states that $\boldsymbol{\alpha}$ is non-singular if and only if

$\exists \delta > 0$ and \exists infinitely many $q \in \mathbb{Z}_+$ such that $\forall \boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$

$$\exists k \in \mathbb{Z}_+ : k \leq q, \quad \max_{1 \leq j \leq n} \|k\alpha_j - \eta_j\| \leq \frac{\delta}{q^{1/n}}.$$

The last statement means that for $C = 2\delta$ in any box of the form (2) there exists a point of the sequence (5) with $k \leq q$.

3. General statement

In this short communication we prove the following result.

THEOREM 3.1. *Let sequence (1) be well distributed. Then for almost all $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in [0, 1]^n$ in the sense of Lebesgue measure in $[0, 1]^n$ one has*

$$\liminf_{k \rightarrow \infty} k^{1/n} \max_{1 \leq j \leq n} \|\xi_{j,k} - \eta_j\| = 0.$$

REMARK 1. For the special case of sequence (4) this result coincide with a statement from [4], which was proved there by means of Ostrowski numerical systems. Multidimensional results which generalise one-dimensional theorem from [4] and deal with systems of n linear forms in m variables were obtained in [5]. The results from [5] deal with a setting in terms of Dynamical systems on space of lattices. Our definition of well distribution is close to the definition of local ubiquity from [3]. Another proof of a statement similar to our Theorem 3.1 can be deduced from Lemma 1 from [1].

Proof of Theorem 3.1. We take positive reals

$$\psi_\nu < 1, \quad \lim_{\nu \rightarrow \infty} \psi_\nu = 0$$

such that the series $\sum_{\nu=1}^{\infty} \psi_\nu^n$ diverges and small $\varepsilon > 0$. Below $\mu(\cdot)$ stands for the Lebesgue measure.

By the conditions of Theorem 3.1, there exists $C > 0$ and arbitrary large integers q such that in any box of the form (2) there exists a point of the sequence (1) with $k \leq q$. We cover the cube $[0, 1]^n$ by

$$W = \left[\frac{q^{1/n}}{C} \right]^n \text{ boxes } B_{C,q}(\boldsymbol{\eta}_{i_1, \dots, i_n}) \text{ of the form (2)}$$

with

$$\boldsymbol{\eta}_{i_1, \dots, i_n} = \left(\frac{i_1 C}{q^{1/n}}, \dots, \frac{i_n C}{q^{1/n}} \right),$$

$$0 \leq i_1, \dots, i_n < \left[\frac{q^{1/n}}{C} \right]^n,$$

$$i_1, \dots, i_n \in \mathbb{Z}$$

which do not intersect by inner points. At least of $W' = \left(\left[\frac{q^{1/n}}{C} \right] - 1 \right)^n$ of these boxes lies inside the cube $[0, 1]^n$. By well approximability, in each of these last boxes there is a point of the form (1) with

$$1 \leq k \leq q.$$

For positive $\psi < 1$ we take q' boxes of the form

$$I_l = \left[\xi_{1,k_l} - \frac{C\psi}{q^{1/n}}, \xi_{1,k_l} + \frac{C\psi}{q^{1/n}} \right] \times \dots \\ \dots \times \left[\xi_{n,k_l} - \frac{C\psi}{q^{1/n}}, \xi_{n,k_l} + \frac{C\psi}{q^{1/n}} \right], \quad 1 \leq l \leq q'$$

with the centres at certain points ξ_{k_l} , $1 \leq k_l \leq q$ which belong to boxes

$$B_{C,q}(\mathbf{\eta}_{i_1, \dots, i_n}) \quad \text{with} \quad i_1 \equiv \dots \equiv i_n \equiv 1 \pmod{3}. \quad (6)$$

In each of such boxes $B_{C,q}(\mathbf{\eta}_{i_1, \dots, i_n})$ we take just one point ξ_{k_l} . Then

$$I_l \cap I_{l'} = \emptyset, \quad l \neq l', \quad \text{and} \quad q' \asymp_C q.$$

Now from these integers q, q' we construct sequences q_ν and q'_ν by an inductive procedure. As q_1 we take an integer q from the definition of the well distribution. It correspond to a certain q'_1 and can be chosen to be arbitrary large.

Now we suppose that the numbers q_1, \dots, q_ν satisfying the definition of the well approximability are defined as well as the numbers q'_1, \dots, q'_ν satisfying

$$q'_\nu \asymp_C q_\nu. \quad (7)$$

and the boxes

$$I_l(\nu) = \left[\xi_{1,k_l^\nu} - \frac{C\psi_\nu}{q_\nu^{1/n}}, \xi_{1,k_l^\nu} + \frac{C\psi_\nu}{q_\nu^{1/n}} \right] \times \dots \\ \dots \times \left[\xi_{n,k_l^\nu} - \frac{C\psi_\nu}{q_\nu^{1/n}}, \xi_{n,k_l^\nu} + \frac{C\psi_\nu}{q_\nu^{1/n}} \right], \quad 1 \leq l \leq q'_\nu,$$

where

$$1 \leq k_l^\nu \leq q_\nu$$

and

$$I_l(\nu) \cap I_{l'}(\nu) = \emptyset, \quad l \neq l'. \quad (8)$$

We can chose $q = q_{\nu+1}$ from the definition of well aproximability large enough to satisfy

$$\#\{r : 1 \leq r \leq q'_{\nu+1}, I_r(\nu+1) \cap I_l(\lambda) \neq \emptyset\} \leq (1 + \varepsilon)\mu(I_l(\lambda))q'_{\nu+1} \quad (9)$$

for every box

$$I_l(\lambda), 1 \leq l \leq q'_\lambda, 1 \leq \lambda \leq \nu$$

which appeared during the previous steps $\lambda \leq \nu$. This is possible by the definition of well approximability because the number of boxes of the form (6) with $q = q_{\nu+1}$ which intersect given box $I_l(\lambda)$ is

$$\sim \mu(I_l(\lambda))q'_{\nu+1} \quad \text{when} \quad q = q_{\nu+1} \rightarrow \infty.$$

Now we define the union

$$E_\nu = \bigcup_{l=1}^{q'_\nu} I_l(\nu).$$

First of all by (7) and (8) we see that

$$\mu(E_\nu) = \sum_l \mu(I_l(\nu)) \gg_C \psi_\nu^n,$$

and so

$$\sum_{\nu=1}^{\infty} \mu(E_\nu) = \infty. \quad (10)$$

Then for $\lambda < \nu$ by (9) we have

$$\mu(E_\lambda \cap E_\nu) \leq (1 + \varepsilon) \mu(E_\lambda) \mu(E_\nu). \quad (11)$$

A famous lemma from metric theory of Diophantine approximation (see [6, Lemma 5, § 3, Ch. 1]) shows that the measure of the set

$$E = \{\boldsymbol{\eta} : \exists \text{ infinitely many } \nu \text{ such that } \boldsymbol{\eta} \in E_\nu\}$$

is

$$\geq \limsup_{t \rightarrow \infty} \frac{(\sum_{\nu=1}^t \mu(E_\nu))^2}{\sum_{\lambda, \nu=1}^t \mu(E_\lambda \cap E_\nu)} \geq \frac{1}{1 + \varepsilon},$$

by (10) and (11). As ε is arbitrary, Theorem 3.1 is proven. \square

REFERENCES

- [1] BERESNEVICH, V.—VELANI, S.: *A note on zero-one laws in metrical Diophantine approximation*, Acta Arithm. **133** (2008), no. 4, 363–374.
- [2] CASSELS, J. W. S.: *An Introduction to Diophantine Approximations*, Cambridge Univ. Press, Cambridge, 1957.
- [3] DICKINSON, D.—BERESNEVICH, V.—VELANI, S.: *Measure theoretic laws for lim sup sets*, Mem. Amer. Math. Soc. **179** (2006), no. 846, 91 pp.
- [4] KIM, D. H.: *The shrinking target property of irrational rotations*, Nonlinearity **20** (2007), 1637–1643.
- [5] SHAPIRA, U.: *Grids with dense values*, Comment. Math. Helv. **88** (2013), 485–506.
- [6] SPRINDZHUK, V. G.: *Metric Theory of Diophantine Approximations*, (Translated from the Russian (Richard A. Silverman ed.) With a foreword by Donald J. Newman.) In: *Scripta Series in Mathematics*. A Halsted Press Book V. H. Winston & Sons, Washington, D. C.; John Wiley & Sons, New York, 1979.

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- [7] MOSHCHEVITIN, N.: *Khinchine's theorem on Chebyshev matrices*, Istoriko-matematicheskie Issledovaniya, Ser. 2, **50** (2014), no. 5, 55–64, Preprint available at arXiv:1212.5662 (in Russian).
- [8] ХИНЧИН, А. Я.: *Регулярные системы линейных уравнений и общая задача Чебышева*, Известия АН СССР, серия математическая, **12** (1948), 249–258. (in Russian)

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