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ON THE DERIVATIVE OF THE MINKOWSKI QUESTION-MARK FUNCTION

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ABSTRACT. The Minkowski question-mark function $?(x)$ is a continuous monotonous function defined on $[0,1]$ interval. It is well known fact that the derivative of this function, if exists, can take only two values: 0 and $+\infty$. It is also known that the value of the derivative $?'(x)$ at the point $x=[0;a_1,a_2,\dots,a_t,\dots]$ is connected with the limit behaviour of the arithmetic mean $(a_1+a_2+\dots+a_t)/t$. Particularly, N. Moshchevitin and A. Dushistova showed that if

$$a_1+a_2+\dots+a_t<\kappa_1,$$

where $\kappa_1=2\log\left(\frac{1+\sqrt{5}}{2}\right)/\log 2=1.3884\dots$, then $?'(x)=+\infty$. They also proved that the constant κ_1 is non-improvable. We consider a dual problem: how small can be the quantity $a_1+a_2+\dots+a_t-\kappa_1 t$ if we know that $?'(x)=0$? We obtain the non-improvable estimates of this quantity.

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1. Introduction

1.1. The Minkowski function $?(x)$

For an arbitrary $x\in[0,1]$ we consider its continued fraction expansion

$$x=[0;a_1,a_2,\dots,a_n,\dots]=\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}},\qquad a_j\in\mathbb{Z}_+$$

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with natural partial quotients a_t. This representation is infinite when $x \notin \mathbb{Q}$ and finite for rational x. For irrational numbers the continued fraction representation is unique, however each rational x has two different representations

$$x = [0; a_1, a_2, \ldots, a_{n-1}, a_n] \quad \text{and} \quad x = [0; a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1],$$

where $a_n \geq 2$. By B_n we denote the n-th level of the Stern-Brocot tree, that is

$$B_n := \{x = [0; a_1, \ldots, a_k] : a_1 + \cdots + a_k = n + 1\}.$$

In [13] Minkowski introduced the function $?(x)$ which may be defined as the limit distribution function of sets B_n. This function was rediscovered several times and studied by many authors (see [1], [4], [11], [12], [14]). For irrational $x = [0; a_1, a_2, \ldots, a_n, \ldots]$ the formula

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\cdots+a_k-1}} \tag{1}$$

introduced by Denjoy [2,3] and Salem [16] may be considered as one of the equivalent definitions of the function $?(x)$. If x is rational, then the infinite series in (1) is replaced by a finite sum. Note that $?([0; a_1, \ldots, a_t + 1]) = ?([0; a_1, \ldots, a_t, 1])$ and hence $?(x)$ is well-defined for rational numbers too. It is known that $?(x)$ is a continuous strictly increasing function, also its derivative $?'(x)$, if exists, can take only two values – 0 and $+\infty$. Almost everywhere in [0; 1] in the sense of Lebesgue measure the derivative exists and equals 0. Also, if $x \in \mathbb{Q}$, then $?'(x) = 0$.

1.2. Notation and parameters

We will denote the sequences by capital letters A, B, C and their elements by the corresponding small letters a_i, b_j, c_k. All sequences of the present paper contain positive integers unless otherwise stated. For an arbitrary finite sequence $B = (b_1, b_2, \ldots, b_n)$ we denote

$$\overleftarrow{B} = (b_n, b_{n-1}, \ldots, b_1), \quad S(B) = \sum_{i=1}^{n} b_i, \quad \Pi(B) = \prod_{i=1}^{n} b_i.$$

By $\langle A \rangle$ we denote the *continuant* of (possibly empty) finite sequence $A = (a_1, \ldots, a_t)$. It is defined as follows: the continuant of the empty sequence $\langle \cdot \rangle$ equals 1, $\langle a_1 \rangle = a_1$, if $t \geq 2$ one has

$$\langle a_1, a_2, \ldots, a_t \rangle = a_t \langle a_1, a_2, \ldots, a_{t-1} \rangle + \langle a_1, a_2, \ldots, a_{t-2} \rangle. \tag{2}$$

Note that the finite continued fraction $[0; a_1, \ldots, a_t]$ can be expressed using continuants

$$[0; a_1, \ldots, a_t] = \frac{\langle a_2, \ldots, a_t \rangle}{\langle a_1, a_2, \ldots, a_t \rangle}. \tag{3}$$

Rule (2) can be generalized as follows

$$\begin{aligned}\langle a_1,a_2,\dots,a_t,a_{t+1},\dots,a_s\rangle &= \langle a_1,a_2,\dots,a_t\rangle\langle a_{t+1},\dots,a_s\rangle\\ &\quad+\langle a_1,a_2,\dots,a_{t-1}\rangle\langle a_{t+2},\dots,a_s\rangle\\ &= \langle a_1,a_2,\dots,a_t\rangle\langle a_{t+1},a_{t+2},\dots,a_s\rangle\\ &\quad\times(1+[0;a_t,a_{t-1},\dots,a_1][0;a_{t+1},a_{t+2},\dots,a_s]).\end{aligned} \tag{4}$$

One can find more about the properties of continuants in [9].

For an irrational $x = [0; a_1, a_2, \dots, a_n, \dots]$ we consider the sum $S_x(t)$ of its partial quotients up to tth

$$S_x(t) = a_1 + a_2 + \dots + a_t.$$

Throughout the paper we always denote the sequence of partial quotients of x by $a_1, a_2, \dots, a_t, \dots$ unless otherwise stated. We will also denote the sequence of t first elements of this infinite sequence by A_t. Thus, $S_x(t) = S(A_t)$. We use subscripts to indicate the repetition of a certain integer number: in particular,

$$1_n = \underbrace{1, 1, \dots, 1}_{n \text{ numbers}}.$$

We also need the following constants

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.618034\dots, \qquad \kappa_1 = \frac{2\log\Phi}{\log 2} = 1.3884838\dots, \tag{5}$$

$$\lambda_n = \frac{n+\sqrt{n^2+4}}{2}, \tag{6}$$

$$\kappa_2 = \frac{4\log\lambda_5 - 5\log\lambda_4}{\log\lambda_5 - \log\lambda_4 - \log\sqrt{2}} = 4.4010487\dots, \tag{7}$$

$$\kappa_4 = \sqrt{\frac{\kappa_1 - 1}{\log 2}} = 0.7486412\dots \tag{8}$$

For an arbitrary sequence A of length t we denote $S(A) - \kappa_1 t$ by $\varphi^{(1)}(A)$. For $x = [0; a_1, \dots, a_t, \dots]$ we denote $\varphi^{(1)}(A_t)$ by $\varphi^{(1)}_x(t)$.

1.3. Critical values

In the paper [15] it was shown by J. Paradis, P. Viader, L. Bibiloni that the value of the derivative of the function $?(x)$ is connected with the limit behaviour of $S_x(t)/t$ – the arithmetic mean of t first partial quotients of x. They showed that if for some irrational number x the inequality $S_x(t)/t < \kappa_1$ holds and the derivative $?'(x)$ exists, then $?'(x) = +\infty$. To formulate their second result, let us introduce the constant $z_0 \approx 5.319$ – the positive root of the equation $2\log(1+z) = z\log 2$. It was shown in [15] that if $S_x(t)/t \geq z_0$ and $?'(x)$ exists, then $?'(x) = 0$.

In the paper [4] A. Dushistova and N. Moshchevitin improved the results of [15] and formulated the following two theorems.

Theorem A.

(i) *Let for real irrational $x \in (0,1)$ the inequality $S_x(t) < \kappa_1 t$ holds for all t large enough. Then the derivative $?'(x)$ exists and equals $+\infty$.*

(ii) *For any positive ε there exists an irrational number $x \in (0,1)$, such that $?'(x) = 0$ and the inequality $S_x(t) < (\kappa_1 + \varepsilon)t$ holds for all t large enough.*

Theorem B.

(i) *Let for real irrational $x \in (0,1)$ the inequality $S_x(t) > \kappa_2 t$ holds for all t large enough. Then the derivative $?'(x)$ exists and equals 0.*

(ii) *For any positive ε there exists an irrational number $x \in (0,1)$, such that $?'(x) = +\infty$ and the inequality $S_x(t) > (\kappa_2 - \varepsilon)t$ holds for all t large enough.*

One can see that the constants κ_1 and κ_2 in theorems A and B are non-improvable.

1.4. The dual problem

In the paper [5] the dual problem was considered. Suppose that $?'(x) = 0$. How small can be the difference $\varphi_x^{(1)}(t) = S_x(t) - \kappa_1 t$? Statement (ii) of Theorem A implies that $\varphi_x^{(1)}(t)$ can be less than εt for any positive ε. The first non-trivial estimate of $\varphi_x^1(t)$ was obtained in [5] by A. Dushistova, I. Kan and N. Moshchevitin.

Theorem C.

(i) *Let for irrational $x \in (0,1)$ the derivative $?'(x)$ exists and $?'(x) = 0$. Then for any $\varepsilon > 0$ for all t large enough one has*

$$\max_{u \leq t} \varphi_x^{(1)}(u) = \max_{u \leq t} (S_x(u) - \kappa_1 u) \geq (\kappa_4 - \varepsilon)\sqrt{t \log t}. \tag{9}$$

(ii) *There exists an irrational $x \in (0,1)$, such that $?'(x) = 0$ and or any $\varepsilon > 0$ for all t large enough one has*

$$\varphi_x^{(1)}(t) = S_x(t) - \kappa_1 t \leq (2\sqrt{2} + \varepsilon)\kappa_4 \sqrt{t \log t}. \tag{10}$$

In the paper [8] a strengthened version of the inequality (9) was obtained. It was shown that within the same condition for all t large enough one has

$$\max_{u \leq t} \varphi_x^{(1)}(u) \geq \frac{2}{\sqrt{3}}\kappa_4 \sqrt{t \log t}. \tag{11}$$

Of course, one can ask the same question when $?'(x) = +\infty$. How small can be the difference $\kappa_2 t - S_x(t)$? The first result in this area was also obtained in [5]. It was improved several times and for now the best known estimates are the following.

Theorem D.

(i) *Let for real irrational $x \in (0,1)$ the derivative $?'(x)$ exists and $?'(x) = +\infty$. Then for all t large enough one has*

$$\max_{u \le t}\big(\kappa_2 u - S_x(u)\big) \ge 0.06222\sqrt{t}. \tag{12}$$

(ii) *There exists an irrational $x \in (0,1)$, such that $?'(x) = +\infty$ and for all t large enough one has*

$$\kappa_2 t - S_x(t) \le 0.26489\sqrt{t}. \tag{13}$$

2. Main results

Note that there are different quantities on the left-hand sides of the inequalities (9) and (10). One can say that the inequality (9) considers the uniform behaviour of $\varphi_x^{(1)}(t)$, whereas the inequality (10) deals with the local behaviour of this quantity. In the present paper we consider upper and lower estimates in both cases. Our first theorem states that the constant in (10) is non-improvable.

Theorem 1. *Let for irrational $x \in (0,1)$ the derivative $?'(x)$ exists and $?'(x)=0$. Then for any $\varepsilon > 0$ for infinitely many t one has*

$$\varphi_x^{(1)}(t) \ge (2\sqrt{2} - \varepsilon)\kappa_4\sqrt{t\log t}. \tag{14}$$

Our second theorem provides optimal estimates of the uniform behaviour of $\varphi_x^{(1)}(t)$.

Theorem 2.

(i) *Let for irrational $x \in (0,1)$ the derivative $?'(x)$ exists and $?'(x) = 0$. Then for any $\varepsilon > 0$ for all t large enough one has*

$$\max_{u \le t}\varphi_x^{(1)}(u) \ge (\sqrt{2} - \varepsilon)\kappa_4\sqrt{t\log t}. \tag{15}$$

(ii) *For any $\varepsilon > 0$ there exists an irrational $x \in (0,1)$ such that $?'(x) = 0$ and for infinitely many t one has*

$$\max_{u \le t}\varphi_x^{(1)}(u) \le (\sqrt{2} + \varepsilon)\kappa_4\sqrt{t\log t}. \tag{16}$$

3. Auxiliary lemmas

3.1. Increment lemmas and their corollaries

The following two lemmas shed light on the connection between the behaviour of the sum of partial quotients of x and the value of the derivative $?'(x)$.

Lemma 3.1 ([5], Lemma 1). *For an irrational $x = [0; a_1, a_2, \ldots, a_t, \ldots]$ and for δ small in absolute value, there exists a natural $t = t(x, \delta)$[1] such that*

$$\frac{?(x+\delta)-?(x)}{\delta} \geq \frac{\langle A_t\rangle\langle A_{t-1}\rangle}{2^{S_x(t)+4}}. \tag{17}$$

Lemma 3.2 ([5], Lemma 2). *For an irrational $x = [0; a_1, a_2, \ldots, a_t, \ldots]$ and for δ small in absolute value, there exists a natural $t = t(x, \delta)$[2] such that*

$$\frac{?(x+\delta)-?(x)}{\delta} \leq \frac{\langle A_t\rangle^2}{2^{S_x(t)-2}}. \tag{18}$$

It is not convenient for us that the numerators of the right-hand sides of (17) and (18) do not coincide. That is why we prove a "symmetric" corollary of Lemmas 3.1 and 3.2.

Lemma 3.3. *For an irrational $x = [0; a_1, a_2, \ldots, a_t, \ldots]$ the derivative $?'(x)$ equals zero if and only if*

$$\lim_{t\to\infty} \frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}} = 0. \tag{19}$$

In the proof of this lemma we will use the following statement from [10].

Lemma 3.4 ([10], Lemma 2.1). *Let for irrational $x \in (0,1)$ the derivative $?'(x)$ exists and $?'(x) = 0$. Then*

$$\lim_{t\to\infty} \frac{\langle A_{t-1}\rangle}{\sqrt{2}^{S_x(t)}} = 0. \tag{20}$$

One can consider this lemma as the reversed version of Lemma 3.1. Now we are ready to prove Lemma 3.3.

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[1] It was also shown in [5], however not included in the statement of the lemma, that $t(x,\delta) \to \infty$ as $\delta \to 0$. We will use the fact in our future argument.
[2] Again, $t(x,\delta) \to \infty$ as $\delta \to 0$.

Proof.

($\Leftarrow$) Follows immediately from Lemma 3.2.

($\Rightarrow$) It follows from Lemma 3.4 that (20) holds. Suppose that $\frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}}$ does not tend to zero as $t \to \infty$. That is, there exists a positive constant c such that

$$\frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}} > c \tag{21}$$

for infinitely many t. Consider $N \in \mathbb{N}$ such that for any $n \geq N$ one has

$$\frac{\langle A_{n-1}\rangle}{\sqrt{2}^{S_x(n)}} < \frac{c}{100}.$$

Consider an arbitrary integer $t > N + a_N$ such that the inequality (21) holds. Then

$$\frac{(a_t+1)c}{100} > \frac{(a_t+1)\langle A_{t-1}\rangle}{\sqrt{2}^{S_x(t)}} > \frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}} > c.$$

Thus, $a_t \geq 100$. On the other hand,

$$\frac{\sqrt{2}^{a_t}c}{(a_{t-1}+1)(a_t+1)} < \frac{\sqrt{2}^{a_t}}{a_{t-1}+1}\frac{\langle A_{t-1}\rangle}{\sqrt{2}^{S_x(t)}} < \frac{\langle A_{t-2}\rangle}{\sqrt{2}^{S_x(t-1)}} < \frac{c}{100}.$$

Therefore, we have $100\sqrt{2}^{a_t} < (a_{t-1}+1)(a_t+1)$. As $a_t \geq 100$, one can easily see that $a_{t-1} > a_t + 1$. From (21) one can also derive that

$$\frac{\langle A_{t-1}\rangle}{\sqrt{2}^{S_x(t-1)}} > \frac{a_t+1}{\sqrt{2}^{a_t}}\frac{\langle A_{t-1}\rangle}{\sqrt{2}^{S_x(t-1)}} > \frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}} > c.$$

Repeating the same argument $t - N$ times we obtain that $a_N > a_t + t - N > a_t + a_N$ and therefore we come to a contradiction. □

3.2. Continuant lower estimate

The following lemma is a useful tool to estimate the values of continuants, most of whose elements are equal to 1. We introduce some notation first. Having a sequence A_t, denote by $w(A_t)$ the number of its elements greater than 1. The set of such elements forms the sequence that we denote by

$$d_{A_t}(1), d_{A_t}(2), \ldots, d_{A_t}\big(w(A_t)\big).$$

Lemma 3.5. *For an arbitrary continuant $\langle A_t\rangle = \langle a_1, a_2, \ldots, a_t\rangle$ one has*

$$\langle A_t\rangle \geq \frac{1}{2}\Phi^t \prod_{i=1}^{w(A_t)} \left(\frac{d_{A_t}(i)}{4}\right). \tag{22}$$

P r o o f. We prove by induction on $w(A_t)$. If $w(A_t)$ equals zero, then $\langle A_t\rangle = F_{t+1}$ – $(t+1)$th Fibonacci number and one can easily verify (22) using Binet's formula.

Now suppose that the inequality (22) holds for all A_t such that $w(A_t) = n$. Consider an arbitrary continuant $\langle A_t\rangle$ having $w(A_t) = n+1$. Let us also consider the continuant $\langle A'_t\rangle$, which is obtained from $\langle A_t\rangle$ by replacing one of its elements greater than one by 1. Of course, $w(A'_t) = n$ and one has

$$\langle A_t\rangle = \frac{\langle A_t\rangle}{\langle A'_t\rangle}\langle A'_t\rangle \geq \frac{\langle A_t\rangle}{\langle A'_t\rangle}\frac{1}{2}\Phi^t\prod_{i=1}^{n}\left(\frac{d_{A'_t}(i)}{4}\right). \tag{23}$$

Applying (4), one can easily see that for arbitrary finite sequences A and B one has

$$\begin{aligned}\langle A, x, B\rangle &= \langle A, x\rangle\langle B\rangle(1 + [0; x, \overleftarrow{A}][0; B])\\ &= x\langle A\rangle\langle B\rangle(1 + [0; \overleftarrow{A}][0; x])(1 + [0; x, \overleftarrow{A}][0; B]).\end{aligned} \tag{24}$$

Hence,

$$\frac{\langle A, x, B\rangle}{\langle A, 1, B\rangle} = x\frac{1 + [0; x, \overleftarrow{A}][0; B]}{1 + [0; 1, \overleftarrow{A}][0; B]}\frac{1 + [0; \overleftarrow{A}][0; x]}{1 + [0; \overleftarrow{A}]} \geq x\frac{1}{2}\cdot\frac{1}{2} = \frac{x}{4}. \tag{25}$$

Applying the lower estimate of $\frac{\langle A_t\rangle}{\langle A'_t\rangle}$ from (25) to (23), we prove the induction step. The lemma is proved. □

Lemma 3.6. *Suppose that $?'(x) = 0$. Then $\varphi_x^{(1)}(t) > 0$ for all t large enough.*

P r o o f. From Lemma 3.3 one has

$$\frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}} < \frac{1}{2}$$

for all t large enough. Using an obvious estimate $\langle A_t\rangle > \frac{\Phi^t}{2}$ from (22), one has

$$\frac{1}{2} > \frac{\langle A_t\rangle}{\sqrt{2}^{S_x(t)}} > \frac{1}{2}\frac{\Phi^t}{\sqrt{2}^{\kappa_1 t + \varphi_x^{(1)}(t)}}.$$

As $\sqrt{2}^{\kappa_1} = \Phi$, we obtain the statement of the lemma. □

Lemma 3.7. *Consider three arbitrary real numbers $s \geq \beta > \alpha \geq 3$. Let $R(s, \alpha, \beta)$ be the set of all finite sequences $R = (r_1, \ldots, r_k)$ of real numbers such that $\alpha \leq r_i \leq \beta$ for all $i \leq k$ and $S(R) = s$. Then*

$$\min_{R\in R(s,\alpha,\beta)} \Pi(R) \geq \beta^{\left[\frac{s}{\beta}\right]}. \tag{26}$$

Proof. Note that the number k is not fixed in the definition of $R(s,\alpha,\beta)$. Denote

$$f(s,\alpha,\beta) = \min_{R\in R(s,\alpha,\beta)} \Pi(R).$$

It is clear that the function $f(s,\alpha,\beta)$ is monotonic in the first argument. Hence, without loss of generality one can say that $\frac{s}{\beta} \in \mathbb{Z}$. Consider the sequence

$$R_\beta = (\underbrace{\beta,\beta,\ldots,\beta}_{s/\beta \text{ times}}).$$

Using the compactness argument, one can easily see that there exists a sequence $R_0 = (r_1^0,\ldots,r_m^0) \in R(s,\alpha,\beta)$ such that $\Pi(R_0) = f(s,\alpha,\beta)$. Suppose that $R_0 \neq R_\beta$. One can easily show that R_0 cannot contain more than two elements not equal to α or β. Indeed, if $\alpha < r_i^0 \leq r_j^0 < \beta$, then there exists $\delta > 0$ such that $r_i^0 - \delta > \alpha$ and $r_j^0 + \delta < \beta$. As $(r_i^0 - \delta)(r_j^0 + \delta) < r_i^0 r_j^0$, one can see that

$$\Pi(r_1^0,\ldots,r_{i-1}^0,r_i^0-\delta,r_{i+1}^0,\ldots,r_{j-1}^0,r_j^0+\delta,r_{j+1}^0,\ldots,r_m^0) < \Pi(r_1^0,\ldots,r_{i-1}^0,r_i^0,r_{i+1}^0,\ldots,r_{j-1}^0,r_j^0,r_{j+1}^0,\ldots,r_m^0) \quad (27)$$

and we obtain a contradiction with the definition of R_0.

On the other hand, it follows from the definition of R_0 that

$$\Pi(R_0) \leq \Pi(R_\beta). \quad (28)$$

Without loss of generality one can assume that R_0 does not contain elements equal to β. Indeed, if we remove all such elements from R_0 and the same number of elements from R_β, the inequality (28) will be still satisfied. Thus,

$$R_0 = (x, \underbrace{\alpha,\alpha,\ldots,\alpha}_{(s-x)/\alpha \text{ times}})$$

up to transposition of elements. Here $\alpha \leq x < \beta$.

Denote $n = \frac{s-x}{\alpha}$. Inequality (28) can be written as

$$\alpha^n x \leq \beta^{\frac{\alpha n + x}{\beta}}. \quad (29)$$

But in fact, the opposite inequality is true for all $\alpha \leq x \leq \beta$. Taking into account the fact that $\alpha^\beta > \beta^\alpha$ for $\beta > \alpha \geq 3$, one can easily verify that

$$\alpha^n x > \beta^{\frac{\alpha n + x}{\beta}} \quad (30)$$

for $x = \alpha$ and $x = \beta$. As $\beta^{\frac{\alpha n + x}{\beta}}$ is a convex downward function of x, one can deduce that it lies below the linear function $\alpha^n x$ for all $\alpha \leq x \leq \beta$. Thus, we obtain a contradiction with (28) and the lemma is proved. □

3.3. Elimination of small elements greater than 1

Lemma 3.8. *Let A, C be arbitrary (possibly empty) sequences of positive integers. Let B be a symmetric sequence. Consider two arbitrary integers $p \geq m \geq 1$. Then*

$$\langle A, 1, B, p+m-1, C\rangle \leq \langle A, m, B, p, C\rangle. \tag{31}$$

P r o o f. Denote $q = \frac{p+m}{2}$. Consider the function

$$f(x) = \langle A, q+x, B, q-x, C\rangle,$$

where x run through the set of integers if $p+m$ is even and runs through the set of half-integers otherwise. One can see that $f(x)$ is a quadratic polynomial with negative leading coefficient. The maximum of $f(x)$ is attained at the point[3]

$$x_m = \frac{[0; \overleftarrow{B}] - [0; B] + [0; C] - [0; \overleftarrow{A}]}{2}.$$

As B is a symmetric sequence, $[0; \overleftarrow{B}] = [0; B]$ and therefore $|x_m| \leq \frac{1}{2}$. Hence, as $p \geq m \geq 1$, one can easily see that

$$\langle A, 1, B, p+m-1, C\rangle = f\left(-\frac{p+m-2}{2}\right) \leq f\left(-\frac{p-m}{2}\right) = \langle A, m, B, p, C\rangle. \ \square$$

Lemma 3.9. *Suppose that $?'(x) = 0$. Then there exists an irrational number $y = [0; b_1, b_2, \ldots, b_t, \ldots]$ such that:*

(1) $?'(y) = 0$.

(2) *For all $i \in \mathbb{N}$ either $b_i = 1$ or $b_i \geq 12$.*

(3) *For all $i \in \mathbb{N}$ one has $\varphi_y^{(1)}(i) \leq \varphi_x^{(1)}(i)$.*

P r o o f. First, let us eliminate all elements equal to 2 from the infinite sequence $(a_1, a_2, \ldots)$. Denote by s_1 the smallest index such that $a_{s_1} = 2$. Denote by t_1 the smallest index greater than s_1 such that $a_{t_1} > 1$. Now the procedure is repeated, recursively,

$$s_i = \min\{n : n > t_{i-1}, a_n = 2\}, \qquad t_i = \min\{n : n > s_i, a_n > 1\}$$

Thus, we obtain the two (possibly infinite) growing sequences $s_1 < t_1 < s_2 < t_2 < \ldots$ Note that if $s_j < n < t_j$ for some j, then $a_n = 1$. Define the irrational number $x' = [0; a'_1, a'_2, \ldots, a'_t, \ldots]$ as follows

$$a'_n = \begin{cases} a_n - 1 = 1 & \text{if} \quad n = s_i, \quad \text{for some} \quad i \in \mathbb{N}, \\ a_n + 1 & \text{if} \quad n = t_i, \quad \text{for some} \quad i \in \mathbb{N}, \\ a_n, & \text{otherwise.} \end{cases}$$

[3] see [6], Lemma 5.4 for computational details

One can easily see from the definition of x' that $\varphi_{x'}^{(1)}(t) \le \varphi_x^{(1)}(t)$ for all $t \ge 1$. Let us now show that $?'(x') = 0$.

It follows from Lemma 3.8 that $\langle a_1', a_2', \ldots, a_t' \rangle := \langle A_t' \rangle \le \langle A_t \rangle$ for all $t \ge 1$. On the other hand $|S_{x'}(t) - S_x(t)| \le 1$. Hence, as

$$\lim_{t\to\infty} \frac{\langle A_t \rangle}{\sqrt{2}^{S_x(t)}} = 0$$

by Lemma 3.3, we obtain that

$$\lim_{t\to\infty} \frac{\langle A_t' \rangle}{\sqrt{2}^{S_{x'}(t)}} = 0$$

and therefore $?'(x') = 0$. Using the same argument, one can eliminate the elements equal to $3, 4, \ldots, 11$ from the sequence $a_1', a_2', \ldots$ The lemma is proved. □

Lemma 3.10. *Suppose that $?'(x) = 0$ and all partial quotients of x are either equal to 1 or greater than 11. There exists $T \in \mathbb{N}$ such that for all $t > T$ the inequality $\varphi_x^{(1)}(t) > 3w(A_t)$ holds.*

Proof. Lemma 3.3 implies that there exists an integer T such that $\forall t > T$ one has

$$\frac{\langle A_t \rangle}{\sqrt{2}^{S_x(t)}} < \frac{1}{2}.$$

Hence, as $\frac{x}{4} \ge 3$ for $x \ge 12$ and $\sqrt{2}^{\kappa_1} = \Phi$, one has by Lemma 3.5

$$\frac{1}{2} > \frac{\langle A_t \rangle}{\sqrt{2}^{S_x(t)}} \ge \frac{1}{2} \frac{\Phi^t 3^{w(A_t)}}{\sqrt{2}^{\kappa_1 t + \varphi_x^{(1)}(t)}} = \frac{1}{2} \frac{3^{w(A_t)}}{\sqrt{2}^{\varphi_x^{(1)}(t)}}. \tag{32}$$

Statement of the lemma immediately follows from (32) as $\sqrt{2}^3 < 3$. □

4. Blocks structure and lower estimates

4.1. Parameters introduction

Let $x = [0; a_1, a_2, \ldots, a_t, \ldots]$ be an irrational number such that $?'(x) = 0$. By Lemma 3.9, without loss of generality one can say that either $a_i = 1$ or $a_i \ge 12$ for all $i \in \mathbb{N}$. Throughout the remaining part of the paper we consider ε as a fixed positive real number from the statements of Theorems 1 and 2. Let $\lambda = \lambda(\varepsilon)$ be an arbitrary rational number such that $1 > \lambda > 1 - \varepsilon^6$. Define the following integer constants

$$M = \frac{10 \log \varepsilon}{\log \lambda}, \quad P = \left[\frac{\log 6}{\log(1+\varepsilon^2)} \right] + 1, \quad N = 2M(P+2). \tag{33}$$

Now we select an integer parameter t_0 large enough such that $(1-\lambda)\lambda^N t_0 > \frac{t_0}{\log t_0}$ and

$$\lambda^N t_0 \in \mathbb{Z}. \tag{34}$$

Denote $t_i = \lambda^i t_0$, where $1 \le i \le N$. We define

$$B_i = (a_{t_i+1}, a_{t_i+2}, \ldots, a_{t_{i-1}}) - \text{the } i\text{th block},$$

where $1 \le i \le N$. Denote $t_{N+1} = 0$ and $B_{N+1} = (a_{t_{N+1}+1}, \ldots, a_{t_N})$. Thus, we have

$$[0; A_{t_0}] = [0; a_1, \ldots, a_{t_0}] = [0; B_{N+1}, B_N, \ldots, B_1].$$

One can easily see that $S_x(t_0) = \sum_{i=1}^{N+1} S(B_i)$ and $\varphi_x^{(1)}(t_0) = \sum_{i=1}^{N+1} \varphi^{(1)}(B_i)$.

For each block B_i denote its greatest element by M_i and the index of such element by m_i (if the greatest element is not unique, we take the rightmost one). Thus, $a_{m_i} = M_i$. Denote $c_k = \frac{M_k}{\sqrt{t_{k-1}\log t_0}}$. Let us also consider for each $1 \le i \le N+1$ the short block $B_i' = (a_{t_i+1}, \ldots, a_{m_i-1})$. Note that

$$\varphi_x^{(1)}(m_i) = \varphi^{(1)}(B_{N+1}) + \varphi^{(1)}(B_N) + \cdots \\ \cdots + \varphi^{(1)}(B_{i+1}) + \varphi^{(1)}(B_i') + (M_i - \kappa_1). \tag{35}$$

For each $1 \le i \le N+1$ define the real numbers f_i and f_i' from the following identities

$$\langle B_i \rangle = \sqrt{2}^{S(B_i)+f_i\sqrt{t_{i-1}\log t_0}}, \qquad \langle B_i' \rangle = \sqrt{2}^{S(B_i')+f_i'\sqrt{t_{i-1}\log t_0}}. \tag{36}$$

4.2. Lower estimate of $\varphi^{(1)}(B_k)$

Lemma 4.1. *Suppose that $?'(x) = 0$. Then for all $1 \le i \le N$ one has*

$$f_i'\sqrt{t_{i-1}\log t_0} + \sum_{k=i+1}^{N+1} f_k\sqrt{t_{k-1}\log t_0} < 0. \tag{37}$$

Proof. As $?'(x) = 0$, by Lemma 3.3, without loss of generality one can say that for all $1 \le i \le N$ the inequality

$$\frac{\langle B_{N+1}, B_N, \ldots, B_{i+1}, B_i' \rangle}{\sqrt{2}^{S(B_{N+1})+S(B_N)+\cdots+S(B_{i+1})+S(B_i')}} < 1$$

is satisfied. Using (4) we obtain

$$\frac{\langle B_{N+1}\rangle\langle B_N\rangle \ldots \langle B_{i+1}\rangle\langle B_i'\rangle}{\sqrt{2}^{S(B_{N+1})+S(B_N)+\cdots+S(B_{i+1})+S(B_i')}} < 1. \tag{38}$$

Substituting (36) to (38) and taking logarithm of both parts, we get the statement of the lemma. □

Lemma 4.2. *Suppose that* $?'(x) = 0$. *If the inequality*

$$|\varphi^{(1)}(B_k)| \geq \kappa_1(t_{k-1} - t_k)\varepsilon^5, \tag{39}$$

holds for some $1 \leq k \leq N+1$, *then one has:*

$$\max_{t_N \leq u \leq t_0} \varphi_x^{(1)}(u) \geq t_0^{0.9}. \tag{40}$$

Proof. Suppose that

$$\varphi^{(1)}(B_k) \geq \kappa_1(t_{k-1} - t_k)\varepsilon^5.$$

As,

$$\varphi_x^{(1)}(t_{k-1}) = \varphi_x^{(1)}(t_k) + \varphi^{(1)}(B_k),$$

using the fact that $\varphi_x^{(1)}(t_k) > 0$ by Lemma 3.6, we have

$$\max_{t_N \leq u \leq t_0} \varphi_x^{(1)}(u) \geq \varphi_x^{(1)}(t_{k-1}) > \varphi^{(1)}(B_k) \geq \kappa_1(t_{k-1} - t_k)\varepsilon^5$$
$$= \kappa_1\varepsilon^5\lambda^{k-1}(1-\lambda)t_0 > t_0^{0.9}. \tag{41}$$

On the other hand, if

$$\varphi^{(1)}(B_k) \leq -\kappa_1(t_{k-1} - t_k)\varepsilon^5,$$

we again use the fact that $\varphi_x^{(1)}(t_{k-1}) > 0$ and obtain

$$\max_{t_N \leq u \leq t_0} \varphi_x^{(1)}(u) \geq \varphi_x^{(1)}(t_k)$$
$$= \varphi_x^{(1)}(t_{k-1}) - \varphi^{(1)}(B_k) \geq \kappa_1(t_{k-1} - t_k)\varepsilon^5 > t_0^{0.9}. \qquad \square$$

One can easily deduce from Lemma 4.2 that if the inequality (39) holds, then the inequalities (14) and (15)) are satisfied. Therefore, throughout the remaining part of the paper we will assume that

$$S(B_k) = \kappa_1(t_{k-1} - t_k)\big(1 + o(\varepsilon^4)\big) \tag{42}$$

for all $1 \leq k \leq N+1$.

Lemma 4.3. *Suppose that* $?'(x) = 0$. *Then for all* $1 \leq k \leq N+1$ *one has*

$$\varphi^{(1)}(B_k) \geq \frac{(\kappa_1 - 1)(t_{k-1} - t_k)\sqrt{\log t_0}(1 + o(\varepsilon^4))}{c_k\sqrt{t_{k-1}}\log 2} - f_k\sqrt{t_{k-1}\log t_0}. \tag{43}$$

Proof. We recall that $w(B_k)$ is the number of elements of the block B_k which are greater than 1 and $d_{B_k}(1), \ldots, d_{B_k}(w(B_k))$ is the sequence of such elements. It follows from eqrefsbkass and Lemma 3.10 that

$$\sum_{i=1}^{w(B_k)} d_{B_k}(i) = (\kappa_1 - 1)(t_{k-1} - t_k)\big(1 + o(\varepsilon^4)\big).$$

Therefore, as $d_{B_k}(i) \geq 12$ for all i, one has $\frac{d_{B_k}(i)}{4} \geq 3$. Now we can obtain a lower estimate of

$$\prod_{i=1}^{w(B_k)} \left(\frac{d_{B_k}(i)}{4}\right)$$

applying Lemma 3.7 for $s = \frac{1}{4}(\kappa_1 - 1)(t_{k-1} - t_k)\big(1 + o(\varepsilon^4)\big)$, $\alpha = 3$, and $\beta = c_k\sqrt{t_{k-1}\log t_0}$. We have

$$\prod_{i=1}^{w(B_k)} \left(\frac{d_{B_k}(i)}{4}\right) \geq \left(\frac{c_k\sqrt{t_{k-1}\log t_0}}{4}\right)^{\frac{(\kappa_1-1)(t_{k-1}-t_k)(1+o(\varepsilon^4))}{c_k\sqrt{t_{k-1}\log t_0}}}. \tag{44}$$

Substituting the estimate (44) to (22) we obtain

$$\langle B_k\rangle \geq \Phi^{t_{k-1}-t_k}\left(\frac{c_k\sqrt{t_{k-1}\log t_0}}{4}\right)^{\frac{(\kappa_1-1)(t_{k-1}-t_k)(1+o(\varepsilon^4))}{c_k\sqrt{t_{k-1}\log t_0}}}. \tag{45}$$

Taking into account that $\sqrt{2}^{\kappa_1} = \Phi$, from (36), (45) and Lemma 3.3 we get

$$\left(\frac{c_k\sqrt{t_{k-1}\log t_0}}{4}\right)^{\frac{(\kappa_1-1)(t_{k-1}-t_k)(1+o(\varepsilon^4))}{c_k\sqrt{t_{k-1}\log t_0}}} \leq \sqrt{2}^{\varphi^{(1)}(B_k)+f_k\sqrt{t_{k-1}\log t_0}}. \tag{46}$$

Taking logarithms of both parts of (46), after some transformations we obtain the inequality

$$\frac{(\kappa_1-1)(t_{k-1}-t_k)(1+o(\varepsilon^4))}{c_i\sqrt{t_{k-1}\log t_0}}\frac{\log t_0}{2} \leq (\varphi^{(1)}(B_k) + f_k\sqrt{t_{k-1}\log t_0})\log\sqrt{2} \tag{47}$$

which is equivalent to (43). Lemma is proved. □

4.3. Main lower estimate

For $k \leq N$ one can write (43) as

$$\varphi^{(1)}(B_k) \geq \left(\frac{(\kappa_1-1)(1-\lambda)(1+o(\varepsilon^4))}{c_k\log 2} - f_k\right)\sqrt{t_{k-1}\log t_0}. \tag{48}$$

For $k = N+1$ we have

$$\varphi^{(1)}(B_{N+1}) \geq \left(\frac{(\kappa_1-1)(1+o(\varepsilon^4))}{c_{N+1}\log 2} - f_{N+1}\right)\sqrt{t_N\log t_0} > -f_{N+1}\sqrt{t_N\log t_0}. \tag{49}$$

One can also deduce the following trivial lower estimate of $\varphi^{(1)}(B'_k)$ using (36)

$$\varphi^{(1)}(B'_k) \geq -f'_k\sqrt{t_{k-1}\log t_0}. \tag{50}$$

Lemma 4.4. *Suppose that* $?'(x) = 0$. *Then for all* $1 \le i \le N$ *one has*

$$\varphi_x^{(1)}(m_i) > \left(\frac{(\kappa_1 - 1)(1-\lambda)}{\log 2} \sum_{k=i+1}^{N} \left(\frac{(\sqrt{\lambda})^{k-i}}{c_k} \right) + c_i \right) \sqrt{t_{i-1} \log t_0} \big(1 + o(\varepsilon^4)\big). \quad (51)$$

P r o o f. Substituting the estimates (43) for $k = N, N-1, \ldots, i+1$, (49), and eqrefgenest2 to (35) and taking into account the inequality (37) we obtain

$$\varphi_x^{(1)}(m_i) > \big(1 + o(\varepsilon^4)\big) \left(\frac{(\kappa_1 - 1)\sqrt{\log t_0}}{\log 2} \sum_{k=i+1}^{N} \frac{t_{k-1} - t_k}{c_k \sqrt{t_{k-1}}} \right.$$
$$\left. + \left(c_i \sqrt{t_{i-1} \log t_0} - \kappa_1 \right) \right). \quad (52)$$

Taking into account the fact that $t_k = \lambda^k t_0$, we immediately obtain (51). Lemma is proved. □

Inequality (51) is the key tool that we will use in proofs of Theorem 1 and the first statement of Theorem 2. Let us simplify it using a new notation. Denote

$$\alpha = \frac{(\kappa_1 - 1)}{\log 2}, \qquad \eta = \frac{1}{\alpha}. \quad (53)$$

Then, one can rewrite (51) as follows

$$\varphi_x^{(1)}(m_i) \ge \left((1-\lambda) \sum_{k=i+1}^{N} \frac{(\sqrt{\lambda})^{k-i}}{c_k} + \eta c_i \right) \alpha \sqrt{t_{i-1} \log t_0}. \big(1 + o(\varepsilon^4)\big). \quad (54)$$

5. Key lemmas

The inequality (54) reduces the estimation of $\max_{1 \le i \le N} \varphi_x^{(1)}(m_i)$ to the problem of finding maximum of the following quantity

$$(1-\lambda) \sum_{k=i+1}^{N} \frac{(\sqrt{\lambda})^{k-i}}{c_k} + \eta c_i, \qquad i = 1, 2, \ldots, N.$$

Note that this problem does not deal with the Minkowski function, continued fractions etc. It is purely combinatorial. The following lemma allows us to estimate the desired maximum.

Lemma 5.1. *Let $\eta, c_1, c_2, \ldots, c_N$ be arbitrary positive real numbers. Define the real numbers φ_i as follows*

$$\varphi_i := (1-\lambda)\sum_{k=i+1}^{N}\frac{\sqrt{\lambda}^{k-i}}{c_k} + \eta c_i. \tag{55}$$

Then the following inequality holds

$$\max_{1\le i\le N}\varphi_i \ge \sqrt{8\eta}\big(1+o(\varepsilon)\big). \tag{56}$$

P r o o f. The proof of the lemma will be splitted into several steps. We also recall that the constants M, N, and P used in our argument are defined in (33).

Lemma 5.2. *Suppose that the inequality* (56) *is not satisfied. Then there exists a natural number $i_1 \le M$ such that $c_{i_1} \ge \frac{1}{2\sqrt{\eta}}$. Moreover, for all $i \le N$ one has*

$$c_i < \frac{3}{\sqrt{\eta}}. \tag{57}$$

P r o o f. Suppose the contrary. Let us estimate φ_1 from below

$$\begin{aligned}\varphi_1 &\ge (1-\lambda)\sum_{k=1}^{N}\frac{\sqrt{\lambda}^{k}}{c_{k+1}} + \eta c_1 \ge (1-\lambda)\sum_{k=1}^{M}\frac{\sqrt{\lambda}^{k}}{c_{k+1}} \ge 2\sqrt{\eta}(1-\lambda)\sum_{k=1}^{M}\sqrt{\lambda}^{k}\\ &= 2\sqrt{\eta}(1-\lambda)\sqrt{\lambda}\frac{1-\sqrt{\lambda}^{M}}{1-\sqrt{\lambda}} = 2\sqrt{\eta}(1+\sqrt{\lambda})\big(1+o(\varepsilon^4)\big)\\ &> \sqrt{8\eta}\big(1+o(\varepsilon^4)\big).\end{aligned} \tag{58}$$

We obtain a contradiction with (56). The estimate (57) comes from the trivial inequality $\varphi_i > \eta c_i$. Lemma is proved. □

Lemma 5.3. *Suppose that the inequality* (56) *is not satisfied. Then for all $i_m < N - M$, $m \ge 1$ there exists a number $i_m < i_{m+1} < i_m + M$ such that $c_{i_{m+1}} > (1+\varepsilon^2)c_{i_m}$.*

P r o o f. Suppose the contrary. Let the inequality $c_{i_m+j} < c_{i_m}(1+\varepsilon^2)$ be satisfied for all $1 \le j \le M$. Then, using the argument from Lemma 5.2 we obtain

$$\begin{aligned}\varphi_{i_m} &\ge (1-\lambda)\sum_{j=1}^{M}\frac{\sqrt{\lambda}^{j}}{c_{i_m+j}} + \eta c_{i_m} \ge \frac{1-\lambda}{1+\varepsilon^2}\sum_{j=1}^{M}\frac{\sqrt{\lambda}^{j}}{c_{i_m}} + \eta c_{i_m}\\ &\ge \left(\frac{2}{(1+\varepsilon^2)c_{i_m}} + \eta c_{i_m}\right)\big(1+o(\varepsilon^4)\big) \ge \sqrt{\frac{8\eta}{1+\varepsilon^2}}\big(1+o(\varepsilon^4)\big).\end{aligned} \tag{59}$$

In the last "$\ge$" of (59) we use the Cauchy-Schwarz inequality. We come to a contradiction with (56). Lemma is proved. □

Now we are ready to prove Lemma 5.1. Suppose that the inequality (56) is not satisfied. By Lemma 5.2 there exists $1 \leq i_1 < M$ such that $c_i \geq \frac{1}{2\sqrt{\eta}}$. Applying Lemma 5.3 $P = \left[\frac{\log 6}{\log(1+\varepsilon^2)}\right] + 1$ times, we obtain the number $c_{i_{P+1}}$ such that

$$c_{i_{P+1}} > \frac{1}{2\sqrt{\eta}}(1+\varepsilon^2)^P\eta > \frac{3}{\sqrt{\eta}}.$$

We come to a contradiction with (57). Lemma is proved. □

One can rewrite (54) as

$$\varphi_x^{(1)}(m_i) \geq \sqrt{\lambda}^i\left((1-\lambda)\sum_{k=i+1}^{N}\frac{(\sqrt{\lambda})^{k-i}}{c_k} + \eta c_i\right)\alpha\sqrt{t_0\log t_0}\big(1+o(\varepsilon^4)\big). \qquad (60)$$

Note that $\sqrt{\lambda} = 1 + o(\varepsilon^5)$. As the first statement of Theorem 2 requires us to show that $\max_{1\leq i\leq N}\varphi_x^{(1)}(m_i)$ is greater than $\sqrt{t_0\log t_0}$ multiplied by some constant factor, we need to estimate the maximum of the following quantity

$$\sqrt{\lambda}^i\left((1-\lambda)\sum_{k=i+1}^{N}\frac{(\sqrt{\lambda})^{k-i}}{c_k} + \eta c_i\right), \qquad i = 1, 2, \ldots, N.$$

This estimate is provided by the following lemma.

Lemma 5.4. *Let $C = (c_1, c_2, \ldots, c_N)$ be an arbitrary sequence of non-negative real numbers. Let η be an arbitrary positive real number. Define the numbers φ_i using (55). Define the numbers φ_i' as follows*

$$\varphi_i'(C) := \sqrt{\lambda}^i\varphi_i = (1-\lambda)\sum_{k=i+1}^{N}\frac{\sqrt{\lambda}^k}{c_k} + \sqrt{\lambda}^i\eta c_i. \qquad (61)$$

Then one has

$$\max_{1\leq i\leq N}\varphi_i'(C) \geq \sqrt{2\eta}\big(1+o(\varepsilon)\big). \qquad (62)$$

P r o o f. The proof of the lemma will be also splitted into several steps. First, using the substitution $d_j = \sqrt{\lambda}^j c_j$, we write (61) as

$$\tilde{\varphi}_i'(D) := \varphi_i'(C) = (1-\lambda)\sum_{k=i+1}^{N}\frac{\lambda^k}{d_k} + \eta d_i. \qquad (63)$$

Here $D = (d_1, d_2, \ldots, d_N)$. Denote $\tilde{\varphi}'_{max}(D) = \max\limits_{1\leq k\leq N}\tilde{\varphi}_k'(D)$. In order to prove the lemma it is enough for us to show that

$$\min_{D\in\mathbb{R}_+^N}\tilde{\varphi}'_{max}(D) = \sqrt{2\eta}\big(1+o(\varepsilon)\big). \qquad (64)$$

Denote the left-hand side of (64) by y_{min}.

Lemma 5.5. *Suppose that* $\tilde{\varphi}'_{max}(D) = y_{min}$ *for some* $D \in \mathbb{R}^N_{\geq 0}$. *Then for all* $1 \leq k \leq N$ *one has* $\tilde{\varphi}'_k(D) = y_{min}$.

P r o o f. Suppose that $\tilde{\varphi}'_{max}(D) = y_{min}$, but for some i one has $\tilde{\varphi}'_i(D) < y_{min}$. We call the index n *minimizing* if $\tilde{\varphi}'_n(D) = y_{min}$. Let k be the largest non-minimizing index. Without loss of generality one can say that all indices less than k are not minimizing too. Indeed, consider the sequence

$$D' = (d_1, \dots, d_{k-1}, d_k + \delta, d_{k+1}, \dots, d_N),$$

where $\delta > 0$ is some small parameter. One can easily see that $\tilde{\varphi}'_i(D') < \tilde{\varphi}'_i(D)$ for $i < k$, $\tilde{\varphi}'_i(D') > \tilde{\varphi}'_i(D)$ for i= k and $\tilde{\varphi}'_i(D') = \tilde{\varphi}'_i(D)$ for $i > k$. As k is a non-minimizing index, there exists $\delta > 0$ such that $\tilde{\varphi}'_i(D') < y_{min}$ for all $i \leq k$.

Thus, $k+1$ is the smallest minimizing index. As all indices less than $k+1$ are not minimizing, there exists $\delta > 0$ such that for the sequence

$$D'' = (d_1, \dots, d_k, d_{k+1} - \delta, d_{k+2}, \dots, d_N)$$

one has

$$y_{min} > \tilde{\varphi}'_i(D'') > \tilde{\varphi}'_i(D) \text{ for } i \leq k \quad \text{and} \quad \tilde{\varphi}'_i(D'') < \tilde{\varphi}'_i(D) = y_{min} \text{ for } i = k+1.$$

Thus, we obtained the sequence whose smallest minimizing index is at least $k+2$. Repeating this argument we obtain the sequence $D^{(N)} = \left(d_1^{(N)}, d_2^{(N)}, \dots, d_N^{(N)}\right)$ with the smallest minimizing index equal to N. One can easily see that for the sequence

$$D'^{(N)} = \left(d_1^{(N)}, d_2^{(N)}, \dots, d_N^{(N)} - \delta\right)$$

for $\delta > 0$ small enough one has $\tilde{\varphi}'_{max}(D'^{(N)}) < \tilde{\varphi}'_{max}(D^{(N)}) = y_{min}$ and we obtain a contradiction with the definition of y_{min}. Lemma is proved. □

Lemma 5.6. *There exists a unique sequence* $D = (d_1, d_2, \dots, d_N)$ *such that* $\tilde{\varphi}'_{max}(D) = y_{min}$. *The elements of this sequence satisfy the recurrent equation*

$$d_{k+1} = \frac{d_k + \sqrt{d_k^2 + \frac{4(1-\lambda)\lambda^{k+1}}{\eta}}}{2} \tag{65}$$

with the initial condition $d_1 = 0$.

P r o o f. Lemma 5.5 implies that $\tilde{\varphi}'_N(D) = y_{min}$. This fact yields a linear equation $\eta d_N = y_{min}$ from which d_N is uniquely defined. Then we substitute d_N to the equation $\tilde{\varphi}'_{N-1}(D) = y_{min}$ and find d_{N-1} etc. Therefore the sequence D is uniquely defined.

Suppose that $d_1 > 0$. But if we decrease d_1, we would also decrease $\tilde{\varphi}'_1(D)$ and not change $\tilde{\varphi}'_i(D)$ for $i \geq 2$, but $\tilde{\varphi}'_{max}(D)$ would be still equal to y_{min}.

We obtain a contradiction with the uniqueness of D. Finally, from the equation

$$\tilde{\varphi}'_{k+1}(D) = \tilde{\varphi}'_k(D)$$

one can derive

$$\eta(d_{k+1} - d_k) = (1 - \lambda)\frac{\lambda^{k+1}}{d_{k+1}}. \tag{66}$$

Considering (66) as the quadratic equation on d_{k+1} and choosing the positive root, we obtain (65). Lemma is proved. □

Thus, $y_{min} = \eta d_N$ where d_N can be evaluated from the recurrent equations (65). Multiplying both sides of (65) by $\sqrt{\eta}$, we obtain

$$\sqrt{\eta}d_{k+1} = \frac{\sqrt{\eta}d_k + \sqrt{(\sqrt{\eta}d_k)^2 + 4(1-\lambda)\lambda^{k+1}}}{2}. \tag{67}$$

Put $e_k = \sqrt{\eta}d_k$. Using the introduced notation, one can write (67) as follows

$$e_{k+1} = \frac{e_k + \sqrt{e_k^2 + 4(1-\lambda)\lambda^{k+1}}}{2}. \tag{68}$$

Denote[4] $\delta_k = (1-\lambda)\lambda^k$ and $X_n = \sum\limits_{k=1}^{n} \delta_n$.

Lemma 5.7. *For $n \geq 1$ one has*

$$e_n < \sqrt{2X_n} \quad \textit{and} \quad e_{n+1} - e_n > \sqrt{2X_{n+2}} - \sqrt{2X_{n+1}}.$$

Proof. The first statement is proved by induction. For $n = 1$ one can easily verify the inequality. As $\sqrt{x}$ is a convex function, one can easily see that

$$e_{n+1} = \frac{e_n + \sqrt{e_n^2 + 4\delta_{n+1}}}{2} < \frac{\sqrt{2X_n} + \sqrt{2X_n + 4\delta_{n+1}}}{2}$$
$$< \sqrt{2X_n + 2\delta_{n+1}} = \sqrt{2X_{n+1}}. \tag{69}$$

Now we prove the second statement. Note that

$$e_{n+1} - e_n = \frac{\sqrt{e_n^2 + 4\delta_{n+1}} - e_n}{2} = \frac{2\delta_{n+1}}{e_n + \sqrt{e_n^2 + 4\delta_{n+1}}} = \frac{\delta_{n+1}}{e_{n+1}}. \tag{70}$$

[4]Equation (68) is equivalent to $\frac{e_{k+1} - e_k}{\delta_{k+1}} = \frac{1}{e_{k+1}}$ (see (70)). Thus, (68) might be considered as numerical integration of the differential equation $y' = \frac{1}{y}$ on the non-uniform grid X_i. I am thankful to I. Mitrofanov who drew my attention to this fact.

On the other hand, as we already showed, $\sqrt{2X_{n+2}} > \sqrt{2X_{n+1}} > e_{n+1}$ and therefore

$$\sqrt{2X_{n+2}} - \sqrt{2X_{n+1}} = \frac{2\delta_{n+2}}{\sqrt{2X_{n+2}} + \sqrt{2X_{n+1}}} < \frac{\delta_{n+1}}{e_{n+1}} = e_{n+1} - e_n. \tag{71}$$

Lemma is proved. □

Now we are ready to prove Lemma 5.4. By Lemma 5.7, $\sqrt{2X_{n+1}} - e_n$ forms a decreasing sequence of positive real numbers. Hence

$$\begin{aligned} 0 < \sqrt{2X_N} - e_{N-1} < \sqrt{2X_2} - e_1 < 2\sqrt{X_2} \\ = \sqrt{2(1-\lambda)(\lambda+\lambda^2)} < 2\sqrt{1-\lambda} = o(\varepsilon^2). \end{aligned} \tag{72}$$

On the other hand,

$$2X_N = 2\sum_{k=1}^{N}(1-\lambda)\lambda^k = 2\bigl(1-\lambda^{N+1}\bigr) = 2 + o(\varepsilon^5).$$

Thus, $e_{N-1} = \sqrt{2} + o(\varepsilon^2)$. It also follows from (68) that $e_N = \sqrt{2} + o(\varepsilon^2)$. Hence,

$$y_{min} = \eta d_N = \sqrt{\eta} e_N = \sqrt{2\eta} + o(\varepsilon^2).$$

Lemma is proved. □

6. Proof of Theorem 1 and the first statement of Theorem 2

Now we are ready to prove Theorem 1.

P r o o f. From (54) and Lemma 5.1 one can deduce that for all t_0 large enough there exist integer numbers i and m_i satisfying $1 \le i \le N$ and $\frac{t_0}{\log t_0} < m_i \le t_0$ such that

$$\varphi_x^{(1)}(m_i) \ge \sqrt{8\eta}\alpha\sqrt{t_{i-1}\log t_0}\bigl(1+o(\varepsilon)\bigr) = 2\sqrt{2}\kappa_4\sqrt{t_{i-1}\log t_0}\bigl(1+o(\varepsilon)\bigr). \tag{73}$$

As $t_i < m_i \le t_{i-1}$ and $\frac{t_i}{t_{i-1}} = \lambda = 1 + o(\varepsilon^5)$, from (73) one has

$$\varphi_x^{(1)}(m_i) \ge 2\sqrt{2}\kappa_4\sqrt{m_i \log m_i}\bigl(1+o(\varepsilon)\bigr).$$

Theorem is proved. □

Let us now prove the first statement of Theorem 2.

P r o o f. It follows from (60) and Lemma 5.4 that for all t_0 large enough there exist integer numbers i and m_i satisfying $1 \le i \le N$ and $\frac{t_0}{\log t_0} < m_i \le t_0$ such that

$$\sqrt{\lambda}^i \varphi_x^{(1)}(m_i) \ge \sqrt{2\eta}\alpha\sqrt{t_i \log t_0}\big(1+o(\varepsilon)\big) = \sqrt{2}\kappa_4\sqrt{t_i \log t_0}\big(1+o(\varepsilon)\big).$$

In other words,

$$\max_{u\le t_0} \varphi_x^{(1)}(u) \ge \varphi_x^{(1)}(m_i) \ge \sqrt{2}\kappa_4\sqrt{t_0 \log t_0}\big(1+o(\varepsilon)\big).$$

The first statement of Theorem 2 is proved. □

7. Proof of the second statement of Theorem 2

7.1. Superblocks definition

P r o o f. In this chapter we will construct an irrational number

$$x = [0; a_1, \dots, a_n, \dots]$$

that will satisfy the conditions of the second statement of Theorem 2. The continued fraction of x will have the form

$$x = \Big[0; \mathfrak{B}^{(0)}, \mathfrak{B}^{(1)}, \mathfrak{B}^{(2)}, \dots, \mathfrak{B}^{(n)}, \dots\Big], \tag{74}$$

where the segments $\mathfrak{B}^{(i)}$ will be defined later. We will call these segments *superblocks*. Recall that the sequence d_k is defined from the equations (65) with the initial condition $d_1 = 0$. It will be convenient for us to modify the first element of this sequence. We set $d_1 = d_2 = \sqrt{\frac{(1-\lambda)\lambda}{\eta}} = o(\varepsilon^5)$.

Let us now define the sequence T_i that plays the key role in our construction. We choose an arbitrary integer T_1, satisfying $(1-\lambda)^N\lambda^N T_1 > \frac{T_1}{\log T_1}$. Then, if the number T_{i-1} is defined, we put $T_i = \left[\frac{T_{i-1}}{\lambda^N}\right]$. For each $i \ge 1$ put $t_0^{(i)} = T_i$.

Now we describe the construction of the superblock $\mathfrak{B}^{(i)}$ for an arbitrary positive integer i. For each k such that $1 \le k \le N$ we select three natural numbers $m_k^{(i)}, n_k^{(i)}$, and $t_k^{(i)}$ from the following conditions:

$$\begin{gathered} d_k\sqrt{T_i \log T_i} \le m_k^{(i)} \le d_k\sqrt{T_i \log T_i}(1+\varepsilon^4), \\ \frac{d_k}{\kappa_1 - 1}\sqrt{T_i \log T_i} \le n_k^{(i)} \le \frac{d_k}{\kappa_1 - 1}\sqrt{T_i \log T_i}(1+\varepsilon^4), \\ \frac{\log T_i}{\log 2}\left(1+\frac{\varepsilon}{8}-\varepsilon^3\right) \le m_k^{(i)} + n_k^{(i)} - \kappa_1(n_k^{(i)}+1) \le \frac{\log T_i}{\log 2}\left(1+\frac{\varepsilon}{8}+\varepsilon^3\right), \\ t_k^{(i)} = \lambda^k t_0^{(i)} + \theta\big(n_k^{(i)}+1\big), \quad \text{where} \quad |\theta| \le \frac{1}{2}, \qquad \big(n_k^{(i)}+1\big) \mid \big(t_{k-1}^{(i)} - t_k^{(i)}\big). \end{gathered} \tag{75}$$

One can easily see that the numbers $m_k^{(i)}$, $n_k^{(i)}$, $t_k^{(i)}$, satisfying conditions (7.1) always exist. Consider the block $B_k^{(i)} = (a_{t_k^{(i)}+1}, \dots, a_{t_{k-1}^{(i)}})$ having the following structure

$$B_k^{(i)} = \left(m_k^{(i)},\, 1_{n_k^{(i)}}, m_k^{(i)},\, 1_{n_k^{(i)}}, \dots, m_k^{(i)},\, 1_{n_k^{(i)}}\right). \tag{76}$$

We recall that

$$1_n = \underbrace{1, 1, \dots, 1}_{n \text{ numbers}}.$$

Denote the sequence of N blocks

$$\left(B_N^{(i)}, B_{N-1}^{(i)}, \dots, B_1^{(i)}\right)$$

by $\mathfrak{B}^{(i)}$. For the initial superblock $\mathfrak{B}^{(0)} = \left(a_1, a_2, \dots, a_{t_N^{(1)}}\right)$ we set all its elements to be equal to 1. Thus, the construction of the continued fraction (74) is fully described. Note that the initial superblock $\mathfrak{B}^{(0)}$ has fixed length and therefore does not affect neither the value of $?'(x)$ nor the behaviour of $\varphi_x^{(1)}(t)$ as t grows.

7.2. $?'(x) = 0$

As we already mentioned, it is enough to show that for $x' = [0; \mathfrak{B}^{(1)}, \mathfrak{B}^{(2)}, \dots, \mathfrak{B}^{(n)}, \dots])$ one has $?'(x') = 0$. Denote the elements of continued fraction expansion of x' by $a'_1, a'_2, \dots$ By Lemma 3.3 it is enough to show that the function

$$f_{x'}(t) = \frac{\langle a'_1, a'_2, \dots, a'_t\rangle}{\sqrt{2}^{a'_1+a'_2+\dots+a'_t}} \tag{77}$$

tends to 0 as $t \to \infty$. One can easily see that $f_{x'}(t) > f_{x'}(t-1)$ if $a'_t = 1$ and $f_{x'}(t) < f_{x'}(t-1)$ if $a'_t \geq 12$. Of course, all partial quotients of x', that are not equal to one, are greater than 12. Thus it is enough to consider $f_{x'}(t)$ only in the case when $a'_t = 1$, but $a'_{t+1} > 1$. In this case, the continuant $\langle a'_1, a'_2, \dots, a'_t\rangle$ consists of the sequences of the form $(m_l^{(i)}, 1_{n_l^{(i)}})$. From the fact that $\langle A, B\rangle \leq 2\langle A\rangle\langle B\rangle$ and Lemma 3.3 one can easily see that if we show that

$$\frac{2\langle m_l^{(i)}, 1_{n_l^{(i)}}\rangle}{\sqrt{2}^{m_l^{(i)}+n_l^{(i)}}} < \frac{1}{2}, \tag{78}$$

we will prove the fact that $?'(x) = 0$. Note that

$$4\left\langle m_l^{(i)}, 1_{n_l^{(i)}}\right\rangle < 8 m_l^{(i)} \Phi^{n_l^{(i)}}.$$

From the definition (7.1) one can easily see that

$$m_l^{(i)} + n_l^{(i)} = \kappa_1 n_l^{(i)} + \frac{\log T_i}{\log 2}\left(1 + \frac{\varepsilon}{8} + o(\varepsilon^2)\right).$$

Thus,

$$\frac{4\langle m_l^{(i)}, 1_{n_l^{(i)}}\rangle}{\sqrt{2}^{m_l^{(i)}+n_l^{(i)}}} < \frac{8m_l^{(i)}\Phi^{n_l^{(i)}}}{\sqrt{2}^{\kappa_1 n_l^{(i)}+\frac{\log T_i}{\log 2}(1+\frac{\varepsilon}{8}+o(\varepsilon^2))}}$$

$$= \frac{8m_l^{(i)}}{\sqrt{2}^{\frac{\log T_i}{\log 2}(1+\frac{\varepsilon}{8}+o(\varepsilon^2))}} \tag{79}$$

And it is enough to show that

$$\frac{8m_l^{(i)}}{\sqrt{2}^{\frac{\log T_i}{\log 2}(1+\frac{\varepsilon}{8}+o(\varepsilon^2))}} < 1. \tag{80}$$

Taking logarithm of both sides of (80) and substituting $m_l^{(i)}$ from (7.1), we obtain that

$$\frac{\log T_i}{2}\Big(1+o(\varepsilon^2)\Big) < \frac{\log T_i}{\log 2}\Big(1+\frac{\varepsilon}{8}+o(\varepsilon^2)\Big)\log\sqrt{2}$$

$$= \frac{\log T_i}{2}\Big(1+\frac{\varepsilon}{8}+o(\varepsilon^2)\Big). \tag{81}$$

Therefore, $?'(x) = 0$.

7.3. The inequality (16) is satisfied

We will show that for x, that we built in Section 7.1, for all $t = T_i$, $i \geq 1$ the inequality (16) is satisfied. This inequality is equivalent to the following

$$\varphi_x^{(1)}(\nu) = S_x(\nu) - \kappa_1\nu \leq (\sqrt{2}+\varepsilon)\kappa_4\sqrt{t\log t} \quad \forall \nu \leq t. \tag{82}$$

We will prove (82) by induction on i. Suppose that $t = T_1 = t_0^{(1)}$. As the segment $\mathfrak{B}^{(0)} = (a_1, a_2, \ldots, a_{t_N^{(1)}})$ has length $\big(o(\varepsilon^6)\big)t$, we will not take it into account in our further argument.

One can easily see that $\varphi_x^{(1)}(\nu) > \varphi_x^{(1)}(\nu-1)$ if and only if $a_\nu > 1$. Thus, it is enough to verify the inequality (82) only in the case when $a_\nu > 1$. Suppose that $t_i^{(1)} < \nu \leq t_{i-1}^{(1)}$ It follows from the definition (7.1) that

$$m_i^{(1)} + n_i^{(1)} - \kappa_1\big(n_i^{(1)}+1\big) > 0.$$

Hence for any finite sequence B and for any $1 \leq i \leq N$ one has

$$\varphi^{(1)}\big(B, 1_{n_i^{(1)}}, m_i^{(1)}\big) > \varphi^{(1)}(B).$$

Thus, it is enough to verify (82) only for the largest $t_k^{(1)} < \nu \le t_{k-1}^{(1)}$ such that $a_\nu > 1$. In this case we have

$$\varphi_x^{(1)}(\nu) \le \sum_{i=k}^{N} \varphi^{(1)}(B_i^{(1)}) + m_k^{(1)}$$

$$= \sum_{i=k}^{N} \left(\frac{T_1(\lambda^{i-1} - \lambda^i)}{n_i^{(1)} + 1} \frac{\log T_1}{\log 2} \left(1 + \frac{\varepsilon}{8}\right)\right) \big(1 + o(\varepsilon^2)\big) + m_k^{(1)}. \qquad (83)$$

Substituting $n_i^{(1)}$ and $m_k^{(1)}$ from (7.1), we obtain that

$$\varphi_x^{(1)}(\nu) \le \left(\frac{(1-\lambda)(\kappa_1 - 1)}{\log 2} \left(1 + \frac{\varepsilon}{8}\right) \sum_{i=k+1}^{N} \frac{\lambda^{i-k}}{d_i} + d_k \right) \sqrt{T_1 \log T_1} \big(1 + o(\varepsilon^2)\big). \quad (84)$$

Denote

$$\alpha = \frac{\kappa_1 - 1}{\log 2} \left(1 + \frac{\varepsilon}{8}\right) = (\kappa_4)^2 \left(1 + \frac{\varepsilon}{8}\right), \qquad \eta = \frac{1}{\alpha}.$$

Using the introduced notation we have

$$\varphi_x^{(1)}(\nu) \le \left((1-\lambda) \sum_{i=k+1}^{N} \frac{\lambda^{i-k}}{d_i} + \eta d_k \right) \alpha \sqrt{T_1 \log T_1} \big(1 + o(\varepsilon^2)\big). \qquad (85)$$

We recall that $d_1, d_2, \ldots, d_N$ is the minimizing sequence for (63). Thus from Lemma 5.5 we obtain that

$$\begin{aligned} \varphi_x^{(1)}(\nu) &\le \alpha \sqrt{2\eta} \sqrt{T_1 \log T_1} \big(1 + o(\varepsilon)\big) \\ &= \sqrt{2\alpha} \sqrt{T_1 \log T_1} \big(1 + o(\varepsilon)\big) \\ &\le \left(\sqrt{2} + \frac{\varepsilon}{2}\right) \kappa_4 \sqrt{T_1 \log T_1}. \end{aligned} \qquad (86)$$

Thus the inequality (82) is satisfied for $t = T_1$. Using the same argument one can derive that for all $n \in \mathbb{N}$ one has

$$\varphi^{(1)}(\mathfrak{B}^{(n)}) \le \left(\sqrt{2} + \frac{\varepsilon}{2}\right) \kappa_4 \sqrt{T_n \log T_n}. \qquad (87)$$

And for all $1 \le k \le N$, $n \in \mathbb{N}$ one also has

$$\sum_{i=k}^{N} \varphi^{(1)}(B_i^{(n)}) + m_k^{(n)} \le \left(\sqrt{2} + \frac{\varepsilon}{2}\right) \kappa_4 \sqrt{T_n \log T_n}. \qquad (88)$$

Now we show that (82) is satisfied for $t = T_n$ when $n \geq 2$. Using the same argument, one can deduce

$$\varphi_x^{(1)}(\nu) \leq \sum_{i=1}^{n-1} \varphi^{(1)}(\mathfrak{B}^{(i)}) + \sum_{i=k}^{N} \varphi^{(1)}(B_i^{(n)}) + m_k^{(n)} \tag{89}$$

for some $1 \leq k \leq N$.
From (87) it follows that

$$\begin{aligned}\sum_{i=1}^{n-1} \varphi^{(1)}(\mathfrak{B}^{(i)}) &\leq \left(\sqrt{2} + \frac{\varepsilon}{2}\right)\kappa_4 \sum_{i=1}^{n-1} \sqrt{T_i \log T_i} \leq \left(\sqrt{2} + \frac{\varepsilon}{2}\right)\kappa_4 \sqrt{\log T_n} \sum_{i=1}^{n-1} \sqrt{T_i} \\ &= \left(\sqrt{2} + \frac{\varepsilon}{2}\right)\kappa_4 \sqrt{T_n \log T_n} \sum_{i=1}^{n-1} \sqrt{\lambda}^{Ni} \\ &\leq \left(\sqrt{2} + \frac{\varepsilon}{2}\right)\kappa_4 \sqrt{T_n \log T_n} \frac{\sqrt{\lambda}^N}{1 - \sqrt{\lambda}^N} = \left(\sqrt{T_n \log T_n}\right) o(\varepsilon^4).\end{aligned} \tag{90}$$

Substituting the estimates (88) and (90) to (89) we obtain that

$$\varphi_x^{(1)}(\nu) \leq \left(\sqrt{2} + \frac{2\varepsilon}{3}\right)\kappa_4 \sqrt{T_n \log T_n} \tag{91}$$

and the second statement of Theorem 2 is proved. □

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