EQUIDISTRIBUTION
OF CONTINUOUS FUNCTIONS
ALONG MONOTONE COMPACT COVERS

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ABSTRACT. We give a necessary and sufficient condition for equidistribution of continuous functions along monotone compact covers on locally compact spaces. We show the existence of equidistributed mappings along Bohr nets arising from group actions. Using almost periodic means, we give an analogue of Weyl's equidistribution criterion for continuous functions with values in arbitrary topological groups. We prove van der Corput's inequality on the lattice $\mathbb{N}^m$ for vectors in Hilbert spaces, and use this inequality to extend Hlawka's equidistribution theorem to functions on the lattice $\mathbb{N}^m$ ($m \geq 1$) with values in arbitrary topological groups.

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1. Introduction and preliminaries

The classical theory of equidistribution (also known as 'uniform distribution') of sequences was initiated by Weyl [32] following earlier works by Bohl and Sierpiński. For a survey of this theory and its developments we refer to Kuipers-Niederreiter [22] and Limic-Limić [24], and the references therein. The purpose of this paper is to study a concept of equidistribution for continuous functions on locally compact spaces that includes equidistribution of sequences as a special case.

In addition to wider applicability, the generality of our study serves several purposes. First, in view of the considerable interest in Banach function algebras [5, 21], it is of interest to investigate equidistribution of functions with respect to linear functionals on such algebras. Second, using almost periodic means, we formulate some of the key results of the theory (e.g., Weyl's criterion Theorem 2.6) for functions with values in arbitrary topological groups. Third, we shall give an extension of Hlawka's equidistribution theorem, to functions defined on the lattice $\mathbb{N}^m$ with values in topological groups (Theorem 3.4).

Since integration and regularity of measures play important roles in some of the key results (e.g., Theorem 2.2), we shall base our study on continuous functions on locally compact spaces. In Section 2, we first define the equidistribution of continuous functions (Definition 2.1), and then give a necessary and sufficient condition for this phenomenon in Theorem 2.2. In Theorem 2.4 we show that continuous group actions naturally give rise to examples of equidistributed functions along Bohr nets (for definition of Bohr nets, see below). Of particular interest for functions with values in topological groups is the notion of almost periodic (a.p.) equidistribution (Definition 2.5). We give an analogue of the Weyl's criterion for a.p.-equidistribution in Theorem 2.6. Some of the applications of Weyl's criterion are given in Corollary 2.8, Theorem 3.4, and Theorem 3.5.

In Section 3 we prove van der Corput's inequality on the lattice $\mathbb{N}^m$ for vectors in Hilbert spaces (Theorem 3.2). This inequality is used to give a sufficient condition for the a.p.-equidistribution of functions $\varphi\colon \mathbb{N}^m \longrightarrow H$, where $m \geq 1$, H is a topological group, and the coordinates $n_1, \ldots, n_m$ of $(n_1, \ldots, n_m) \in \mathbb{N}^m$ vary over arithmetic progressions (Theorem 3.4). The theorem extends a result of Hlawka [20, Satz 6, p. 42] from sequences to functions $\varphi\colon \mathbb{N}^m \longrightarrow H$. We end the paper with an application of Theorem 3.4 to multivariate polynomials $p(x_1, \ldots, x_m)$, where p has a nonconstant term with an irrational coefficient, and where the variables $x_1, \ldots, x_m$ run over arithmetic progressions in $\mathbb{N}$ (Theorem 3.5).

In the remaining of this section we review some terminology and notation. Throughout this paper all topological spaces are assumed to be Hausdorff, unless otherwise stated. If X is a topological space, then a Banach function algebra on X is a subalgebra A of $C^b(X)$ (continuous bounded functions on X) that is a Banach algebra under a norm $\| \cdot \|$. The topological dual of A (i.e., the set of all continuous linear functionals on A) is denoted by A^*. For each $x \in X$, the evaluation functional τ_x on A is defined by $\tau_x(f) = f(x)$ ($f \in A$). Since τ_x is multiplicative, it follows that $\tau_x \in A^*$ with $\|\tau_x\| \leq 1$ ([19, Theorem C. 21, p. 477]), and therefore $\|f\|_{\sup} \leq \|f\|$ for every $f \in A$. We denote the characteristic function of a set $E \subset X$ by 1_E. We say that A is unital if the

constant function $1_X \in A$, in which case we also assume $\|1_X\| = 1$. If X is a locally compact space, and $M(X)$ is the Banach space of all complex regular Borel measures on X (with the norm $\|\mu\| = |\mu|(X)$), then each $\mu \in M(X)$ gives rise to an element $\tau_\mu \in A^*$ by defining $\langle \tau_\mu, f \rangle = \int_X f \, d\mu$ ($f \in A$). The functional τ_μ is continuous since $|\langle \tau_\mu, f \rangle| = |\int_X f \, d\mu| \le \|\mu\| \|f\|_{\sup} \le \|\mu\| \|f\|$.

A Banach function algebra A on a topological space X is called *completely normal* if for every two closed disjoint subsets A, B of X, there is a function $f \in A$ such that $f(X) \subset [0, 1]$, $f = 1$ on A and $f = 0$ on B. By Urysohn's lemma, $C^b(X)$ is completely normal if and only if X is normal. On a compact topological group G, the Herz–Figà-Talamanca algebras $A_p(G)$ ($1 < p < \infty$), and in particular the Fourier algebra $A(G)$, are completely normal (for more on these algebras see [6, 10, 18, 21]).

For the definitions of nets, subnets, and their convergences we refer to Willard [33, pp. 73–75]. For properties of the space of almost periodic functions $AP(G)$ on topological groups G, we refer to von Neumann [25] or Dixmier [7, § 16, p. 296]. The unique invariant mean on $AP(G)$ will be denoted by M_G, or just M when there is no fear of confusion. If G is a locally compact group and λ is a left Haar measure on G, then a *Bohr net* on G is a net $\{K_\alpha\}_{\alpha \in I}$ of compact subsets of G such that (i) $K_\alpha \subset K_\beta$ if $\alpha \le \beta$, (ii) $G = \bigcup_{\alpha \in I} K_\alpha^\circ$ (where K_α° is the interior of K_α) and (iii) $M(f) = \lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} f \, d\lambda$, for all $f \in AP(G)$. Davis [4, Theorems 2.1, 3.4] has shown the existence of Bohr nets for all locally compact groups.

By a *monotone compact cover* on a topological space S we mean a net $\{K_\alpha\}_{\alpha \in I}$ of compact subsets of S such that (i) $S = \bigcup_{\alpha \in I} K_\alpha$ and (ii) $K_\alpha \subset K_\beta$ if $\alpha \le \beta$. Every space S has at least one such cover obtained by taking all compact subsets of S directed by inclusion. Unless otherwise stated, we shall assume that the set of natural numbers $\mathbb{N}$ is equipped with its canonical monotone cover $\mathbb{N}_N = \{1, 2, \ldots N\}$ ($N \in \mathbb{N}$). On locally compact groups, we shall be mainly interested in covers provided by Bohr nets.

Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_i\}_{i \in I}$. For a continuous linear operator $T \in \mathcal{B}(\mathcal{H})$, we let $T_{ij} = (Te_j | e_i)$ ($i, j \in I$). An operator $T \in \mathcal{B}(\mathcal{H})$ is called Hilbert-Schmidt if

$$\|T\|_2 = \left(\sum_{i,j} |T_{ij}|^2 \right)^{1/2} < \infty.$$

The space of all Hilbert-Schmidt operators on $\mathcal{H}$ denoted by $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space under the inner product defined by $(T|S) = \operatorname{tr}(S^*T)$ ($S, T \in \mathcal{B}_2(\mathcal{H})$). The norm induced by this inner product is the Hilbert-Schmidt norm

$$\|T\|_2 = (T|T)^{1/2}.$$

2. Equidistribution of continuous functions

DEFINITION 2.1. Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha \in I}$. Let X be a topological space, A a Banach function algebra on X, and $\mathfrak{m} \in A^*$. We say that a continuous function $\varphi \colon S \longrightarrow X$ is $\mathfrak{m}$-*equidistributed along* $\{K_\alpha\}_{\alpha \in I}$ if for every $f \in A$:

$$\langle \mathfrak{m}, f \rangle = \lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s) \, d\lambda(s). \tag{1}$$

REMARK 1.

(a) This definition extends other notions of equidistribution for continuous functions given in Hewitt and Ross [19, Theorem 26.17, p. 432], or Bergelson and Moreira [1, Definition 2.9] (see also Theorem 2.2 below). In the special case that $S = \mathbb{N}$ is the set of natural numbers equipped with the counting measure and the monotone cover $\{\mathbb{N}_N\}_{N=1}^\infty$, letting $x_n = \varphi(n)$, we can rewrite (1) as

$$\langle \mathfrak{m}, f \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \qquad (f \in A). \tag{2}$$

This is consistent with the definition of equidistribution of sequences, usually stated when X is compact, $A = C(X)$, and $\mathfrak{m}$ is a probability measure on X ([22, Definition 3.1.1, p. 171]).

(b) $\mathfrak{m}$-equidistribution can be interpreted as 'weak convergence' of measures. Suppose φ is injective, and $\varphi_* \lambda$ is the image (or, the push-forward) measure of λ on X. Thus for every Borel set $E \subset X$, $(\varphi_* \lambda)(E) = \lambda(\varphi^{-1}(E))$. For each $\alpha \in I$, we define a Borel measure μ_α on X by $\mu_\alpha = \frac{1}{\lambda(K_\alpha)} 1_{\varphi(K_\alpha)} \varphi_* \lambda$. Then the identity (1) can be written as $\langle \mathfrak{m}, f \rangle = \lim_{\alpha \in I} \langle \mu_\alpha, f \rangle$ $(f \in A)$. Thus φ is $\mathfrak{m}$-equidistributed if and only if μ_α converges weakly (relative to functions in A) to $\mathfrak{m} \in A^*$ (cf., [27, p. 40]).

Our first theorem is a characterization of equidistributed functions, when $\mathfrak{m} = \tau_\mu$ is induced by a finite regular Borel measure μ on X.

THEOREM 2.2. *Let S be a locally compact space equipped with a regular Borel measure λ, and $\{K_\alpha\}_{\alpha \in I}$ be a monotone compact cover of S. Let X be a locally compact space, A be a completely normal Banach function algebra on X, and let $\mathfrak{m} = \tau_\mu \in A^*$, for some finite regular Borel measure μ on X. Then a continuous function $\varphi \colon S \longrightarrow X$ is τ_μ-equidistributed along $\{K_\alpha\}_{\alpha \in I}$ if and only if for every Borel set $E \subset X$ with $\mu(\partial E) = 0$, we have*

$$\mu(E) = \lim_{\alpha} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (1_E \circ \varphi)(s) \, d\lambda(s). \tag{3}$$

P r o o f. First we prove the necessity of (3). Since μ is finite and regular, for each $n \in \mathbb{N}$ there is an open set $V_n \supset \overline{E}$ with $\mu(V_n \setminus \overline{E}) < 1/2n$, and a compact set $W_n \subset E^\circ$ with $\mu(E^\circ \setminus W_n) < 1/2n$. Since A is completely normal, there are $f_n, g_n \in A$, $f_n, g_n \colon X \longrightarrow [0,1]$, such that

$$f_n|_{W_n} = 1, \quad f_n|_{X \setminus E^\circ} = 0; \qquad g_n|_{\overline{E}} = 1, \quad g_n|_{X \setminus V_n} = 0.$$

(If $E^\circ = \varnothing$, we put $W_n = \varnothing$ and $f_n = 0$; furthermore, we drop the condition $g_n|_{X \setminus V_n} = 0$ if $X = V_n$.) Therefore,

$$1_{W_n} \leq f_n \leq 1_E \leq 1_{\overline{E}} \leq g_n \leq 1_{V_n}. \tag{4}$$

Integrating with respect to μ we get

$$\mu(W_n) \leq \int_X f_n \, d\mu \leq \mu(E) \leq \int_X g_n \, d\mu \leq \mu(V_n).$$

If $n \to \infty$, then $\mu(W_n) \to \mu(E^\circ)$ and $\mu(V_n) \to \mu(\overline{E})$, and since by assumption $\mu(\partial E) = 0$, we have $\mu(E) = \mu(E^\circ) = \mu(\overline{E})$, thus

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu = \mu(E). \tag{5}$$

Using (1) and (4), we can also write

$$
\begin{aligned}
\int_X f_n \, d\mu &= \lim_\alpha \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f_n \circ \varphi)(s) \, d\lambda(s) \\
&\leq \liminf_\alpha \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (1_E \circ \varphi)(s) \, d\lambda(s) \\
&\leq \limsup_\alpha \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (1_E \circ \varphi)(s) \, d\lambda(s) \\
&\leq \limsup_\alpha \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (1_{\overline{E}} \circ \varphi)(s) \, d\lambda(s) \\
&\leq \lim_\alpha \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (g_n \circ \varphi)(s) \, d\lambda(s) = \int_X g_n \, d\mu. \tag{6}
\end{aligned}
$$

Combining (5) and (6) implies (3).

Next, we prove the converse statement. Suppose that (3) holds for every Borel set E with $\mu(\partial E) = 0$, and we shall prove that φ is τ_μ-equidistributed along $\{K_\alpha\}_{\alpha \in I}$. In fact, we can prove the stronger statement that

$$\int_X f \, d\mu = \lim_\alpha \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s) \, d\lambda(s) \quad \text{for all } f \in C^b(X). \tag{7}$$

Let $\mathscr{S}$ be the set of all simple functions $h = \sum_{i=1}^n c_i 1_{E_i}$, where $n \in \mathbb{N}$, $c_i \in \mathbb{C}$, and E_i are Borel sets with $\mu(\partial E_i) = 0$.

CLAIM. *Every $f \in C^b(X)$ can be uniformly approximated to arbitrary degree by functions in $\mathscr{S}$.*

We may assume without loss of generality that $f \geq 0$. Let $a > 0$ such that $0 \leq f(x) < a$ for all $x \in X$. For each $r \in [0, a]$, the set $E_r = \{x \in X \colon f(x) \geq r\}$ is a Borel set with $\partial E_r \subset \{x \in X \colon f(x) = r\}$. Since $X = \bigcup_{r \in [0,a]} \{x \in X \colon f(x) = r\}$ and $\mu(X) < \infty$, it follows that for all but possibly countably many $r \in [0, a]$, $\mu(\{x \in X \colon f(x) = r\}) = 0$. Thus, for all but possibly countably many of E_r ($r \in [0, a]$), we have $\mu(\partial E_r) = 0$. Let us now choose $\epsilon > 0$ arbitrary but fixed. It follows from above that there exist numbers $0 = r_0 < r_1 < \cdots < r_n = a$, such that $r_{i+1} - r_i \leq \epsilon$, and $\mu(\partial E_{r_i}) = 0$, for $i = 1, 2, \ldots, n-1$. Now, for each $x \in X$ there exists some integer k, $0 \leq k \leq n-1$, with $r_k \leq f(x) < r_{k+1}$, hence

$$\left| \sum_{i=0}^{n-1} (r_{i+1} - r_i) 1_{E_{r_i}}(x) - f(x) \right| = \left| \sum_{i=0}^{k} (r_{i+1} - r_i) - f(x) \right| = |r_{k+1} - f(x)| \leq \epsilon.$$

This proves the above claim.

Now, for a given $\epsilon > 0$, we choose $h = \sum_{i=1}^{n} c_i 1_{E_i} \in \mathscr{S}$ such that

$$\|h - f\|_{\sup} \leq \epsilon. \tag{8}$$

By our assumption in (3), for $h = \sum_{i=1}^{n} c_i 1_{E_i}$,

$$\int_X h \, d\mu = \sum_{i=1}^{n} c_i \mu(E_i) = \sum_{i=1}^{n} c_i \lim_{\alpha} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (1_{E_i} \circ \varphi)(s) \, d\lambda(s)$$

$$= \lim_{\alpha} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (h \circ \varphi)(s) \, d\lambda(s).$$

Thus we can choose β large enough so that for all $\alpha \geq \beta$,

$$\left| \int_X h \, d\mu - \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (h \circ \varphi)(s) \, d\lambda(s) \right| \leq \epsilon. \tag{9}$$

Now, combining (8) and (9), we find that for all $\alpha \geq \beta$,

$$\left| \int_X f \, d\mu - \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s) \, d\lambda(s) \right|$$

$$\leq \left| \int_X f \, d\mu - \int_X h \, d\mu \right| + \left| \int_X h \, d\mu - \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (h \circ \varphi)(s) \, d\lambda(s) \right|$$

$$+ \frac{1}{\lambda(K_\alpha)} \left| \int_{K_\alpha} ((h - f) \circ \varphi)(s) \, d\lambda(s) \right| \leq \epsilon \mu(X) + \epsilon + \epsilon = (\mu(X) + 2)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, (7) follows. $\qquad\square$

REMARK 2.

(a) For the case of sequences in compact groups the above result is due Eckmann [8, p. 258] (it should be noted, however, that the necessary condition $\mu(\partial M) = 0$ is missing in this reference); our proof has been inspired by the excellent exposition given in [22, Theorem 3.1.2, p. 175]. In the special case that X is a metric space, φ is injective, and $A = C^b(X)$, the theorem follows also from Portemanteau's theorem (cf., Remark 1 (b) and [27, Theorem 6.1, p. 40]).

(b) Theorem 2.2 gives a characterization of $\mathfrak{m}$-equidistribution in the special case that $\mathfrak{m} = \tau_\mu$ is induced by some finite regular Borel measure μ on X. In general, not every element of A^* is induced by a measure on X, and therefore, for such elements of A^*, Theorem 2.2 does not hold. We shall use Theorem 2.2 only in Remark 7 (b).

For the next result we keep the assumptions and notation of Definition 2.1. The support of an element $\mathfrak{m} \in A^*$, denoted by $\mathrm{supp}\,(\mathfrak{m})$, is defined as the set of all $a \in X$ such that for each open neighborhood U of a, there is a function $f \in A$ with $\mathrm{supp}\,(f) \subset U$ and $\langle \mathfrak{m}, f \rangle \neq 0$.

THEOREM 2.3. *If a continuous function $\varphi \colon S \longrightarrow X$ is $\mathfrak{m}$-equidistributed along a monotone compact cover $\{K_\alpha\}_{\alpha \in I}$, then $\mathrm{supp}\,(\mathfrak{m}) \subset \overline{\varphi(S)}$.*

P r o o f. Let $a \in \mathrm{supp}\,(\mathfrak{m})$, U be an open neighborhood of a, and $f \in A$ be such that $\mathrm{supp}\,(f) \subset U$ and $\langle \mathfrak{m}, f \rangle \neq 0$. Since φ is $\mathfrak{m}$-equidistributed along $\{K_\alpha\}_{\alpha \in I}$, we have

$$\lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s)\, d\lambda(s) = \langle \mathfrak{m}, f \rangle.$$

It follows that $\varphi(s) \in U$ for some $s \in S$ (otherwise all the integrals on the left side will be 0). Since the neighborhood U of a is arbitrary, it follows that $a \in \overline{\varphi(S)}$. Hence $\mathrm{supp}\,(\mathfrak{m}) \subset \overline{\varphi(S)}$. $\qquad\square$

REMARK 3. Theorem 2.3 can also be proved using Portemanteau's theorem (see, e.g., Parthasarathy [27, Theorem 6.1, p. 40]). To apply the theorem, let us assume also that X is a metric space and $\mathfrak{m} = \tau_\mu$ for some probability measure μ on X. Now let $V = [\varphi(S)^c]^\circ$. Using the Remark (b) following Definition 2.1, Portemanteau's theorem gives

$$\liminf_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} 1_V \circ \varphi\, d\lambda \geq \mu(V).$$

Since the integrand in the left side is identically zero, it follows that $\mu(V) = 0$. Thus $\mathrm{supp}\,(\mu) \subset \overline{\varphi(S)}$. Now it is easy to check that $\mathrm{supp}\,(\tau_\mu) \subset \mathrm{supp}\,(\mu)$, and

hence the theorem follows for the case that $\mathfrak{m} = \tau_\mu$. We thank the referee for suggesting this alternative proof.

We recall that a *mean* on a unital Banach function algebra A is a positive linear functional $\mathfrak{m} \in A^*$ such that $\langle \mathfrak{m}, 1_X \rangle = \|\mathfrak{m}\| = 1$. The following result shows the existence of equidistributed functions arising from group actions.

THEOREM 2.4. *Let G be a locally compact group acting continuously on a locally compact space X, and let A be a Banach function algebra on X. Suppose that $x_0 \in X$ and $\varphi \colon G \longrightarrow X$ is defined by $\varphi(s) = sx_0$. Then there exists a Bohr net $\{K_\alpha\}_{\alpha \in I}$ on G and a positive linear functional $\mathfrak{m} \in A^*$ such that φ is $\mathfrak{m}$-equidistributed along $\{K_\alpha\}_{\alpha \in I}$. Moreover if A is unital, then $\mathfrak{m}$ is a mean.*

P r o o f. Let $\{F_\beta\}_{\beta \in J}$ be an arbitrary Bohr net on G, and for each $\beta \in J$, the consider the function $\frac{1}{\lambda(F_\beta)} 1_{F_\beta} \in L^1(G)$ (where λ is a left Haar measure on G). Using the natural linear isometric embedding $f \mapsto \widehat{f}$, of $L^1(G)$ into its second dual $L^\infty(G)^*$ ([3, p. 89]), consider $\left\{ \frac{1}{\lambda(F_\beta)} \widehat{1}_{F_\beta} \right\}_{\beta \in J}$ as a net in the closed unit ball of $L^\infty(G)^*$. By Alaoglu's theorem ([3, Theorem V.3.1, p. 130]), the closed unit ball of $L^\infty(G)^*$ is compact in the w^*-topology $\sigma(L^\infty(G)^*, L^\infty(G))$, and hence the net $\left\{ \frac{1}{\lambda(F_\beta)} \widehat{1}_{F_\beta} \right\}_{\beta \in J}$ has a w^*-cluster point $\mathfrak{n} \in L^\infty(G)^*$ ([33, Theorem 17.4, p. 118]). Therefore, there exists a (monotone) subnet $\{F_{\beta_\alpha}\}_{\alpha \in I}$ of $\{F_\beta\}_{\beta \in J}$, such that $\mathfrak{n} = w^*\text{-}\lim_\alpha \frac{1}{\lambda(F_{\beta_\alpha})} \widehat{1}_{F_{\beta_\alpha}}$ ([33, Definition 11.2, p. 73, Theorem 11.5, p.;75]). Letting $K_\alpha = F_{\beta_\alpha}$ ($\alpha \in I$), we find from the definition of w^*-convergence and the inclusion $C^b(G) \subset L^\infty(G)$ that

$$\langle \mathfrak{n}, g \rangle = \lim_{\alpha \in I} \left\langle \frac{1}{\lambda(K_\alpha)} \widehat{1}_{K_\alpha}, g \right\rangle = \lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} g(s) \, d\lambda(s) \quad \left(g \in C^b(G) \right). \quad (10)$$

Let $\mathfrak{m} = \varphi_* \mathfrak{n} \in A^*$ be defined by

$$\langle \mathfrak{m}, f \rangle = \langle \varphi_* \mathfrak{n}, f \rangle = \langle \mathfrak{n}, f \circ \varphi \rangle = \lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s) \, d\lambda(s) \quad (f \in A). \quad (11)$$

It is easy to check that $\mathfrak{m}$ is a positive linear functional on A and $\|\mathfrak{m}\| \leq 1$. The fact that φ is $\mathfrak{m}$-equidistributed along $\{K_\alpha\}_{\alpha \in I}$ follows from (11). If A is unital, then $\langle \mathfrak{m}, 1_X \rangle = \langle \mathfrak{n}, 1_X \circ \varphi \rangle = \langle \mathfrak{n}, 1_G \rangle = 1$, and hence $\mathfrak{m}$ is a mean. $\square$

REMARK 4.

(a) In the above theorem, if G is amenable ([26, p. 5]) and A is unital and invariant under left translations by elements of G, then a simple modification of the proof shows that $\{K_\alpha\}_{\alpha \in I}$ can be chosen to be a Følner net on G and $\mathfrak{m} \in A^*$ to be a left invariant mean.

(b) The proof of the above theorem simplifies greatly when G is compact. In that case, $AP(G) = C(G)$, and the almost periodic mean M coincides with the normalized left Haar measure λ on G. Thus, if we define $\mathfrak{m} = \varphi_*\lambda \in A^*$ by $\langle \mathfrak{m}, f \rangle = \langle \lambda, f \circ \varphi \rangle$, then it follows directly from the condition (iii) of a Bohr net, that the function φ is $\mathfrak{m}$-equidistributed along any Bohr net of G.

For the remainder of this paper, we shall be mainly interested in equidistribution of functions with respect to almost periodic means.

DEFINITION 2.5. Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha \in I}$. Let H be a topological group and M the invariant mean on $AP(H)$. A continuous mapping $\varphi \colon S \longrightarrow H$ is called *almost periodic equidistributed* (*a.p.-equidistributed*) along $\{K_\alpha\}_{\alpha \in I}$ if it is M-equidistributed along $\{K_\alpha\}_{\alpha \in I}$.

REMARK 5.

(a) For a related concept, without the use of monotone covers, see Helmberg [17, p. 78]. We remark that there are at least two reasons for preferring the study of a.p.-equidistribution to a study of equidistribution via the Bohr compactification. First, the structure of a Bohr compactification (even for abelian groups such as $\mathbb{Q}$ or $\mathbb{R}$) is often more complex than the structure of the original group. Second, our results on a.p.-equidistributions (in their full generality) are new even for functions with values in compact topological groups, and as it turns out, it is simpler and more direct to use almost periodic means than to transfer the functions to Bohr compactifications.

(b) Uniform distribution of sequences in locally compact groups with respect to the almost periodic mean, and with respect to Godement's mean on positive definite functions [14, p. 59], was introduced in Veech [31, p. 777]. A sequence $\{r_n\}_{n=1}^\infty$ in $\mathbb{N}$ is called *uniformly distributed sequence generators* if for every sequence $\{x_n\}_{n=1}^\infty$ which generates a dense subgroup of a locally compact group G, the sequence $y_n := x_{r_1} x_{r_2} \cdots x_{r_n}$, is uniformly distributed in G in the sense that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(y_n) = M(f)$, for every continuous almost periodic or positive definite function f on G. Here M denotes the almost periodic mean if $f \in AP(G)$ and it denotes the Godement's mean if $f \in PD(G)$ (continuous positive definite functions). Veech showed the existence and gave examples of such sequences in [30, 31] (see also [12, 13]), and obtained a Weyl's criterion for this notion of uniform distribution [31, Proposition 1.1.2]. For sequences, uniform distribution defined by Veech implies a.p.-equidistribution in the sense of Definition 2.5. In fact, since $AP(G) \cup PD(G)$ is a subset of the space of *weakly almost periodic* functions $WAP(G)$, Veech's definition can be alternatively formulated with

respect to a mean on $WAP(G)$ (for a related open question, see [31, p. 778]). In view of our results in Section 3, it will be interesting to see if Veech's results can be extended from sequences to functions defined on the lattice $\mathbb{N}^m$. We hope to return to this question in another paper.

(c) For other variations and generalizations of the notion of uniform distribution studied by Niven, Rubel, Hartman, Berg-Rajagopolan-Rubel, Benzinger, and Rindler we refer to the bibliography in [31].

For a topological group H, let $\mathscr{C}_H$ be the family of all (equivalence classes) of finite-dimensional, continuous, irreducible unitary representations of H. The trivial 1-dimensional representation is denoted by 1_H. For $\sigma \in \mathscr{C}_H$, $\mathcal{H}_\sigma$ is the representation space, d_σ is the dimension, and $\sigma_{ij}(s) = \big(\sigma(s)e_j|e_i\big)$ $(1 \leq i, j \leq d_\sigma)$ are the coefficient functions of σ with respect to an orthonormal basis $\{e_i\}_{i=1}^{d_\sigma}$ of $\mathcal{H}_\sigma$.

THEOREM 2.6 (Weyl's criterion). *Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha \in I}$. Let H be a topological group. Then a continuous mapping $\varphi \colon S \longrightarrow H$ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha \in I}$ if and only if*

$$\lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (\sigma_{ij} \circ \varphi)(s) \, d\lambda(s) = 0, \tag{12}$$

for all nontrivial $\sigma \in \mathscr{C}_H$, $1 \leq i, j \leq d_\sigma$.

P r o o f. First, suppose that φ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha \in I}$, and let M be the almost periodic mean on $AP(H)$. Since $\sigma_{ij} \in AP(H)$ $(1 \leq i, j \leq d_\sigma)$, by (1) we can write

$$\lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (\sigma_{ij} \circ \varphi)(s) \, d\lambda(s) = \langle M, \sigma_{ij} \rangle. \tag{13}$$

Consider the inner product on $AP(H)$ defined by $(f|g) = M(f\overline{g})$, $f, g \in AP(H)$. By assumption $\sigma \not\cong 1_H$, and therefore we can use the orthogonality relation between coefficient functions of representations in $\mathscr{C}_H$ ([25, Theorem 24, p. 471], [7, (16.3.4), p. 301]) to write

$$\langle M, \sigma_{ij} \rangle = \langle M, \sigma_{ij} 1_H \rangle = (\sigma_{ij} | 1_H) = 0. \tag{14}$$

The identities (13) and (14) imply (12).

Conversely, suppose that (12) holds. We need to show that

$$\lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s) \, d\lambda(s) = \langle M, f \rangle \qquad \text{for all } f \in AP(H). \tag{15}$$

Since M is a mean on $AP(H)$, we have $\langle M, 1_H \rangle = 1$, and hence both sides of (15) are equal to 1 for $f = 1_H$. If $f = \sigma_{ij}$, $\sigma \in \mathscr{C}_H$, $\sigma \not\cong 1_H$, $1 \leq i, j \leq d_\sigma$,

then (15) holds by (12) and (14). Thus (15) holds for all linear combinations of coefficient functions of representations in $\mathscr{C}_H$. Since such functions are uniformly dense in $AP(H)$ ([25, Theorem 29, p. 475]), (15) follows by a routine 3ϵ-argument. □

REMARK 6.

(a) Weyl's equidistribution criterion ([32, Satz 1, p. 315]) was originally stated for sequences in the circle group $\mathbb{T}$. For sequences in compact topological groups the result is due to Eckmann [8, Satz 7, p. 259].

(b) In Section 3 we will find it more convenient to use the following equivalent statement of the Weyl's criterion (12), formulated in terms of the Hilbert-
-Schmidt norms:
$$\lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \left\| \int_{K_\alpha} (\sigma \circ \varphi)(s)\, d\lambda(s) \right\|_2 = 0 \qquad (\sigma \in \mathscr{C}_H, \sigma \not\cong 1_H). \tag{16}$$

The following two corollaries of Weyl's criterion deal with equidistribution of group homomorphisms along Bohr nets.

COROLLARY 2.7. *Let G be a locally compact group equipped with a left Haar measure λ, and a Bohr net $\{K_\alpha\}_{\alpha \in I}$. Let H be a topological group, M_H the invariant mean on $AP(H)$, and M_G the invariant mean on $AP(G)$. A continuous homomorphism $\varphi \colon G \longrightarrow H$ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha \in I}$ if and only if for all $f \in AP(H)$,*
$$\langle M_H, f \rangle = \langle M_G, f \circ \varphi \rangle. \tag{17}$$

P r o o f. Suppose that φ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha \in I}$. Since M_G and M_H are both means, the identity in (17) certainly holds for $f = 1_H$. If $f = \sigma_{ij}$, where $\sigma \in \mathscr{C}_H$, $\sigma \not\cong 1_H$, then we have seen in (14) that $\langle M_H, \sigma_{ij} \rangle = 0$. In addition, by Weyl's criterion (12), the fact that $\sigma_{ij} \circ \varphi \in AP(G)$, and the assumption that $\{K_\alpha\}_{\alpha \in I}$ is a Bohr net, we have
$$\langle M_G, \sigma_{ij} \circ \varphi \rangle = \lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (\sigma_{ij} \circ \varphi)(s)\, d\lambda(s) = 0.$$

Hence $\langle M_H, \sigma_{ij} \rangle = \langle M_G, \sigma_{ij} \circ \varphi \rangle$. Thus (17) holds for all linear combinations of coefficient functions of representations in $\mathscr{C}_H$. Since such functions are uniformly dense in $AP(H)$, (17) follows by passage to a limit.

Conversely, suppose that (17) holds. Then for every $f \in AP(H)$, $f \circ \varphi \in AP(G)$, and the assumption that $\{K_\alpha\}_{\alpha \in I}$ is a Bohr net implies that
$$\lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (f \circ \varphi)(s)\, d\lambda(s) = \langle M_G, f \circ \varphi \rangle = \langle M_H, f \rangle,$$

which shows that $\varphi \colon G \longrightarrow H$ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha \in I}$. □

COROLLARY 2.8. *Let G be a locally compact group equipped with a left Haar measure, and a Bohr net $\{K_\alpha\}_{\alpha \in I}$. Let H be a topological group. Then every continuous homomorphism $\varphi \colon G \longrightarrow H$ such that $\overline{\varphi(G)} = H$ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha \in I}$.*

Proof. Let M_G and M_H be the invariant means on $AP(G)$ and $AP(H)$, respectively. Consider the push-forward mean $\varphi_* M_G$ on $AP(H)$, defined by

$$\langle \varphi_* M_G, f \rangle = \langle M_G, f \circ \varphi \rangle \qquad \big(f \in AP(H) \big).$$

Using the density of $\varphi(G)$ in H, we can easily verify that $\varphi_* M_G$ is invariant under translations by elements from H. By the uniqueness of the invariant mean on $AP(H)$, it follows that $M_H = \varphi_* M_G$. This shows that (17) holds and hence the result follows from Corollary 2.7. $\square$

REMARK 7.

(a) In Helmberg [17, p. 78], a condition analogous to (17) is given as part of an abstract definition of equidistribution.

(b) Theorem 2.2 and Corollary 2.8 extend Theorem 26.17 in Hewitt–Ross [19, p. 432], by removing the assumptions that G and H are abelian, H is compact, and G is σ-compact.

EXAMPLE. Consider the amenable group $G = (0, \infty) \rtimes \mathbb{R}$ formed by the semidirect product of the multiplicative group $(0, \infty)$ and the additive group $\mathbb{R}$. This group has the multiplication $(x', y')(x, y) = (x'x, y' + x'y)$, and a left Haar measure defined by $\lambda = dx\, dy / x^2$ ([11, p. 41]). A Bohr sequence $\{K_n\}_{n=1}^\infty$ for G is given by the trapezoidal regions K_n defined by

$$K_n = \{(x, y) \in \mathbb{R}^2 \colon 1/n \le x \le n, \ -nx \le y \le nx\}.$$

A direct computation shows that

$$\lambda(K_n) = 4n \log n.$$

Let

$$q \colon G \longrightarrow G/\mathbb{R} \cong (0, \infty),$$

be the canonical quotient homomorphism, and $\psi \colon (0, \infty) \longrightarrow (0, \infty)$ by any continuous homomorphism with dense range. Then $\varphi = \psi \circ q \colon G \longrightarrow (0, \infty)$ is a continuous homomorphism with dense range in $(0, \infty)$, and therefore by Corollary 2.8, φ is a.p.-equidistributed along $\{K_n\}_{n=1}^\infty$. As a result, the equidistribution criterion (1) allows us to obtain various formulas for the invariant mean M on $AP((0, \infty))$. For example, let $\psi(x) = x^\alpha$, where $\alpha > 0$ is fixed. Then $\varphi(x, y) = (\psi \circ q)(x, y) = x^\alpha$, and hence by (1), for every $f \in AP((0, \infty))$

we have

$$\langle M, f \rangle = \lim_{n \to \infty} \frac{1}{\lambda(K_n)} \int_{K_n} (f \circ \varphi)(x, y) \, d\lambda = \lim_{n \to \infty} \frac{1}{4n \log n} \int_{1/n}^{n} \int_{-nx}^{nx} f(x^\alpha) \frac{dydx}{x^2}$$

$$= \lim_{n \to \infty} \frac{1}{2 \log n} \int_{1/n}^{n} \frac{f(x^\alpha)}{x} \, dx.$$

3. A multi-dimensional van der Corput's inequality for vectors in Hilbert spaces and its applications

In this section we study equidistribution of functions defined on $\mathbb{N}^m$ ($m \geq 1$). We shall assume that $\mathbb{N}^m$ is equipped with its discrete topology and the counting measure. To facilitate working with multi-indices, we shall use bold-face letters to denote m-tuple of numbers, such as $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$. If $\mathbf{n}, \mathbf{q}, \mathbf{r} \in \mathbb{N}^m$, we define

$$\mathbf{nq} + \mathbf{r} = (n_1 q_1 + r_1, \ldots, n_m q_m + r_m).$$

The lattice $\mathbb{N}^m$ is ordered by the relation $\mathbf{n} \geq \mathbf{n}'$ if $n_1 \geq n_1', \ldots, n_m \geq n_m'$. Under this relation, $\mathbb{N}^m$ is a directed set, and hence limits can be defined unambiguously for functions f defined on $\mathbb{N}^m$; such a limit will be denoted by $\lim_{\mathbf{n} \to \infty} f(\mathbf{n})$. For $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$, we define

$$\mathbf{n}^* = (n_2, \ldots, n_m) \in \mathbb{N}^{m-1}.$$

If $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$, we define

$$K_{\mathbf{N}} = \mathbb{N}_{N_1} \times \cdots \times \mathbb{N}_{N_m}, \qquad K_{\mathbf{N}^*} = \mathbb{N}_{N_2} \times \cdots \times \mathbb{N}_{N_m}, \tag{18}$$

where $\mathbb{N}_N = \{1, 2, \ldots, N\}$. Throughout this section, unless otherwise noted, we shall assume $\mathbb{N}^m$ is equipped with the monotone cover $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$.

LEMMA 3.1. *If $u_1, \ldots, u_n$ are vectors in a Hilbert space $\mathcal{H}$, then*

$$\left\| \sum_{k=1}^{n} u_k \right\|^2 \leq n \sum_{k=1}^{n} \|u_k\|^2.$$

P r o o f. Let $\{e_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. Let $u_{k,i} = (u_k | e_i)$. Using Parseval's identity and the Cauchy-Schwarz inequality,

$$\left\| \sum_{k=1}^{n} u_k \right\|^2 = \sum_{i \in I} | \, sum_{k=1}^{n} u_{k,i}|^2 \leq n \sum_{i \in I} \sum_{k=1}^{n} |u_{k,i}|^2 = n \sum_{k=1}^{n} \|u_k\|^2. \qquad \square$$

Our first theorem in this section is an analogue of van der Corput's inequality [29, p. 407] for vectors in Hilbert spaces. The original inequality deals with complex-numbers and is formulated over multidimensional lattice points of integers. For our purposes, we shall need a version which is formulated for the lattice points in $\mathbb{N}^m$, and is general enough to apply to vectors in Hilbert spaces (more specifically, to Hilbert-Schmidt operators). A version of such an inequality for the case that $m = 1$ is proved in Hlawka [20, Satz 5, p. 42]. For a recent study of van der Corput's inequality for sequences in Hilbert spaces based on Furstenberg correspondence principle, see [9]. We remark that the elementary proof given below is based on classical lines and should be of some independent interest.

THEOREM 3.2. *Let* $m \geq 1$, $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$, *and* $\{u_{\mathbf{n}}\}_{\mathbf{n} \in K_{\mathbf{N}}}$ *be a family of vectors in a Hilbert space* $\mathcal{H}$*. For each* $0 \leq h \leq N_1 - 1$*, let*

$$v_h = \sum_{j=1}^{N_1 - h} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \left(u_{j\mathbf{n}^*} | u_{(j+h)\mathbf{n}^*} \right),$$

so in particular,

$$v_0 = \sum_{\mathbf{n} \in K_{\mathbf{N}}} \|u_{\mathbf{n}}\|^2.$$

Then for every $1 \leq H \leq N_1$,

$$\frac{H^2}{N_2 \cdots N_m} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} \right\|^2 \leq H(N_1 + H - 1)v_0 +$$
$$2(N_1 + H - 1) \sum_{h=1}^{H-1} (H - h)\Re(v_h), \quad (19)$$

where $\Re$ *denotes the real part.*

REMARK 8. (a) When $H = 1$, the above inequality reduces

$$\left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} \right\|^2 \leq (N_1 \cdots N_m) \sum_{\mathbf{n} \in K_{\mathbf{N}}} \|u_{\mathbf{n}}\|^2,$$

which follows directly from Lemma 3.1.

(b) The choice of $1 \leq H \leq N_1$ simplifies the notation. The lemma still holds with obvious notational modification if we choose $1 \leq H \leq N_\ell$, for some $\ell \in \{1, \ldots, m\}$.

P r o o f. We extend the family $u_{\mathbf{n}}$ by putting $u_{\mathbf{n}} = 0$, whenever $n_1 \leq 0$ or $n_1 > N_1$. Starting from the obvious identity

$$\frac{H}{N_2 \cdots N_m} \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} = \frac{1}{N_2 \cdots N_m} \sum_{i=1}^{N_1} \sum_{h=0}^{H-1} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} u_{i\mathbf{n}^*},$$

if we put $p = i + h$, then $1 \leq p \leq N_1 + H - 1$, and hence

$$\frac{H}{N_2 \cdots N_m} \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} = \frac{1}{N_2 \cdots N_m} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{h=0}^{H-1} \sum_{p=1}^{N_1+H-1} u_{(p-h)\mathbf{n}^*}.$$

An application of Lemma 3.1 leads to

$$\frac{H^2}{N_2^2 \cdots N_m^2} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} \right\|^2 = \frac{1}{N_2^2 \cdots N_m^2} \left\| \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{h=0}^{H-1} \sum_{p=1}^{N_1+H-1} u_{(p-h)\mathbf{n}^*} \right\|^2$$

$$\leq \frac{N_1 + H - 1}{N_2 \cdots N_m} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{p=1}^{N_1+H-1} \left\| \sum_{h=0}^{H-1} u_{(p-h)\mathbf{n}^*} \right\|^2.$$

Using the identity $\|u\|^2 = (u|u)$, we obtain from the last inequality:

$$\frac{H^2}{N_2^2 \cdots N_m^2} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} \right\|^2 \leq$$

$$\frac{N_1 + H - 1}{N_2 \cdots N_m} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{p=1}^{N_1+H-1} \sum_{r,s=0}^{H-1} \left(u_{(p-r)\mathbf{n}^*} | u_{(p-s)\mathbf{n}^*} \right). \qquad (20)$$

A simple counting shows that for each fixed $1 \leq i \leq N_1$ and each fixed $\mathbf{n}^* \in K_{\mathbf{N}^*}$, the number of terms $(u_{i\mathbf{n}^*} | u_{i\mathbf{n}^*})$ occurring in the sum

$$\sum_{p=1}^{N_1+H-1} \sum_{r,s=0}^{H-1} (u_{(p-r)\mathbf{n}^*} | u_{(p-s)\mathbf{n}^*}) \quad \text{is equal to } H.$$

So, using (20), we arrive at

$$\frac{H^2}{N_2^2 \cdots N_m^2} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} \right\|^2 \leq \frac{H(N_1 + H - 1)}{N_2 \cdots N_m} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{i=1}^{N_1} \| u_{i\mathbf{n}^*} \|^2$$

$$+ \frac{N_1 + H - 1}{N_2 \cdots N_m} \sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{p=1}^{N_1 + H - 1} \sum_{\substack{r,s=0 \\ r \neq s}}^{H-1} (u_{(p-r)\mathbf{n}^*} | u_{(p-s)\mathbf{n}^*})$$

$$\leq \frac{H(N_1 + H - 1)}{N_2 \cdots N_m} v_0$$

$$+ \frac{2(N_1 + H - 1)}{N_2 \cdots N_m} \Re \left[\sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{p=1}^{N_1 + H - 1} \sum_{\substack{r,s=0 \\ s<r}}^{H-1} (u_{(p-r)\mathbf{n}^*} | u_{(p-s)n^*}) \right]. \tag{21}$$

For a fixed $\mathbf{n}^* \in K_{\mathbf{N}^*}$, a fixed h with $1 \leq h \leq H - 1$, and a fixed j with $1 \leq j \leq N_1 - h$, the number of terms $(u_{j\mathbf{n}^*} | u_{(j+h)\mathbf{n}^*})$ occurring in the sum

$$\sum_{p=1}^{N_1 + H - 1} \sum_{\substack{r,s=0 \\ s<r}}^{H-1} (u_{(p-r)\mathbf{n}^*} | u_{(p-s)\mathbf{n}^*})$$

is equal to $H - h$. Consequently, we have

$$\sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{p=1}^{N_1 + H - 1} \sum_{\substack{r,s=0 \\ s<r}}^{H-1} (u_{(p-r)\mathbf{n}^*} | u_{(p-s)\mathbf{n}^*}) =$$

$$\sum_{\mathbf{n}^* \in K_{\mathbf{N}^*}} \sum_{h=1}^{H-1} (H - h) \sum_{j=1}^{N_1 - h} (u_{j\mathbf{n}^*} | u_{(j+h)\mathbf{n}^*}) = \sum_{h=1}^{H-1} (H - h) v_h. \tag{22}$$

Substituting (22) into (21) yields

$$\frac{H^2}{N_2^2 \cdots N_m^2} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} u_{\mathbf{n}} \right\|^2 \leq \frac{H(N_1 + H - 1)}{N_2 \cdots N_m} v_0 +$$

$$\frac{2(N_1 + H - 1)}{N_2 \cdots N_m} \sum_{h=1}^{H-1} (H - h) \Re(v_h). \tag{23}$$

Multiplying both sides of (23) by $N_2 \cdots N_m$ yields (19), completing the proof. $\square$

LEMMA 3.3. *Let H be a topological group, $m \geq 1$, and $\varphi \colon \mathbb{N}^m \longrightarrow H$ a function. If φ is a.p.-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$, then for every $\mathbf{r} \in \mathbb{N}^m$, $R_{\mathbf{r}}\varphi$ is a.p.-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$, where $R_{\mathbf{r}}\varphi(\mathbf{n}) = \varphi(\mathbf{n} + \mathbf{r})$, $(\mathbf{n} \in \mathbb{N}^m)$.*

P r o o f. For $\mathbf{N}, \mathbf{r} \in \mathbb{N}^m$, and $1 \leq j \leq m$, let

$$K_{\mathbf{N}+\mathbf{r}}^j = \left\{ \mathbf{n} \in K_{\mathbf{N}+\mathbf{r}} \colon 1 \leq n_i \leq r_i \text{ for exactly } j \text{ choices of } i \in \{1, 2, \ldots, m\} \right\}.$$

Then, by triangle inequality, for every nontrivial $\sigma \in \mathscr{C}_H$,

$$\frac{1}{N_1 \cdots N_m} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} (\sigma \circ \varphi)(\mathbf{n} + \mathbf{r}) \right\|_2 \leq \frac{1}{(N_1 \cdots N_m)} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}+\mathbf{r}}} (\sigma \circ \varphi)(\mathbf{n}) \right\|_2 +$$
$$\frac{1}{N_1 \cdots N_m} \sum_{j=1}^m \sum_{\mathbf{n} \in K_{\mathbf{N}+\mathbf{r}}^j} \|(\sigma \circ \varphi)(\mathbf{n})\|_2. \quad (24)$$

The first term on the right side of (24) approaches to 0 as $\mathbf{N} \to \infty$, by an application of the Weyl's criterion (16) to φ. The remaining terms are easily shown to tend to 0 by replacing $\|(\sigma \circ \varphi)(\mathbf{n})\|_2$ with its value $\sqrt{d_\sigma}$. $\qquad \square$

Next we come to the main result of this section which extends Hlawka's equidistribution result to mappings from $\mathbb{N}^m$ to arbitrary topological groups.

THEOREM 3.4. *Let H be a topological group,*

$$m \geq 1, \quad and \quad \varphi \colon \mathbb{N}^m \longrightarrow H$$

a function. Suppose that there exists some $1 \leq \ell \leq m$, such that for every $h \in \mathbb{N}$, the function

$$\mathbb{N}^m \longrightarrow H, \quad \mathbf{n} \mapsto \varphi(n_1, \ldots, n_\ell + h, \ldots, n_m) \varphi(n_1, \ldots, n_\ell, \ldots, n_m)^{-1}$$

is a.p.-equidistributed along the monotone cover $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$. Then for all $\mathbf{q} \in \mathbb{N}^m$, and all $\mathbf{r} \in \mathbb{Z}^m$ with $r_j \geq 0$ $(1 \leq j \leq m)$, the function

$$\psi \colon \mathbb{N}^m \longrightarrow H, \quad \mathbf{n} \mapsto \varphi(\mathbf{n}\mathbf{q} + \mathbf{r}) \quad (25)$$

is a.p.-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$.

REMARK 9. The case $m = 1$ and H compact is proved by Hlawka [20, Satz 6, p. 42]. The case that $m \geq 1$, $H = \mathbb{T}$, $q_j = 1$, $r_j = 0$ $(j = 1, \ldots, m)$, follows from van der Corput [29, §4, Satz 1, p. 411]. For other generalizations of van der Corput's result see Bergelson and Moreira [1], Cigler [2], and Tao [28, Proposition 1.3.6].

Proof. To simplify the notation, we shall assume without loss of generality that $\ell = 1$, so that by assumption, for every $h \in \mathbb{N}$, the function

$$\mathbb{N}^m \longrightarrow H, \qquad \mathbf{n} \mapsto \varphi(n_1 + h, \mathbf{n}^*)\varphi(\mathbf{n})^{-1} \tag{26}$$

is a.p.-equidistributed along $\{K_\mathbf{N}\}_{\mathbf{N} \in \mathbb{N}^m}$.

Let $\sigma \in \mathscr{C}_H$ be nontrivial. We will show that

$$\lim_{\mathbf{N} \to \infty} \frac{1}{N_1 \cdots N_m} \left\| \sum_{\mathbf{n} \in K_\mathbf{N}} (\sigma \circ \varphi)(\mathbf{n}\mathbf{q} + \mathbf{r}) \right\|_2 = 0. \tag{27}$$

This is equivalent to the Weyl's criterion (16) for the function ψ in (25), and hence implies the a.p.-equidistribution of ψ.

If $k \geq 0$ is any integer, then by using the trigonometric identity

$$\frac{1}{q} \sum_{j=1}^{q} \exp(2\pi i j k / q) = \begin{cases} 1 & k \equiv 0 \mod q, \\ 0 & k \not\equiv 0 \mod q \end{cases} \tag{28}$$

(where $i = \sqrt{-1}$), and letting $\mathbf{q}\mathbf{N} = (q_1 N_1, \ldots, q_m N_m)$, we can write

$$\sum_{\mathbf{n} \in K_\mathbf{N}} (\sigma \circ \varphi)(\mathbf{n}\mathbf{q} + \mathbf{r}) =$$
$$\frac{1}{q_1 \cdots q_m} \sum_{\mathbf{k} \in K_{\mathbf{q}\mathbf{N}}} \sum_{\mathbf{j} \in K_\mathbf{q}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s)(\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}). \tag{29}$$

Changing the order of summations on the right side of (29) and applying the triangle inequality yields

$$\left\| \sum_{\mathbf{n} \in K_\mathbf{N}} (\sigma \circ \varphi)(\mathbf{n}\mathbf{q} + \mathbf{r}) \right\|_2 \leq$$
$$\frac{1}{q_1 \cdots q_m} \sum_{\mathbf{j} \in K_\mathbf{q}} \left\| \sum_{\mathbf{k} \in K_{\mathbf{q}\mathbf{N}}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s)(\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}) \right\|_2 \leq$$
$$\max_{\mathbf{j} \in K_\mathbf{q}} \left\| \sum_{\mathbf{k} \in K_{\mathbf{q}\mathbf{N}}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s)(\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}) \right\|_2. \tag{30}$$

Using Theorem 3.2, for each $\mathbf{j} \in K_\mathbf{q}$ and $1 \leq H \leq q_1 N_1$:

$$\frac{H^2}{(q_2 N_2) \cdots (q_m N_m)} \left\| \sum_{\mathbf{k} \in K_{\mathbf{q}\mathbf{N}}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s) (\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}) \right\|_2^2 \leq$$
$$H(q_1 N_1 + H - 1)v_0 + 2(q_1 N_1 + H - 1) \sum_{h=1}^{H-1} (H - h)|v_h|, \tag{31}$$

where, for $0 \leq h \leq H - 1$,

$$v_h = \sum_{p_1=1}^{q_1 N_1 - h} \sum_{\mathbf{k}^* \in K_{\mathbf{q}^* \mathbf{N}^*}} \left((\sigma \circ \varphi)(p_1 + r_1, \mathbf{k}^* + \mathbf{r}^*) | (\sigma \circ \varphi)(p_1 + r_1 + h, \mathbf{k}^* + \mathbf{r}^*) \right). \quad (32)$$

Since σ is a unitary representation, $\|\sigma(x)\|_2^2 = d_\sigma$ $(x \in H)$, and hence

$$v_0 = \sum_{\mathbf{k} \in K_{\mathbf{q} \mathbf{N}}} \|(\sigma \circ \varphi)(\mathbf{k} + \mathbf{r})\|_2^2 = (q_1 N_1) \cdots (q_m N_m) d_\sigma. \quad (33)$$

Moreover, using (32), the fact that $\|I\|_2 = \sqrt{d_\sigma}$, and the Cauchy–Schwarz inequality

$$|v_h| = \left| \sum_{p_1=1}^{q_1 N_1 - h} \sum_{\mathbf{k}^* \in K_{\mathbf{q}^* \mathbf{N}^*}} \left(I | \sigma \left(\varphi(p_1 + r_1 + h, \mathbf{k}^* + \mathbf{r}^*) \, \varphi(p_1 + r_1, \mathbf{k}^* + \mathbf{r}^*)^{-1} \right) \right) \right|$$

$$\leq \sqrt{d_\sigma} \left\| \sum_{p_1=1}^{q_1 N_1 - h} \sum_{\mathbf{k}^* \in K_{\mathbf{q}^* \mathbf{N}^*}} \sigma \left(\varphi(p_1 + r_1 + h, \mathbf{k}^* + \mathbf{r}^*) \varphi(p_1 + r_1, \mathbf{k}^* + \mathbf{r}^*)^{-1} \right) \right\|_2$$

$$= \sqrt{d_\sigma} \|T_h\|_2, \quad (34)$$

where, T_h is defined as

$$T_h = \sum_{p_1=1}^{q_1 N_1 - h} \sum_{\mathbf{k}^* \in K_{\mathbf{q}^* \mathbf{N}^*}} \sigma \left(\varphi(p_1 + r_1 + h, \mathbf{k}^* + \mathbf{r}^*) \varphi(p_1 + r_1, \mathbf{k}^* + \mathbf{r}^*)^{-1} \right). \quad (35)$$

Combining (31), (33), and (34) we have

$$\frac{H^2}{(q_2 N_2) \cdots (q_m N_m)} \left\| \sum_{\mathbf{k} \in K_{\mathbf{q} \mathbf{N}}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s) \, (\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}) \right\|_2^2 \leq$$

$$H(q_1 N_1 + H - 1)(q_1 N_1) \cdots (q_m N_m) d_\sigma +$$

$$2(q_1 N_1 + H - 1)\sqrt{d_\sigma} \sum_{h=1}^{H-1} (H - h) \|T_h\|_2. \quad (36)$$

If we multiply both sides of (36) by $\dfrac{q_2 \cdots q_m}{H^2 N_1^2 N_2 \cdots N_m}$, then an easy rearrangement of the coefficients leads to

$$\left\| \frac{1}{N_1 \cdots N_m} \sum_{\mathbf{k} \in K_{\mathbf{q} \mathbf{N}}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s) \, (\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}) \right\|_2^2 \leq A + B v_h', \quad (37)$$

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where

$$A := \frac{(q_1 N_1 + H - 1) q_1 q_2^2 \cdots q_m^2 d_\sigma}{H N_1}, \qquad B := \frac{2(q_1 N_1 + H - 1)\sqrt{d_\sigma}(q_2^2 \cdots q_m^2)}{H^2 N_1},$$

$$v'_h := \sum_{h=1}^{H-1} \frac{(q_1 N_1 - h)(H - h)}{N_1} \left\| \frac{1}{(q_1 N_1 - h)(q_2 N_2) \cdots (q_m N_m)} T_h \right\|_2. \qquad (38)$$

Letting $\mathbf{N} = (N_1, \ldots, N_m) \to \infty$,

$$A \to \frac{q_1^2 \cdots q_m^2 d_\sigma}{H}, \qquad B \to \frac{2 q_1 q_2^2 \cdots q_m^2 \sqrt{d_\sigma}}{H^2}, \qquad (39)$$

and

$$\left\| \frac{1}{(q_1 N_1 - h)(q_2 N_2) \cdots (q_m N_m)} T_h \right\|_2 \to 0, \qquad (40)$$

where the last limit follows from Weyl's criterion (16) applied to the a.p.-equidistributed function in (26), and by using Lemma 3.3.

Now, let $\epsilon > 0$. We choose an integer H large enough so that

$$\frac{q_1^2 \cdots q_m^2 d_\sigma}{H} < \epsilon/3, \qquad \frac{2 q_1 q_2^2 \cdots q_m^2 \sqrt{d_\sigma}}{H^2} < 1. \qquad (41)$$

Next, using (39), (40), and (41) we choose $(N_1, \ldots, N_m)$ large enough, with $q_1 N_1 > H$, so that

$$A < \epsilon/3, \qquad B < 1, \qquad (42)$$

and

$$\left\| \frac{1}{(q_1 N_1 - h)(q_2 N_2) \cdots (q_m N_m)} T_h \right\| \leq \frac{\epsilon}{3 q_1 H(H - 1)}. \qquad (43)$$

Using the estimate

$$\frac{(q_1 N_1 - h)(H - h)}{N_1} \leq q_1 H,$$

we get from (38), that for sufficiently large $(N_1, \ldots, N_m)$:

$$v'_h \leq q_1 H(H - 1) \cdot \frac{\epsilon}{3 q_1 H(H - 1)} = \epsilon/3. \qquad (44)$$

Then it follows from (37), (42), and (44) that

$$\left\| \frac{1}{N_1 \cdots N_m} \sum_{\mathbf{k} \in K_{\mathbf{qN}}} \prod_{s=1}^{m} \exp(2\pi i j_s k_s / q_s) (\sigma \circ \varphi)(\mathbf{k} + \mathbf{r}) \right\|_2^2 \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon. \qquad (45)$$

The inequality holds for all values of

$$1 \leq j_1 \leq q_1, \ldots, 1 \leq j_m \leq q_m.$$

Therefore, we conclude from (30) that for $\mathbf{N} = (N_1, \ldots, N_m)$ sufficiently large, we have

$$\frac{1}{N_1 \cdots N_m} \left\| \sum_{\mathbf{n} \in K_{\mathbf{N}}} (\sigma \circ \varphi)(\mathbf{n}\mathbf{q} + \mathbf{r}) \right\|_2 < \sqrt{\epsilon}. \qquad (46)$$

This proves (27), and with it completes the proof of the theorem. $\qquad \square$

As an application of Theorem 3.4, we state the following theorem. In the special case that $q_j = 1$, $r_j = 0$ ($j = 1, \ldots, m$), the result follows from van der Corput [29, §5, Satz 1, p. 413].

THEOREM 3.5. *Let $m \geq 1$ and $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$ be the monotone cover of $\mathbb{N}^m$ defined in (18). Let $p(x_1, \ldots, x_m)$ be a polynomial in m variables and with real coefficients that contains at least one nonconstant term with an irrational coefficient. Then for all integers $q_j \geq 1$, $r_j \geq 0$ ($j = 1, \ldots, m$), the function*

$$\psi \colon \mathbb{N}^m \longrightarrow \mathbb{T}, \qquad (n_1, \ldots, n_m) \mapsto e^{2\pi i p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)}, \qquad (47)$$

is μ-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$, where μ is the normalized Lebesgue measure on $\mathbb{T}$.

P r o o f. Let

$$a_{k_1 \cdots k_m} x_1^{k_1} \cdots x_m^{k_m}, \qquad k := k_1 + \cdots + k_m \geq 1,$$

be a term in $p(x_1, \ldots, x_m)$ where the coefficient $a_{k_1 \cdots k_m}$ is irrational, but all higher degree terms (if any) have rational coefficients. Without loss of generality we may assume that $k_1 \geq 1$. The proof is by induction on k.

Case $k = 1$. In this case, the polynomial $p(x_1, \ldots, x_m)$ has a term, say, $a_1 x_1$, where a_1 is irrational, but all higher degree terms have rational coefficients. If $p(x_1, \ldots, x_m)$ has degree 1, then

$$p(x_1, \ldots, x_m) = \sum_{j=1}^{m} a_j x_j + a_0 \qquad (a_1 \notin \mathbb{Q}),$$

and hence, for every nonzero integer ℓ,

$$\frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} e^{2\pi i \ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} =$$

$$\frac{e^{2\pi i \ell a_0}}{N_1 \cdots N_m} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} \prod_{j=1}^{m} e^{2\pi i \ell a_j (n_j q_j + r_j)} =$$

$$e^{2\pi i \ell a_0} \prod_{j=1}^{m} \left(\frac{1}{N_j} \sum_{n_j=1}^{N_j} e^{2\pi i \ell a_j (n_j q_j + r_j)} \right). \qquad (48)$$

Since a_1 is irrational, so is $\ell a_1 q_1$, and hence

$$\left| \frac{1}{N_1} \sum_{n_1=1}^{N_1} e^{2\pi i \ell a_1 (n_1 q_1 + r_1)} \right| = \left| \frac{e^{2\pi i \ell a_1 q_1}(1 - e^{2\pi i N_1 \ell a_1 q_1})}{N_1(1 - e^{2\pi i \ell a_1 q_1})} \right| \leq \frac{1}{N_1|\sin \pi \ell a_1 q_1|} \to 0,$$

when $N_1 \to \infty$. Since all the other factors in the right side of (48) are bounded by 1, it follows that the entire expression in (48) converges to 0 when $\mathbf{N} \to \infty$. Since the nonzero $\ell \in \mathbb{Z}$ is arbitrary and since $\widehat{\mathbb{T}} \cong \mathbb{Z}$, it follows from Theorem 2.6 that the function ψ defined in (47) is μ-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}^m}$.

Next, we consider the case that $\deg p \geq 2$, so that we can write

$$p(x_1, \ldots, x_m) = q(x_1, \ldots, x_m) + \sum_{j=1}^{m} a_j x_j + a_0 \qquad (a_1 \notin \mathbb{Q}), \qquad (49)$$

where q consists of all terms of p with degrees greater than 1. Since by assumption, $k = 1$, all coefficients of q are rational. Let $\{x\}$ and $[x]$ denote the fractional and the integral parts of a real number x, respectively. Then if $M \geq 1$ is a common multiple of the denominators of the coefficients of q, we have

$$\{q(Mc_1 + d_1, \ldots, Mc_m + d_m)\} = \{q(d_1, \ldots, d_m)\}, \quad \text{for all } c_j, d_j \in \mathbb{Z}. \quad (50)$$

Now, we note that for all nonzero $\ell \in \mathbb{Z}$, and all integers $q_j \geq 1$, $r_j \geq 0$, we can write

$$\frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} e^{2\pi i \ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} =$$

$$\frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{[N_1/M]M} \cdots \sum_{n_m=1}^{[N_m/M]M} e^{2\pi i \ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} + L, \quad (51)$$

where L is the sum of the remaining terms. However, estimating $|L|$ from above, we have

$$|L| \leq \frac{1}{N_1 \cdots N_m} \sum_{j=1}^{m} \frac{M(N_1 \cdots N_m)}{N_j} = \sum_{j=1}^{m} \frac{M}{N_j} \to 0, \qquad (52)$$

as $\mathbf{N} \to \infty$. So in view of Weyl's criterion, in order to complete the induction for the step $k = 1$, all that remains to show is that the first expression on the right side of (51) tends to 0 as well.

For each $1 \leq j \leq m$, let us write

$$n_j = Mc_j + d_j, \quad \text{where} \quad 0 \leq d_j \leq M - 1.$$

Therefore

$$\sum_{n_1=1}^{[N_1/M]M} \cdots \sum_{n_m=1}^{[N_m/M]M} e^{2\pi i\ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} =$$

$$\sum_{d_1=0}^{M-1} \cdots \sum_{d_m=0}^{M-1} \sum_{c_1=0}^{[N_1/M]} \cdots \sum_{c_m=0}^{[N_m/M]} e^{2\pi i\ell p(Mc_1 q_1 + d_1 q_1 + r_1, \ldots, Mc_m q_m + d_m q_m + r_m)}. \quad (53)$$

However, using (49), and putting

$$s_j := d_j q_j + r_j \quad (j = 1, \ldots, m),$$

we get

$$e^{2\pi i\ell p(Mc_1 q_1 + s_1, \ldots, Mc_m q_m + s_m)} =$$

$$e^{2\pi i\ell(q(Mc_1 q_1 + s_1, \ldots, Mc_m q_m + s_m) + \sum_{j=1}^{m} a_j(Mc_j q_j + s_j) + a_0)}.$$

Removing the integral part of $q(Mc_1 q_1 + s_1, \ldots, Mc_m q_m + s_m)$ and using (50), we obtain

$$e^{2\pi i\ell p(Mc_1 q_1 + s_1, \ldots, Mc_m q_m + s_m)} = e^{2\pi i\ell(\{q(s_1, \ldots, s_m)\} + \sum_{j=1}^{m} a_j s_j + a_0)} e^{2\pi i\ell \sum_{j=1}^{m} a_j Mc_j q_j}$$

$$= e^{2\pi i\ell p(s_1, \ldots, s_m)} \prod_{j=1}^{m} e^{2\pi i\ell a_j Mc_j q_j}. \quad (54)$$

Combining (53) with (54), and replacing back $s_j = d_j q_j + r_j$, we get

$$\frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{[N_1/M]M} \cdots \sum_{n_m=1}^{[N_m/M]M} e^{2\pi i\ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} =$$

$$\left(\prod_{j=1}^{m} \frac{1}{N_j} \sum_{c_j=0}^{[N_j/M]} e^{2\pi i\ell a_j Mc_j q_j} \right) \left(\sum_{d_1=0}^{M-1} \cdots \sum_{d_m=0}^{M-1} e^{2\pi i\ell p(d_1 q_1 + r_1, \ldots, d_m q_m + r_m)} \right). \quad (55)$$

Since a_1 is irrational,

$$\lim_{N_1 \to \infty} \frac{1}{N_1} \sum_{c_1=0}^{[N_1/M]} e^{2\pi i\ell a_1 Mc_1 q_1} = 0,$$

and since the remaining factors in (55) are obviously all bounded,

$$\lim_{\mathbf{N} \to \infty} \frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{[N_1/M]M} \cdots \sum_{n_m=1}^{[N_m/M]M} e^{2\pi i\ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} = 0. \quad (56)$$

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Using (56), (52) and (51) we conclude

$$\lim_{\mathbf{N}\to\infty} \frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} e^{2\pi i \ell p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)} = 0. \qquad (57)$$

Now Theorem 2.6 implies that ψ in (47) is μ-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N}\in\mathbb{N}^m}$ for the case that $k=1$ and $\deg p \geq 2$. This completes the induction step for $k=1$.

To complete the induction, let the theorem hold for $k = 1, 2, \ldots, n-1$. If $k = n$, then for every $h \in \mathbb{N}$,

$$p_h(x_1, \ldots, x_m) := p(x_1 + h, x_2, \ldots, x_m) - p(x_1, \ldots, x_m) \qquad (58)$$

is a polynomial whose highest degree term with an irrational coefficient is of degree $n-1$, and therefore by the induction hypothesis the function

$$\mathbb{N}^m \longrightarrow \mathbb{T}, \qquad (n_1, \ldots, n_m) \mapsto e^{2\pi i p_h(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)},$$

is μ-equidistributed along $\{K_{\mathbf{N}}\}_{\mathbf{N}\in\mathbb{N}^m}$. Now our result for $k = n$ follows from Theorem 3.4, and the induction is complete. $\qquad\square$

REMARK 10. The circle group $\mathbb{T}$ is the simplest example of a nilmanifold. A *nilmanifold* is a quotient space $X = G/\Gamma$, where G is a connected, simply connected nilpotent Lie group and Γ is a discrete, cocompact subgroup of G. Examples include the torus $\mathbb{R}^m/\mathbb{Z}^m$, $m \geq 1$, and the 3-dimensional Heisenberg nilmanifold G/Γ, where

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

A *polynomial sequence* in G is a function $g \colon \mathbb{Z} \longrightarrow G$, $g(n) = a_1^{p_1(n)} \cdots a_m^{p_m(n)}$, where $a_1, \ldots, a_m \in G$ and $p_1, \ldots, p_m$ are polynomials taking on integer values on integers. The behaviour of polynomial orbits in nilmanifolds has been studied by several authors. Leibman [23, Theorem A, p. 202] has shown that if g is a polynomial sequence in G, then for any $x \in X$, $f \in C(X)$ and Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}$, $\lim_{N\to\infty}(1/|\Phi_N|) \sum_{n\in\Phi_N} f(g(n)x)$ exists. Furthermore, if g is a polynomial sequence in G and $x \in X$, there exist closed sub-nilmanifolds $Y_1, \ldots, Y_k$ of X, such that for each $j = 1, \ldots, k$, the sequence $\{g(j + nk)x\}_{n\in\mathbb{Z}}$ is μ_j-equidistributed in Y_j, where μ_j is a suitable probability measure on Y_j (Leibman [23, Theorem B, p. 202]). Leibman's deep results are formulated in the general setting of nilmanifolds. Theorem 3.5, on the other hand, deals with equidistribution in the unit circle for functions defined on the lattice $\mathbb{N}^m$, and the polynomial $p(x_1, \ldots, x_m)$ in (47) need not take integer values on integers.

Studies of polynomial orbits in nilmanifolds have led to a proof of the Möbius and Nilsequence conjecture by Green and Tao [15,16], of importance in combinatorial number theory.

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