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REFINEMENT OF THE THEOREM OF VAHLEN

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ABSTRACT. In 1895, Vahlen proved a theorem concerning a simultaneous approximation of a real number by its two consecutive convergent. In this paper, we will provide a sharper bound for the theorem.

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1. Introduction

Following Vahlen [5] we prove the following theorem.

Theorem 1. *Suppose $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Also suppose $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}$ are two consecutive convergents to a. Then at least one of them satisfies the inequality*

$$\left|a - \frac{p}{q}\right| < \frac{1}{(2 + \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)})q^2} . \tag{1}$$

This result has some history, In 1895 [5] proved (1) with the constant 2 instead of

$$\left(2 + \frac{4(q^3-1)}{q\,(q^4+2q^3+5q^2+q+30)}\right).$$

Later, Hančl [1] improved this constant 2 by $\left(2 + \frac{2(q-1)}{q^2(q+1)}\right)$ on the right hand side. After this Hančl and Bahnerová [2] further improved result in [1] with

$$\left(2 + \frac{4(q-1)}{q(q^2+q+6)}\right).$$

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It is an open problem regarding if we can also do it for the theorem of Dirichlet, See [4] for further reference.

2. Notations

Throughout the entire paper, we denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of non-negative integers and integers, respectively. Let

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$$

be the nth partial fraction of the real number $a = [a_0; a_1, a_2, \ldots]$. we have

$$p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_1 a_0 + 1, \quad q_1 = a_1,$$

$$p_{n+2} = a_{n+2} p_{n+1} + p_n, \quad q_{n+2} = a_{n+2} q_{n+1} + q_n, \quad q_{n+1} p_n - p_{n+1} q_n = (-1)^{n+1},$$

$$\begin{aligned}
a &= [a_0; a_1, a_2, \ldots] \\
&= \big[a_0; a_1, a_2, \ldots, a_n, [a_{n+1}; a_{n+2}, a_{n+3}, \ldots]\big] \\
&= \frac{p_n\,[a_{n+1}; a_{n+2}, a_{n+3}, \ldots] + p_{n-1}}{q_n\,[a_{n+1}; a_{n+2}, a_{n+3}, \ldots] + q_{n-1}}, \\
a - \frac{p_n}{q_n} &= \frac{(-1)^n}{q_n^2\left([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, a_{n-1}, \ldots, a_1]\right)} \\
&= \frac{(-1)^n}{q_n^2\left([a_{n+1}; a_{n+2}, \ldots] + \frac{q_{n-1}}{q_n}\right)}, \\
\left|a - \frac{p_n}{q_n}\right| &= \frac{1}{q_n^2\left([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, a_{n-1}, \ldots, a_1]\right)}. \qquad (2)
\end{aligned}$$

All of this can be found in the book of Schmidt [[4] page 7-10] or see [3]. If $a = [a_0; a_1, a_2, \ldots, a_k]$ is finite and $k \geq 1$, then we suppose that $a_k \neq 1$.

3. Proofs of Lemmas

Before, we begin to prove the theorem, we will prove the following lemmas.

Lemma 1. *Let $q \in \mathbb{N}$, then the following inequality holds.*

$$\frac{4(q^3 - 1)}{q(q^4 + 2q^3 + 5q^2 + q + 30)} \geq \frac{4(q-1)}{q(q^2 + q + 6)}. \qquad (3)$$

P r o o f. We have
$$\frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)} - \frac{4(q-1)}{q(q^2+q+6)}$$
$$= \frac{(4q^3-4)(q^2+q+6)-(4q-4)(q^4+2q^3+5q^2+q+30)}{q(q^4+2q^3+5q^2+q+30)(q^2+q+6)}$$
$$= \frac{12q^3+12q^2-120q+96}{q(q^4+2q^3+5q^2+q+30)(q^2+q+6)}$$
$$= \frac{12(q^3+q^2-10q+8)}{q(q^4+2q^3+5q^2+q+30)(q^2+q+6)}$$
$$= \frac{12(q-1)(q-2)(q+4)}{q(q^4+2q^3+5q^2+q+30)(q^2+q+6)}\,.$$
And (3) follows. □

Lemma 2. *Let $q \in \mathbb{N}$, then the following inequality holds*
$$\frac{1}{6} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)} \geq 0 \tag{4}$$
with equality at $q = 2$.

P r o o f.
$$\frac{1}{6} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)}$$
$$= \frac{q^5+2q^4-19q^3+q^2+30q+24}{6q(q^4+2q^3+5q^2+q+30)}$$
$$= \frac{q^3(q^2+2q-19)+q^2+30q+24}{6q(q^4+2q^3+5q^2+q+30)}\,. \tag{5}$$
If $q \geq 4$, then $(q^2+2q-19) \geq 0$ and (4) follows. We can write (5) as
$$= \frac{(q-2)(q^4+4q^3-11q^2-21q-12)}{6q(q^4+2q^3+5q^2+q+30)}$$
if $q = 2$, then
$$\frac{1}{6} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)} = 0\,.$$
(4) holds. If q=1, then (4) holds. For q=3 we can also write (5) as
$$\frac{(q-2)(q(q^3-21)+4q^2(q-3)+q^2-12)}{6q(q^4+2q^3+5q^2+q+30)}\,. \tag{1*}$$

Then, $(q^3 - 21) > 1$ and $q^2 - 12 = -3$, This with the equation (1*) implies (4), which proves the Lemma. □

Lemma 3. *For all $c \geq 2$ we have*

$$\frac{c^2}{q(q+c)} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)} > 0\,. \tag{6}$$

Proof.

$$\frac{c^2}{q(q+c)} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)}$$

$$= \frac{c^2(q^4+2q^3+5q^2+q+30) - (4q^3-4)(q+c)}{q(q^4+2q^3+5q^2+q+30)(q+c)}$$

$$= \frac{c^2q^4+2c^2q^3+5c^2q^2+c^2q+30c^2-4q^4-4cq^3+4q+4c}{q(q^4+2q^3+5q^2+q+30)(q+c)}$$

$$= \frac{(c^2-4)q^4+(2c^2-4c)q^3+5c^2q^2+(c^2+4)q+(30c^2+4c)}{q(q^4+2q^3+5q^2+q+30)(q+c)}\,.$$

And (6) follows. □

Lemma 4. *For every positive integer q and c with $q > c \geq 2$, we have,*

$$\frac{c^2}{q(q-c)} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)} > 0\,. \tag{7}$$

Proof.

$$\frac{c^2}{q(q-c)} - \frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)}$$

$$= \frac{c^2(q^4+2q^3+5q^2+q+30) - (4q^3-4)(q-c)}{q(q^4+2q^3+5q^2+q+30)(q-c)}$$

$$= \frac{c^2q^4+2c^2q^3+5c^2q^2+c^2q+30c^2-4q^4+4cq^3+4q-4c}{q(q^4+2q^3+5q^2+q+30)(q-c)}$$

$$= \frac{(c^2-4)q^4+(2c^2+4c)q^3+5c^2q^2+(c^2+4)q+(30c^2-4c)}{q(q^4+2q^3+5q^2+q+30)(q-c)}\,.$$

And (7) follows. □

4. Proof of Theorem 1

Proof. If $a = [a_0; a_1, a_2, \ldots, a_k]$ and $n = k$, then $\left|a - \frac{p_n}{q_n}\right| = 0$ and (1) follows. So, we will not consider this case.

Case 1. Suppose $a_n \geq 3$ or that $a_{n+1} \geq 3$. Using assumption (2) and (4) we obtain (1).

Case 2. Now assume that $a_n = a_{n+1} = 2$. This assumption and (2) gives us

$$\begin{aligned}\left|a - \frac{p_n}{q_n}\right| &= \frac{1}{q_n^2\left([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, a_{n-1}, \ldots, a_1]\right)} \\ &= \frac{1}{q_n^2\left([2; a_{n+2}, \ldots] + [0; 2, a_{n-1}, \ldots, a_1]\right)} \\ &\leq \frac{1}{\left(2 + \frac{1}{3}\right) q_n^2}.\end{aligned}$$

This and (4) yield (1).

Case 3. Next, Let, $a_n = 2$ and $a_{n+1} = 1$. This assumption and (2) imply that

$$\begin{aligned}\left|a - \frac{p_{n-1}}{q_{n-1}}\right| &= \frac{1}{q_{n-1}^2\left([a_n; a_{n+1}, \ldots] + [0; a_{n-1}, a_{n-2}, \ldots, a_1]\right)} \\ &= \frac{1}{q_{n-1}^2\left([2; 1, a_{n+2}, \ldots] + [0; a_{n-1}, a_{n-2}, \ldots, a_1]\right)} \\ &\leq \frac{1}{\left(2 + \frac{1}{2}\right) q_{n-1}^2}.\end{aligned}$$

This and (4) yield (1).

Case 4. Assume that $a_n = 1$ and $a_{n+1} = 2$. This assumption and (2) gives us

$$\begin{aligned}\left|a - \frac{p_n}{q_n}\right| &= \frac{1}{q_n^2\left([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, a_{n-1}, \ldots, a_1]\right)} \\ &= \frac{1}{q_n^2\left([2; a_{n+2}, \ldots] + [0; 1, a_{n-1}, \ldots, a_1]\right)} \\ &\leq \frac{1}{\left(2 + \frac{1}{2}\right) q_n^2}.\end{aligned}$$

This with (4) implies (1).

Case 5. Next, we assume $a_1 = a_2 = 1$ and $n \in \{1, 2\}$. Then $q_0 = q_1 = 1$ and $\frac{4(q^3-1)}{q(q^4+2q^3+5q^2+q+30)} = 0$. These assumptions and (2) imply that

$$\left|a - \frac{p_1}{q_1}\right| = \frac{1}{q_1^2\left([a_2; a_3, \ldots] + [0; a_1]\right)} = \frac{1}{q_1^2\left(2 + [0; a_3, \ldots]\right)} < \frac{1}{2q_1^2} .$$

And (1) follows.

Case 6. Now, we assume $a_1 = a_2 = a_3 = a_4 = 1$, $n = 3$, and $[0; a_5, \ldots] < \frac{1}{2}$. These assumptions and (2) imply that

$$\left|a - \frac{p_2}{q_2}\right| = \frac{1}{q_2^2\left([a_3; a_4, \ldots] + [0; a_2, a_1]\right)} = \frac{1}{q_2^2\left(\frac{3}{2} + \frac{1}{1+[0; a_5, \ldots]}\right)} < \frac{1}{\left(2 + \frac{1}{6}\right) q_2^2} .$$

This with (4) implies (1).

Case 7. Next, assume $a_1 = a_2 = a_3 = a_4 = 1$, $n = 3$, and $[0; a_5, \ldots] \geq \frac{1}{2}$, then $q_3 = 3$. This assumption and (2) gives us

$$\left|a - \frac{p_3}{q_3}\right| = \frac{1}{q_3^2\left([a_4; a_5, \ldots] + [0; a_3, a_2, a_1]\right)} = \frac{1}{q_3^2\left(\frac{5}{3} + [0; a_5, \ldots]\right)} \leq \frac{1}{\left(2 + \frac{1}{6}\right) q_3^2} .$$

And (1) follows.

Case 8. Assume $a_n = a_{n+1} = 1$, $q_{n-2} \geq 2$ and

$$[0; a_{n-1}, a_{n-2}, \ldots, a_1] \geq [0; a_{n+2}, a_{n+3}, \ldots] .$$

This assumption and (2) imply

$$\begin{aligned}
\left|a-\frac{p_{n-1}}{q_{n-1}}\right| &= \frac{1}{q_{n-1}^2\left([a_n;a_{n+1},\ldots]+[0;a_{n-1},a_{n-2},\ldots,a_1]\right)} \\
&= \frac{1}{q_{n-1}^2\left(1+\frac{1}{1+[0;a_{n+2},a_{n+3},\ldots]}+[0;a_{n-1},a_{n-2},\ldots,a_1]\right)} \\
&\leq \frac{1}{q_{n-1}^2\left(1+\frac{1}{1+[0;a_{n-1},a_{n-2},\ldots,a_1]}+[0;a_{n-1},a_{n-2},\ldots,a_1]\right)} \\
&= \frac{1}{q_{n-1}^2\left(1+\frac{1}{1+\frac{q_{n-2}}{q_{n-1}}}+\frac{q_{n-2}}{q_{n-1}}\right)} = \frac{1}{\left(2+\frac{q_{n-2}^2}{q_{n-1}(q_{n-1}+q_{n-2})}\right)q_{n-1}^2} \\
&< \frac{1}{\left(2+\frac{4\left(q_{n-1}^3-1\right)}{q_{n-1}\left(q_{n-1}^4+2q_{n-1}^3+5q_{n-1}^2+q_{n-1}+30\right)}\right)q_{n-1}^2}.
\end{aligned}$$

And (1) follows.

Case 9. Next, Assume $a_n = a_{n+1} = 1$, $q_{n-2} \geq 2$ and

$$[0;a_{n-1},a_{n-2},\ldots,a_1] < [0;a_{n+2},a_{n+3},\ldots]$$

This assumption and (2) imply

$$\begin{aligned}
\left|a-\frac{p_n}{q_n}\right| &= \frac{1}{q_n^2\left([a_{n+1};a_{n+2},\ldots]+[0;a_n,a_{n-1},\ldots,a_1]\right)} \\
&= \frac{1}{q_n^2\left(1+[0;a_{n+2},\ldots]+[0;a_n,a_{n-1},\ldots,a_1]\right)} \\
&< \frac{1}{q_n^2\left(1+[0;a_{n-1},a_{n-2},\ldots,a_1]+[0;a_n,a_{n-1},\ldots,a_1]\right)} \\
&= \frac{1}{q_n^2\left(\frac{q_n}{q_{n-1}}+\frac{q_{n-1}}{q_n}\right)} = \frac{1}{q_n^2\left(2+\frac{q_{n-2}^2}{(q_n-q_{n-2})q_n}\right)} \\
&< \frac{1}{\left(2+\frac{4(q_n^3-1)}{q_n(q_n^4+2q_n^3+5q_n^2+q_n+30)}\right).q_n^2}.
\end{aligned}$$

And we obtain (1).

This completes the proof of Theorem (1) . □

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