

DENSITY OF OSCILLATING SEQUENCES IN THE REAL LINE

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ABSTRACT. In this paper we study the density in the real line of oscillating sequences of the form

$$(g(k) \cdot F(k\alpha))_{k \in \mathbb{N}},$$

where g is a positive increasing function and F a real continuous 1-periodic function. This extends work by Berend, Boshernitzan and Kolesnik [Distribution Modulo 1 of Some Oscillating Sequences I-III] who established differential properties on the function F ensuring that the oscillating sequence is dense modulo 1.

More precisely, when F has finitely many roots in $[0, 1)$, we provide necessary and also sufficient conditions for the oscillating sequence under consideration to be dense in \mathbb{R} . All the results are stated in terms of the Diophantine properties of α , with the help of the theory of continued fractions.

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1. Introduction

Given a real number x , denote by $\{x\}_2$ the signed fractional part of x , which is the unique real number in $[-\frac{1}{2}, \frac{1}{2})$ such that $x - \{x\}_2 \in \mathbb{Z}$. Similarly, $\{x\}$ stands for the fractional part of x . Denote by $\|x\|$ its distance from the nearest integer: $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. Let also $\llbracket a, b \rrbracket$ be the integer interval with end points determined by the real numbers a and b ; that is $\llbracket a, b \rrbracket = \{n \in \mathbb{Z} : a \leq n \leq b\}$.

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Finally, we will make use of Landau's little- o notation: if, given two real functions $w, v : \mathbb{R}^+ \mapsto \mathbb{R}^+$, it holds that $w(x)/v(x) \rightarrow 0$ as $x \rightarrow +\infty$ (respectively, as $x \rightarrow 0$), then one may write $w(x) = o(v(x))$ as $x \rightarrow +\infty$ (respectively, as $x \rightarrow 0$).

It is asked in [7] whether the sequence $(k \cdot \sin(k))_{k \in \mathbb{N}}$ is dense in \mathbb{R} . More generally, it is natural to determine the values of the parameters $\beta > 0$ and $\alpha \in \mathbb{R}$ for which the oscillating sequence $(k^\beta \cdot \sin(2\pi \cdot k\alpha))_{k \in \mathbb{N}}$ is dense in \mathbb{R} . In this paper we answer this question by studying the density properties in \mathbb{R} of the more general class of oscillating sequences of the form

$$(g(k) \cdot F(k\alpha))_{k \in \mathbb{N}}, \tag{1}$$

where

$$g(t) = t^\beta + o(t^\beta) \quad \text{as } t \rightarrow +\infty \tag{2}$$

for some $\beta > 0$, and where the function F is a real, 1-periodic, continuous function with only isolated roots. We assume further that, if $r \in \mathbb{R}$ is a root of F , then F has the form

$$F(r+x) = c_r \cdot \epsilon(x) \cdot |x|^{\gamma(r)} + o(|x|^{\gamma(r)}) \quad \text{as } x \rightarrow 0 \tag{3}$$

for some $\gamma(r) > 0$ and some $c_r \in \mathbb{R} \setminus \{0\}$. Here, the function $\epsilon : \mathbb{R} \mapsto \{-1, 0, 1\}$ stands for the sign function

$$\epsilon(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

A study of the density of oscillating sequences in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has been made by Berend, Boshernitzan and Kolesnik (see [2–4]). In this body of work, the authors consider oscillating sequences of the form

$$(P(k) \cdot f(Q(k)))_{k \in \mathbb{N}}, \tag{4}$$

where P, Q are polynomials and f is a (highly differentiable) periodic function with period $T > 0$. In particular, they consider three aspects of the problem: the problem of small values modulo 1 of such sequences, that of their density modulo 1, and that of their uniform distribution.

More precisely, in [2], the authors deal with the above-stated problems by providing in each case sufficient conditions on the degree of differentiability of the function f at the point $Q(0)$ for the related properties to hold. In [3], they generalise the results regarding the small values and the density of the sequence (4) in two directions. On the one hand, they allow the function f to be quasi-periodic, that is $f(x) = f_0(x, x, \dots, x)$, where $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is a periodic function

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of several variables. On the other hand, they study a more general family of sequences, namely sequences of the form

$$\left(P(k) \cdot f(Q(k)) \cdot g(R(k))\right)_{k \in \mathbb{N}},$$

where $R(k)$ is a polynomial and the function g is periodic. For instance, they prove that, given integers d and l , there exists $r = r(d, l)$ having the following properties:

for any polynomial P of degree d , any function f with $f^{(s)}(0) \neq 0$ for some $s \geq r$ and any real number α with $\frac{\alpha}{T}$ irrational, the sequence $\left(P(k) \cdot f\left(k^l \cdot \frac{\alpha}{T}\right)\right)_{k \in \mathbb{N}}$ is dense modulo 1.

Other results regarding the distribution of the sine function in the real line are given for instance in [1]. In this paper, Adiceam exploits a result concerning rational approximations of irrationals with the numerators and denominators of the rational approximants restricted to prescribed arithmetic progressions, and proves that for every $\rho \in \mathbb{R}$ and irrational α , it holds that

$$\limsup_{k \rightarrow +\infty} (\sin(2\pi k\alpha + \rho))^k = -\liminf_{k \rightarrow +\infty} (\sin(2\pi k\alpha + \rho))^k = 1.$$

Our approach to study the sequence (1) makes a connection between its density properties in \mathbb{R} and the density properties of auxiliary sequences of the form

$$(k^\beta \cdot \{k\alpha - \rho\}_2)_{k \in \mathbb{N}}, \tag{5}$$

where ρ is a real number (see Proposition 2.1 in Section 2 below for details). Working with the signed fractional part instead of the distance from the nearest integer, which may seem more natural, is a consequence of working in the real line as we have to consider separately the positive and the negative values of the function (3).

Little seems to be known regarding the density of oscillating sequences in the real line. One of the goals of this paper is to relate the density of (1) with the approximation properties of α . Here, by approximation properties we are referring to the irrationality measure $\mu(\alpha)$ of α :

$$\mu(\alpha) = \sup \left\{ v > 0 : \left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^v} \text{ holds for infinitely many rationals } \frac{p}{q} \right. \\ \left. \text{with } \gcd(p, q) = 1 \right\}.$$

It can readily be checked that every rational number r has irrationality measure $\mu(r) = 1$ while, from Dirichlet's theorem in Diophantine approximation, for every irrational x it holds that $\mu(x) \geq 2$. We consider more precisely some additional quantities which refine the notion of irrationality measure.

To define them, we first recall the continued fraction expansion and the Ostrowski expansion of a real number. Throughout this paper, the continued fraction expansion [5, Section 3.1] of every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is denoted by

$$\alpha = [a_0; a_1, \dots, a_n \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

and the sequence of the denominators of the convergents of α by $(q_n)_{n \in \mathbb{N}}$. Given an irrational $\alpha = [a_0; a_1, a_2, \dots]$ and a real number ρ , the Ostrowski expansion [5, Section 3.2, Lemma 3.2] of ρ with base α is the unique choice of natural numbers $\{e_n(\rho)\}_{n \in \mathbb{N}_0}$ and of an integer ρ_0 such that

$$\rho = \rho_0 + e_0(\rho) \cdot \{\alpha\} + \sum_{n=1}^{+\infty} e_n(\rho) \cdot \{q_n \alpha\}_2, \quad (6)$$

where $\rho - \rho_0 \in [-\alpha, 1 - \alpha)$, $e_0(\rho) \in \llbracket 0, a_1 - 1 \rrbracket$ and $e_n(\rho) \in \llbracket 0, a_{n+1} \rrbracket$ for every $n \geq 1$, with $e_n(\rho) = 0$ whenever $e_{n+1}(\rho) = a_{n+2}$ for $n \geq 1$.

DEFINITION 1.1 (Signed Irrationality Evaluation). Given an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, a positive real number $\beta > 0$ and a real number $\rho \in \mathbb{R}$, denote by $\mu_+(\alpha, \beta, \rho)$ and $\mu_-(\alpha, \beta, \rho)$ the quantities

$$\mu_+(\alpha, \beta, \rho) = \liminf_{\substack{k \rightarrow +\infty, \\ \{k\alpha - \rho\}_2 > 0}} k^\beta \cdot \{k\alpha - \rho\}_2 \geq 0$$

and

$$\mu_-(\alpha, \beta, \rho) = \liminf_{\substack{k \rightarrow +\infty, \\ \{k\alpha - \rho\}_2 < 0}} -k^\beta \cdot \{k\alpha - \rho\}_2 \geq 0.$$

Moreover, denote by $\mu(\alpha, \beta, \rho) = \min\{\mu_+(\alpha, \beta, \rho), \mu_-(\alpha, \beta, \rho)\}$ the minimum of the above two quantities. When $\rho = 0$, we may write $\mu_+(\alpha, \beta)$, $\mu_-(\alpha, \beta)$ and $\mu(\alpha, \beta)$ to simplify notation.

Given the Ostrowski expansion (6) of ρ , set further

$$\begin{aligned} \tau_+(\alpha, \beta, \rho) &= \liminf_{n \rightarrow +\infty} \max \left\{ 1, \min \left\{ e_{2n}(\rho)^\beta, (a_{2n+1} - e_{2n}(\rho))^{\frac{\beta+1}{2}} \right\} \right\} \cdot q_{2n}^\beta \cdot \{q_{2n} \alpha\}_2 \geq 0 \end{aligned}$$

and

$$\begin{aligned} \tau_-(\alpha, \beta, \rho) &= \liminf_{n \rightarrow +\infty} -\max \left\{ 1, \min \left\{ e_{2n-1}(\rho)^\beta, (a_{2n} - e_{2n-1}(\rho))^{\frac{\beta+1}{2}} \right\} \right\} \cdot q_{2n-1}^\beta \cdot \{q_{2n-1} \alpha\}_2 \geq 0. \end{aligned}$$

Our main result provides necessary and also sufficient conditions on the oscillating sequence (1) to be dense in \mathbb{R} .

THEOREM 1.2. *Denote by $(y_k)_{k \in \mathbb{N}}$ the sequence defined in (1). Let the function F satisfy assumption (3) and let g satisfy assumption (2).*

- (1) *If the sequence (1) is dense in \mathbb{R}^+ (resp. in \mathbb{R}^-), then there exists a root r of F such that either $c_r > 0$ (resp. $c_r < 0$) and $\mu_+(\alpha, \frac{\beta}{\gamma(r)}, r) = 0$, or else $c_r < 0$ (resp. $c_r > 0$) and $\mu_-(\alpha, \frac{\beta}{\gamma(r)}, r) = 0$. Moreover, if the root r is rational then this condition is also sufficient.*
- (2) *If there exists a root r of F such that either $c_r > 0$ (resp. $c_r < 0$) and $\tau_+(\alpha, \frac{\beta}{\gamma(r)}, r) = 0$, or else $c_r < 0$ (resp. $c_r > 0$) and $\tau_-(\alpha, \frac{\beta}{\gamma(r)}, r) = 0$, then the sequence (1) is dense in \mathbb{R}^+ (resp. in \mathbb{R}^-).*

Under the assumptions of Theorem 1.2, the density of the oscillating sequence (1) depends only on the local properties of F around its isolated roots. In order to establish Theorem 1.2, we first prove the results for the auxiliary sequence (5). Thus, in Section 3, we prove that if ρ is rational, then the sequence (5) is dense in \mathbb{R}^+ (resp. \mathbb{R}^-) if and only if $\mu_+(\alpha, \beta, \rho) = 0$ (resp. $\mu_-(\alpha, \beta, \rho) = 0$). In Section 4, we will exploit the Ostrowski expansion in order to prove that, if $\tau_+(\alpha, \beta, \rho) = 0$ (resp. $\tau_-(\alpha, \beta, \rho) = 0$), then the sequence (5) is dense in \mathbb{R}^+ (resp. in \mathbb{R}^-).

In the special case where $F(x) = \sin(2\pi \cdot x)$, we obtain the following corollary answering the opening question of the paper.

COROLLARY 1.3. *Given $\beta > 0$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the sequence*

$$(k^\beta \cdot \sin(2\pi \cdot k\alpha))_{k \in \mathbb{N}}$$

is dense in \mathbb{R} if and only if at least one of the following holds:

- (1) $\mu_+(\alpha, \beta) = 0$ and $\mu_-(\alpha, \beta) = 0$,
- (2) $\mu_+(\alpha, \beta) = 0$ and $\mu_+(\alpha, \beta, \frac{1}{2}) = 0$,
- (3) $\mu_-(\alpha, \beta) = 0$ and $\mu_-(\alpha, \beta, \frac{1}{2}) = 0$.

For instance, we can apply Corollary 1.3 when α is badly approximable; that is, when there exists $c > 0$ such that for every $k \in \mathbb{N}$, it holds

$$k \cdot \|k\alpha\| \geq c.$$

In this case, for every $\beta \geq 1$, it holds that $\mu(\alpha, \beta) > 0$ and therefore

$$(k^\beta \cdot \sin(2\pi \cdot k\alpha))_{k \in \mathbb{N}}$$

is not dense in \mathbb{R} . Similarly, if $\beta < 1$ it holds that $\mu_\pm(\alpha, \beta) = 0$ and the same sequence is dense in \mathbb{R} .

REMARK 1. From Definition 1.1, it follows immediately that

$$\mu(\alpha, \beta, r) = \liminf_{k \rightarrow +\infty} k^\beta \cdot \|k\alpha - r\|.$$

However more natural this quantity may seem, as proved in Section 5, it does not hold that $\mu_+(\alpha, \beta, r) = 0$ if and only if $\mu_-(\alpha, \beta, r) = 0$. This is the reason why the results are not stated in terms of the quantity $\mu(\alpha, \beta, r)$ alone.

Theorem 1.2 also yields the following corollary stating some cases where the sequence (1) is trivially dense in \mathbb{R} .

COROLLARY 1.4. *Let $(y_k)_{k \in \mathbb{N}}$ be the sequence defined in (1) with the function F satisfying assumption (3) and g satisfying assumption (2). If there exists a root $r \in \mathbb{R}$ of F such that $\frac{\beta}{\gamma(r)} \in (0, 1)$, then the sequence (1) is dense in \mathbb{R} .*

Note that the sufficient condition stated in Theorem 1.2 is not necessary. This is proved in Section 4 by explicitly constructing a suitable sequence $(e_n)_{n \geq 0}$ in the Ostrowski expansion (6).

In addition to Theorem 1.2, we prove the following result which, in the case where $\rho \in \mathbb{Q}$, characterizes the quantities $\mu_\pm(\alpha, \beta, \rho)$ in terms of the sequence of denominators of the convergents to the irrational α .

THEOREM 1.5. *Given $\beta \geq 1$, an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and a rational number $\theta \in \mathbb{Q}$, where $\theta = \frac{p}{q}$ for some $p \in \mathbb{Z}, q \in \mathbb{N}$ with $(p, q) = 1$, it holds that*

$$\mu_+\left(\alpha, \beta, \frac{p}{q}\right) = 0 \quad \left(\text{resp.} \quad \mu_-\left(\alpha, \beta, \frac{p}{q}\right) = 0\right)$$

if and only if

$$\liminf_{\substack{n \rightarrow +\infty, \\ q|q_n}} q_{2n}^\beta \cdot \{q_{2n}\alpha\}_2 = 0 \quad \left(\text{resp.} \quad \liminf_{\substack{n \rightarrow +\infty, \\ q|q_n}} q_{2n-1}^\beta \cdot \{q_{2n-1}\alpha\}_2 = 0.\right)$$

Finally, we provide results regarding the density of oscillating sequences (1) in \mathbb{R} when the parameters α and β satisfy $\mu(\alpha, \beta) = +\infty$. Note that the inequalities $\mu_+(\alpha, \beta) \leq \tau_+(\alpha, \beta, \rho)$ and $\mu_-(\alpha, \beta) \leq \tau_-(\alpha, \beta, \rho)$ hold for every choice of α, β and ρ (see Lemma 5.1). The aforementioned assumption therefore implies that, for every real ρ , $\tau_+(\alpha, \beta, \rho) = \tau_-(\alpha, \beta, \rho) = +\infty$. Thus, the sufficient condition in the statement of Theorem 1.2 does not hold. Before stating the result, recall the definition of inhomogeneous Bohr sets (see [6, 8] for more details): given a real number ρ , an irrational number α , a natural number N and a positive number $\epsilon > 0$, let

$$\mathcal{N}_\rho(N, \alpha, \epsilon) = \{k \in \mathbb{N} : k \leq N, \|k\alpha - \rho\| \leq \epsilon\}. \tag{7}$$

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We use Bohr sets in order to capture the terms of the sequence (5) which affect its density properties in \mathbb{R} . Given the Ostrowski expansion (6) of ρ , define a sequence of natural numbers by setting

$$\kappa_n = \sum_{j=0}^n e_j(\rho) \cdot q_j \quad \text{for all } n \geq 0. \quad (8)$$

THEOREM 1.6. *Let α be an irrational number and let $\beta > 0$ be such that $\mu(\alpha, \beta) = +\infty$. Denote by $(w_k)_{k \in \mathbb{N}}$ the sequence defined in (5). Let ρ be a real number and let $(e_j(\rho))_{j \geq 0}$ be the digits in its Ostrowski expansion. Also, let $(\kappa_n)_{n \geq 0}$ be the sequence defined in (8). Then, the sequence $(w_k)_{k \in \mathbb{N}}$ is dense in \mathbb{R} if and only if the subsequence $(w_k)_{k \in \mathfrak{D}}$ is dense in \mathbb{R} , where*

$$\mathfrak{D} = \bigcup_{n=1}^{+\infty} (\mathcal{N}_\rho(n) \cup \mathcal{N}'_\rho(n)) \quad (9)$$

with $\mathcal{N}_\rho(n) = \mathcal{N}_\rho(\kappa_n, \alpha, \lceil q_n \alpha \rceil)$ and $\mathcal{N}'_\rho(n) = \mathcal{N}_\rho(\kappa_{n-1} + q_n, \alpha, \frac{\lceil q_n - 1 \alpha \rceil}{1 + e_{n-1}^\beta})$. Moreover, the inclusions

$$\{\kappa_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{D} \quad \text{and} \quad \mathfrak{D} \subseteq \bigcup_{n=0}^{+\infty} (\mathcal{M}_\rho(n) \cup \mathcal{M}'_\rho(n)) \quad (10)$$

hold, where

$$\mathcal{M}_\rho(n) = \bigcup_{l=0}^2 \{\kappa_n + (e_n - l) \cdot q_{n+1}\}$$

and

$$\mathcal{M}'_\rho(n) = \bigcup_{l=0}^1 \{\kappa_n + (l + 1)q_n, \kappa_n + q_{n+1} - lq_n\}.$$

Throughout this paper we use Vinogradov's asymptotic notation: if, given two real functions $w, v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, there exists a positive constant $C > 0$ such that for every $x \in \mathbb{R}^+$ it holds that $w(x) \leq C \cdot v(x)$, then we write $w(x) \ll v(x)$. Equivalently, one may use Landau's Big- O notation and write $w(x) = O(v(x))$. The constant C is referred to as the implicit constant. If the implicit constant depends on some parameter, say t , then we index the notation as $w(x) \ll_t v(x)$ (equivalently, as $w(x) = O_t(v(x))$). If for two functions w, v it holds that $w(x) \ll v(x)$ and $v(x) \ll w(x)$ for all admissible values of x , then we write $w(x) \asymp v(x)$. Two real sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ are called *asymptotically equal* if

$$a_n/b_n \xrightarrow{n \rightarrow +\infty} 1.$$

The paper is organized as follows. In Section 2, we first reduce the study of (1) to that of the auxiliary sequences (5). In Section 3, we study the case where ρ is rational and establish in this case the first statement in Theorem 1.2. In Section 4, we use the Ostrowski expansion (6) to prove sufficient conditions for (5) to be dense in \mathbb{R} when the root r is irrational. Moreover, given parameters α and β and a prescribed positive quantity γ , we provide an effective construction of the sequence $(e_n)_{n \geq 0}$ in the expansion (6) ensuring that oscillating sequences of the form (1) are dense in \mathbb{R} and satisfy $\gamma(r) = \gamma$, for some root r of F . In Section 5, we use the results from the previous sections to complete the proof of Theorem 1.2 and to prove Theorem 1.5. In Section 6, we prove Theorem 1.6.

2. Some auxiliary results

The goal of this section is to reduce the study of the density of sequence (1) to that of the sequence (5).

PROPOSITION 2.1. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a 1-periodic function satisfying assumption (3). Let also $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfy assumption (2) and let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers. Then, a real number $h \in \mathbb{R}$ is a limit point of the sequence $(g(k) \cdot F(a_k))_{k \in \mathbb{N}}$ if and only if there exists a root r of F such that h lies in the closure of the set*

$$\left\{ \epsilon \left(\{a_k - r\}_2 \right) \cdot c_r \cdot k^\beta \cdot \|a_k - r\|^{\gamma(r)} \right\}_{k \in \mathbb{N}}.$$

To prove Proposition 2.1, we need the following lemma which allows us to remove the error terms from the definitions of the growth rate function in (2) and the periodic function in (3). Its proof, which is elementary, is left to the reader.

LEMMA 2.2 (Removing the Error Terms from Periodic Functions and Growth Rates). *Let $\mathbf{f} = (f_k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{R} such that*

$$f_k \xrightarrow[k \rightarrow +\infty]{} 0.$$

Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing function such that

$$g(t) \xrightarrow[t \rightarrow +\infty]{} +\infty.$$

Let also u, v be real functions such that

$$\lim_{t \rightarrow +\infty} u(t) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} v(x) = 0.$$

Then, the sequences

$$(g(k) \cdot f_k)_{k \in \mathbb{N}}, \quad \left((g(k) + u(k) \cdot g(k)) \cdot f_k \right)_{k \in \mathbb{N}}$$

and

$$\left(g(k) \cdot (f_k + v(f_k) \cdot f_k) \right)_{k \in \mathbb{N}}$$

are pairwise asymptotically equal and have therefore the same limit points.

We now deduce Proposition 2.1.

Proof of Proposition 2.1: By assumption, the function F is 1-periodic, continuous in \mathbb{R} and has only isolated roots in $[0, 1)$. Therefore, it admits only finitely many roots in the interval $[0, 1)$. Let $r_0 < r_1 < \dots < r_m$ be the finitely many distinct roots of F in $[0, 1)$. Fix $h \in \mathbb{R}$, where h is a limit point of the sequence $(g(k) \cdot F(a_k))_{k \in \mathbb{N}}$. Thus, there exists a sequence of natural numbers $(k_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} g(k_n) \cdot F(a_{k_n}) = h$. This implies that $\lim_{n \rightarrow +\infty} F(a_{k_n}) = 0$ because $g(t) \xrightarrow{t \rightarrow +\infty} +\infty$. By passing to a subsequence if necessary, the sequence $(a_{k_n})_{n \in \mathbb{N}}$ converges modulo 1 to some $r \in [0, 1)$ which, by continuity, is a root of F . In particular,

$$\{a_{k_n} - r\}_2 \xrightarrow{n \rightarrow +\infty} 0.$$

Set

$$u(t) = \frac{t^\beta - g(t)}{g(t)}$$

and

$$v(x) = \frac{c_r \cdot \epsilon(x) \cdot |x|^{\gamma(r)} - F(r+x)}{F(r+x)}.$$

Assumptions (2) and (3) imply that $\lim_{t \rightarrow +\infty} u(t) = 0$ and $\lim_{x \rightarrow 0} v(x) = 0$, respectively. Applying Lemma 2.2 to $\mathbf{f} = (f_{k_n})_{n \in \mathbb{N}} = (F(a_{k_n}))_{n \in \mathbb{N}}$, u and v yields that h lies in the closure of the set

$$\left\{ \epsilon \left(\{a_k - r\}_2 \right) \cdot c_r \cdot k^\beta \cdot \|a_k - r\|^{\gamma(r)} \right\}_{k \in \mathbb{N}}.$$

The converse follows similarly from Lemma 2.2 and assumption (3). The proof is complete. □

3. Rational values of the parameter ρ

In this section, we study the sequence (5) in the case where $\rho \in \mathbb{Q}$. To this end, we prove the following proposition which relates the quantities $\mu_{\pm}(\alpha, \beta, \rho)$ with the density in \mathbb{R} of the sequence (5).

PROPOSITION 3.1. *Let $\beta > 0$ be a positive real number. Given an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and a rational number ρ , it holds that the sequence (5) is dense in \mathbb{R}^+ (resp. in \mathbb{R}^-) if and only if*

$$\mu_+(\alpha, \beta, \rho) = 0 \quad (\text{resp. } \mu_-(\alpha, \beta, \rho) = 0). \quad (11)$$

Proof. We prove the claim concerning the quantity $\mu_+(\alpha, \beta, \rho)$ and the density of the sequence (5) in \mathbb{R}^+ , as the claim related to $\mu_-(\alpha, \beta, \rho)$ and \mathbb{R}^- is established in the same way.

Assume that $\mu_+(\alpha, \beta, \rho) = 0$. From assumption (11), we have that, for every $n \in \mathbb{N}$, there exists $m = m(n) \in \mathbb{N}$ such that

$$0 \leq \{m\alpha - \rho\}_2 = \frac{\epsilon_n}{m^\beta} < \frac{1}{2} \quad \text{for some } 0 \leq \epsilon_n \leq \frac{1}{n}.$$

Without loss of generality, assume that

$$\{m\alpha\}_2 = \{\rho\}_2 + \frac{\epsilon_n}{m^\beta}$$

as otherwise

$$\{m\alpha\}_2 = -1 + \{\rho\}_2 + \frac{\epsilon_n}{m^\beta},$$

in which case we work similarly. Let us assume that $\{\rho\}_2 = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(p, q) = 1$. Then, for every $l \in \mathbb{N}_0$ such that

$$(lq + 1) \cdot \frac{\epsilon_n}{n^\beta} < \frac{1}{2}, \quad (12)$$

it holds that $\{(lq + 1) \cdot m\alpha - \frac{p}{q}\}_2 = (lq + 1) \cdot \frac{\epsilon_n}{m^\beta}$. For those $l \in \mathbb{N}$ which satisfy inequality (12), set

$$Q_\beta(m, l) = (lq + 1)^\beta \cdot m^\beta \cdot \left\{ (lq + 1) \cdot m\alpha - \frac{p}{q} \right\}_2 = (lq + 1)^{1+\beta} \cdot \epsilon_n, \quad (13)$$

where we recall that ϵ_n depends on the choice m . Fix $h > 0$. Notice that, for n large enough, that is for $\epsilon_n \leq \frac{1}{n}$ sufficiently small and $m = m(n)$ sufficiently large, the natural number

$$l_h = \left\lfloor \frac{h^{\frac{1}{1+\beta}} \cdot \epsilon_n^{-\frac{1}{1+\beta}} - 1}{q} \right\rfloor$$

satisfies inequality (12). The quantity $Q_\beta(m, l_h)$ is therefore a term in the sequence (5).

The density of sequence (5) follows upon noticing that

$$h = \left(\left(\frac{h^{\frac{1}{1+\beta}} \cdot \epsilon_n^{-\frac{1}{1+\beta}} - 1}{q} \right) \cdot q + 1 \right)^{1+\beta} \cdot \epsilon_n = Q_\beta(m(n), l_h) + O\left(h^\beta \cdot \epsilon_n^{\frac{1}{1+\beta}}\right)$$

and upon letting $n \rightarrow +\infty$.

In the other direction, assume that $\mu_+(\alpha, \beta, \rho) > 0$. From Definition 1.1 of the quantity $\mu_+(\alpha, \beta, \rho)$, we have that for every $k \gg_\alpha 1$ such that $\{k\alpha - \rho\}_2 > 0$, it holds that

$$k^\beta \cdot \{k\alpha - \rho\}_2 \geq C$$

for some positive constant C . Therefore, sequence (5) cannot be dense in \mathbb{R}^+ .

The proof is complete. □

An immediate consequence of Proposition 3.1 is the following corollary which deals with the case when the exponent β takes values in $(0, 1)$.

COROLLARY 3.2. *Given $\beta \in (0, 1)$, α irrational and ρ a rational number, the sequence*

$$(k^\beta \cdot \{k\alpha - \rho\}_2)_{k \in \mathbb{N}}$$

is dense in \mathbb{R} . Equivalently,

$$\mu_+(\alpha, \beta, \rho) = \mu_-(\alpha, \beta, \rho) = 0.$$

Proof. Let β, α, ρ as in the statement of the corollary. By the theory of continued fractions, if $\frac{p_n}{q_n}$ is one of the convergents of α , then it holds that

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}.$$

Thus, we obtain easily that the finite sequence $(k\alpha)_{k=1}^{q_n}$ is $\frac{2}{q_n}$ -dense in \mathbb{T} . This implies that

$$\mu_+(\alpha, 1, \rho) = \liminf_{\substack{k \rightarrow +\infty, \\ \{k\alpha - \rho\}_2 > 0}} k \cdot \{k\alpha - \rho\}_2 \leq 2,$$

which in turn implies that

$$\mu_+(\alpha, \beta, \rho) = \liminf_{\substack{k \rightarrow +\infty, \\ \{k\alpha - \rho\}_2 > 0}} k^\beta \cdot \{k\alpha - \rho\}_2 = 0.$$

We work similarly with the quantities $\mu_-(\alpha, 1, \rho)$ and $\mu_-(\alpha, \beta, \rho)$. Proposition 3.1 now implies the result. □

4. Real values of the parameter ρ

The goal of this section is to use the Ostrowski expansion of a real number ρ in order to obtain sufficient conditions for the sequence (5) to be dense in \mathbb{R} . This will lead us to the proof of the second statement in Theorem 1.2.

4.1. Sufficient Conditions for Density in \mathbb{R}

We now prove that, if $\tau_+(\alpha, \beta, \rho) = \tau_-(\alpha, \beta, \rho) = 0$, then the sequence (5) is dense in \mathbb{R} . Moreover, in the case where $\mu_+(\alpha, \beta) = \mu_-(\alpha, \beta) = 0$, the proof provides an effective way to construct the coefficients in the Ostrowski expansion (6), and thus the parameter ρ , for the sequence (5) to enjoy the density property.

PROPOSITION 4.1. *Given $\beta > 0$ and $\alpha, \rho \in \mathbb{R} \setminus \mathbb{Q}$, consider the Ostrowski expansion of ρ as defined in (6). If*

$$\tau_+(\alpha, \beta, \rho) = 0 \quad (\text{resp.} \quad \tau_-(\alpha, \beta, \rho) = 0),$$

then the sequence (5) is dense in \mathbb{R}^+ (resp. in \mathbb{R}^-).

Before we continue, recall some facts which will be used extensively in the forthcoming proofs. For every $x \in \mathbb{R}$, it holds that $-||x|| \leq \{x\}_2 \leq ||x||$. Also, given an irrational α and the Ostrowski expansion (6) of a real number ρ , we have that, for every $n \in \mathbb{N}$

$$\left| \sum_{j=n+1}^{+\infty} e_j(\rho) \cdot \{q_j \alpha\}_2 \right| \leq ||q_n \alpha||. \quad (14)$$

Indeed, by the definition of the continued fraction expansion of a real number α (see [5, Section 3]), we have that $a_1 \cdot \{\alpha\} - 1 = \{q_1 \alpha\}_2$, $\{\alpha\} + a_2 \cdot \{q_1 \alpha\}_2 = \{q_2 \alpha\}_2$ and, for every $n \geq 1$, it holds that $\{q_n \alpha\}_2 + a_{n+2} \cdot \{q_{n+1} \alpha\}_2 = \{q_{n+2} \alpha\}_2$. This implies that

$$\{q_n \alpha\}_2 = - \sum_{j \geq 1} a_{n+2j} \cdot \{q_{n+2j-1} \cdot \alpha\}_2 \quad \text{for every } n \geq 1. \quad (15)$$

In turn, this implies that

$$\left| \sum_{j=n+1}^{+\infty} e_j(\rho) \cdot \{q_j \alpha\}_2 \right| \leq \left| \sum_{i=1}^{+\infty} a_{2i+n} \cdot \{q_{2i-1+n} \alpha\}_2 \right| = ||q_n \alpha||,$$

whence the claim.

Proof of Proposition 4.1: We prove the result regarding the quantity $\tau_+(\alpha, \beta, \rho)$ and the density of (5) in \mathbb{R}^+ . The other case follows in the same way. To this end, assume that

$$\tau_+(\alpha, \beta, \rho) = 0.$$

Given $j \geq 0$, set $e_j = e_j(\rho)$.

Case 1: Assume that

$$\liminf_{j \rightarrow +\infty} \max \left\{ 1, e_{2j}^\beta \right\} \cdot q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 = 0. \quad (16)$$

Fix $n \in \mathbb{N}$. There exists $m = m(n) \in 2\mathbb{N}$ such that

$$\epsilon_n := q_m^\beta \cdot \{q_m\alpha\}_2 \leq \frac{1}{n} \quad \text{and} \quad e_m^\beta \cdot \epsilon_n \leq \frac{1}{n}. \quad (17)$$

Since $m \in 2\mathbb{N}$, one has that $\{q_m\alpha\}_2 = \|q_m\alpha\|$ and thus inequality (14) yields that

$$\left| \sum_{j=m+1}^{+\infty} e_j \cdot \{q_j\alpha\}_2 \right| \leq \{q_m\alpha\}_2.$$

Set

$$\eta = q_m^\beta \cdot \sum_{j=m+1}^{+\infty} e_j \cdot \{q_j\alpha\}_2, \quad (18)$$

so that $|\eta| \leq \epsilon_n$. Given $l \in \mathbb{N}$ such that $l\epsilon_n - \eta < \frac{1}{2}$, set

$$\begin{aligned} Q_\beta(m, l) &= \left(\sum_{j=0}^{m-1} e_j \cdot q_j + e_m \cdot q_m + l \cdot q_m \right)^\beta \\ &\quad \times \left\{ \left(\sum_{j=0}^{m-1} e_j \cdot q_j + e_m \cdot q_m + l \cdot q_m \right) \cdot \alpha - \rho \right\}_2 \\ &\stackrel{(6), (17), (18)}{=} \left(\frac{\sum_{j=0}^{m-1} e_j \cdot q_j}{q_m} + e_m + l \right)^\beta \cdot (l \cdot \epsilon_n - \eta). \end{aligned}$$

It easily follows from the Ostrowski expansion of a natural number (see [5, Lemma 3.1]) that

$$\frac{\sum_{j=0}^{m-1} e_j \cdot q_j}{q_m} \leq 1.$$

In turn, from inequalities (17) one infers that $|Q_\beta(m, 0)| \ll 1/n$.

Fix $h > 0$. Note that for

$$l' = 2 \cdot \left[h^{\frac{1}{\beta+1}} \cdot \epsilon_n^{-\frac{1}{\beta+1}} \right]$$

and for n large enough, it holds that $l'\epsilon_n - \eta \leq \frac{1}{4}$ thanks to relations (17) and (18). Thus, for every $l \in \llbracket 0, l' \rrbracket$, the quantity $Q_\beta(m, l)$ is a term of the sequence (5). Moreover, it holds that

$$Q_\beta(m, l') > h.$$

We can then use the relations in (17) in order to prove that, for every $l \in \llbracket 0, l' \rrbracket$,

$$\begin{aligned} |Q_\beta(m(n), l+1) - Q_\beta(m(n), l)| &\ll h^{\frac{1}{1+\beta}} \cdot \left(\frac{1}{n}\right)^{\frac{\beta}{\beta+1}} \\ &\quad + h^{\frac{\beta}{\beta+1}} \cdot \left(\frac{1}{n}\right)^{\frac{1}{\beta+1}} + \frac{1}{n} \\ &= \eta(n, h). \end{aligned}$$

Since

$$|Q_\beta(m(n), 0)| \ll 1/n \quad \text{and} \quad Q_\beta(m(n), l') \geq h,$$

the last inequality yields that the terms $\{Q_\beta(m(n), l)\}_{l \in \llbracket 0, l' \rrbracket}$ partition the interval $[0, h]$ into subintervals with length at most $O(\eta(n, h))$. Since $\eta(n, h) \rightarrow 0$ when $n \rightarrow +\infty$ and the choice of $h > 0$ was arbitrary, one infers that the sequence (5) is dense in \mathbb{R}^+ .

Case 2: Let us assume that

$$\liminf_{j \rightarrow +\infty} \max \left\{ 1, (a_{2j+1} - e_{2j})^{\frac{\beta+1}{2}} \right\} \cdot q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 = 0. \quad (19)$$

Without loss of generality, assume that $\beta \geq 1$ as otherwise assumption (16) dealt with in Case 1 holds. We will follow arguments similar to the first case. Fix $n \in \mathbb{N}$. Then, there exists $m = m(n) \in 2\mathbb{N}$ such that

$$\epsilon_n := q_m^\beta \cdot \{q_m\alpha\}_2 \leq \frac{1}{n} \quad \text{and} \quad (a_{m+1} - e_m)^{\frac{\beta+1}{2}} \cdot \epsilon_n \leq \frac{1}{n}. \quad (20)$$

Define η as in (17) in such a way that $|\eta| \leq \epsilon_n$. Also, set $\eta' = q_m^\beta \cdot \{q_{m+1}\alpha\}_2$, wherefrom it follows that $|\eta'| \leq \epsilon_n$.

Given $l \geq 1$ such that $l \cdot \epsilon_n + (a_{m+1} - e_m) \cdot \epsilon_n - \eta' - \eta < \frac{1}{2}$, let

$$\begin{aligned}
 P_\beta(m, l) &= \left(\sum_{j=0}^{m-1} e_j \cdot q_j + l \cdot q_m - q_{m-1} \right)^\beta \\
 &\quad \times \left\{ \left(\sum_{j=0}^{m-1} e_j \cdot q_j + l \cdot q_m - q_{m-1} \right) \cdot \alpha - \rho \right\}_2 \\
 &\stackrel{(6)}{=} \left(\sum_{j=0}^{m-1} e_j \cdot q_j + l \cdot q_m - q_{m-1} \right)^\beta \\
 &\quad \times \left\{ (l + a_{m+1} - e_m) \cdot \{q_m \alpha\}_2 - \{q_{m+1} \alpha\}_2 - \sum_{j=m+1}^{+\infty} e_j \cdot \{q_j \alpha\}_2 \right\}_2 \\
 &\stackrel{(18), (20)}{=} \left(\frac{\sum_{j=0}^{m-1} e_j \cdot q_j - q_{m-1}}{q_m} + l \right)^\beta \cdot (l \epsilon_n + (a_{m+1} - e_m) \cdot \epsilon_n - \eta' - \eta).
 \end{aligned}$$

As in the previous case, the Ostrowski expansion of a natural number yields

$$\left| \frac{\sum_{j=0}^{m-1} e_j \cdot q_j - q_{m-1}}{q_m} \right| \leq 1.$$

In turn, from inequalities (20) one infers that $|P_\beta(m, 0)| \ll 1/n$.

Fix $h > 0$. For $l' = 4 \cdot \left[h^{\frac{1}{\beta+1}} \cdot \epsilon_n^{-\frac{1}{\beta+1}} \right]$ and n large enough, inequalities (20) and (18) imply that

$$l' \cdot \epsilon_n + (a_{n+1} - e_n) \cdot \epsilon_n - \eta' - \eta \leq \frac{1}{4}.$$

Thus, given $l \in \llbracket 0, l' \rrbracket$, the quantity $P_\beta(m, l)$ is a term in the sequence (5). Moreover, it holds that $P_\beta(m, l') > h$. We can then use the relations in (20) in order to prove that, for every $l \in \llbracket 0, l' \rrbracket$,

$$|P_\beta(m(n), l+1) - P_\beta(m(n), l)| \ll h^{\frac{\beta}{\beta+1}} \cdot \left(\frac{1}{n} \right)^{\frac{1}{\beta+1}} + h^{\frac{\beta-1}{\beta+1}} \cdot \left(\frac{1}{n} \right)^{\frac{2}{\beta+1}} =: \eta(n, h).$$

Since $|P_\beta(m(n), 0)| \ll 1/n$ and $P_\beta(m(n), l') > h$, the last inequality yields that the terms $\{P_\beta(m(n), l)\}_{l \in \llbracket 0, l' \rrbracket}$ partition the interval $[0, h]$ into subintervals with length at most $O(\eta(n, h))$. Since $\eta(n, h) \rightarrow 0$ when $n \rightarrow +\infty$ and the choice of $h > 0$ was arbitrary, one infers that the sequence (5) is dense in \mathbb{R}^+ .

The proof is complete. \square

The following corollary is a straightforward consequence of Proposition 4.1.

COROLLARY 4.2. *Let $\beta \in (0, 1)$ be a real number. Let also $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational and let ρ be a real number. Then, the sequence*

$$(k^\beta \cdot \{k\alpha - \rho\}_2)_{k \in \mathbb{N}}$$

is dense in \mathbb{R} .

Proof. Let β, α, ρ be as in the statement. Assume that $(e_j)_{j \geq 0}$ is the sequence of the digits in the Ostrowski expansion of ρ . From the theory of continued fractions, for every $n \in \mathbb{N}$, it holds that $\|q_n \alpha\| \leq \frac{1}{a_{n+1} q_n}$. Consequently,

$$\begin{aligned} \liminf_{j \rightarrow +\infty} q_{2j}^\beta \cdot \max \left\{ e_{2j}^\beta, 1 \right\} \cdot \{q_{2j} \alpha\}_2 &\leq \liminf_{j \rightarrow +\infty} \max \left\{ \frac{e_{2j}^\beta \cdot q_{2j}^\beta}{a_{2j+1} \cdot q_{2j}}, \frac{q_{2j}^\beta}{a_{2j+1} \cdot q_{2j}} \right\} \\ &\leq \liminf_{j \rightarrow +\infty} \frac{1}{q_{2j}^{1-\beta}} = 0. \end{aligned}$$

Similarly, we can show that $\liminf_{j \rightarrow +\infty} -q_{2j+1}^\beta \cdot \max \left\{ 1, e_{2j+1}^\beta \right\} \cdot \{q_{2j+1} \alpha\}_2 = 0$. Therefore, Proposition 4.1 implies that the sequence (5) is dense in \mathbb{R} . The proof is complete. \square

Proposition 2.1 and Corollary 4.2 immediately imply Corollary 1.4.

4.2. Effective Construction of the Parameter ρ

The sufficient condition in the statement of Proposition 4.1 is not necessary. Indeed, in the following proposition we construct real numbers $\rho \in \mathbb{R}$ such that the sequence (5) is dense in \mathbb{R} but with $\tau_\pm(\alpha, \beta, \rho) = +\infty$.

PROPOSITION 4.3. *Let $\beta > 0$ be a positive number and α be an irrational such that $\mu_+(\alpha, \beta)$ and $\mu_-(\alpha, \beta)$ equal either zero or infinity. Then, there exists an effectively constructible sequence of digits $(e_j)_{j \geq 0}$ in the Ostrowski expansion (6) of the real number ρ such that the sequence $(w_k)_{k \in \mathbb{N}}$ defined in (5) is dense in \mathbb{R} . Moreover, there exist uncountably many such numbers ρ .*

Proof. We split the proof into three cases depending on the values of $\mu_\pm(\alpha, \beta)$.

Case 1: Assume that $\mu_+(\alpha, \beta) = \mu_-(\alpha, \beta) = 0$. Then, the result follows easily from Proposition 4.1. For instance, for every $j \geq 0$, we can choose $e_j \in \{0, 1\}$ so that the resulting sequence is dense in \mathbb{R} .

Case 2: Assume that $\mu_+(\alpha, \beta) = \mu_-(\alpha, \beta) = +\infty$; that is, that

$$\liminf_{n \rightarrow +\infty} q_{2n}^\beta \cdot \{q_{2n}\alpha\}_2 = \liminf_{n \rightarrow +\infty} -q_{2n+1}^\beta \cdot \{q_{2n+1}\alpha\}_2 = +\infty. \quad (21)$$

Fix a sequence $\mathbf{b} = (b_j)_{j \in \mathbb{N}}$ of real numbers which is dense in \mathbb{R} . The goal is to define the sequence $(e_j)_{j \geq 0}$ recursively in such a way that

$$|b_j - w_{\kappa_{m(j)}}| \xrightarrow{j \rightarrow +\infty} 0, \quad (22)$$

where $(\kappa_{m(j)})_{j \in \mathbb{N}}$ is a proper subsequence of the sequence (8) defined in the course of the proof below. Relation (22) then yields the density of sequence (5).

Choose $\epsilon_0 \in \llbracket 0, a_1 - 1 \rrbracket$ arbitrary and fix $j \in \mathbb{N}$. If $j = 1$, then, without loss of generality, assume that $b_1 > 0$. From equation (21), there exists $m(1) \in 2\mathbb{N}$ such that $q_{m(1)}^\beta \cdot \{q_{m(1)}\alpha\}_2 \geq 5b_1$. Given $n \in \llbracket 1, m(1) - 1 \rrbracket$, set $e_n = 0$ and choose $e_{m(1)} \in \llbracket 1, a_{m(1)+1} \rrbracket$ arbitrary. Fix $l(1) \in \mathbb{N}$ large enough. From equation (15) and the choice of $m(1)$, for every $n \in \llbracket m(1) + 1, m(1) + l(1) \rrbracket$, the digits $e_n \in \llbracket 0, a_{n+1} \rrbracket$ can be chosen in such a way that

$$\left| b_1 + \kappa_{m(1)}^\beta \cdot \sum_{n=m(1)+1}^{m(1)+l(1)} e_n \cdot \{q_n\alpha\}_2 \right| \leq \frac{1}{2}.$$

If $j \geq 2$, then assume that the numbers $m(j-1), l(j-1) \in \mathbb{N}$ have been chosen in such a way that, for every $n \in \llbracket 1, m(j-1) + l(j-1) \rrbracket$, the digits $e_n \in \llbracket 0, a_{n+1} \rrbracket$ are such that for every $j' \in \llbracket 1, j-1 \rrbracket$,

$$\left| b_{j'} + \kappa_{m(j')}^\beta \cdot \sum_{n=m(j')+1}^{m(j')+l(j')} e_n \cdot \{q_n\alpha\}_2 \right| \leq \frac{1}{2j'}$$

and

$$\kappa_{m(j'-1)}^\beta \cdot \|q_{m(j'-1)-2} \cdot \alpha\| \leq \frac{1}{2(j'-1)}.$$

Without loss of generality, assume that $b_j \geq 0$. From equation (21), there exists $m_j \in 2\mathbb{N}$ such that $m(j) \geq m(j-1) + l(j-1) + 1$,

$$q_{m(j)}^\beta \cdot \{q_{m(j)}\alpha\}_2 \geq 5b_j \quad \text{and} \quad \kappa_{m(j-1)}^\beta \cdot \|q_{m(j)-2} \cdot \alpha\| \leq \frac{1}{2(j-1)}, \quad (23)$$

where the last inequality holds if $m(j)$ is chosen large enough. Here, the constant 5 in the left inequality ensures that the choice of the digits e_n in the next step of the induction satisfies the properties of the Ostrowski expansion (as given in relation (6)).

Given $n \in \llbracket m(j-1) + l(j-1) + 1, m(j) - 1 \rrbracket$, set $e_n = 0$ and choose $e_{m(j)} \in \llbracket 1, a_{m(j)+1} \rrbracket$ arbitrary. Fix $l(j) \in \mathbb{N}$ large enough. From equation (15) and the left inequality of (23), for every $n \in \llbracket m(j) + 1, m(j) + l(j) \rrbracket$, the digits $e_n \in \llbracket 0, a_{n+1} \rrbracket$ can be chosen in such a way that

$$\left| b_j + \kappa_{m(j)}^\beta \cdot \sum_{n=m(j)+1}^{m(j)+l(j)} e_n \cdot \{q_n \alpha\}_2 \right| \leq \frac{1}{2j}.$$

In the case where $b_j < 0$, one works in a similar way by choosing $m(j) \in 2\mathbb{N} + 1$ large enough. Therefore, we have defined the sequence $(e_n)_{n \in \mathbb{N}}$ and can thus set

$$\rho = e_0 \cdot \{\alpha\} + \sum_{n=1}^{+\infty} e_n \cdot \{q_n \alpha\}_2.$$

It is not hard to check that for every $j \in \mathbb{N}$, it holds that

$$\left| b_j - w_{\kappa_{m(j)}} \right| \leq \frac{1}{j}.$$

The claim is thus proved.

Case 3: Assume that one of the quantities $\mu_\pm(\alpha, \beta)$ equals zero and the other one equals infinity. For instance, assume that $\mu_+(\alpha, \beta) = +\infty$ and $\mu_-(\alpha, \beta) = 0$. Fix a sequence $\mathbf{b} = (b_j)_{j \in \mathbb{N}}$ of real numbers which is dense in \mathbb{R}^+ . We follow the steps in the proof of the second case but this time we choose $m(j) \in 2\mathbb{N}$ large enough so that

$$q_{m(j)-1}^\beta \cdot \left| |q_{m(j)-1} \alpha| \right| \xrightarrow{j \rightarrow +\infty} 0 \quad \text{and} \quad e_{m(j)-1} \in \{0, 1\}.$$

The density in \mathbb{R}^+ follows from the arguments presented in the second case, and the density in \mathbb{R}^- follows from Proposition 4.1. When $\mu_+(\alpha, \beta) = 0$ and $\mu_-(\alpha, \beta) = +\infty$, one works similarly.

The arguments in all three cases imply easily the construction of uncountably many such numbers ρ . The proof is complete. \square

REMARK 2. Given $\beta > 0$ and an irrational α , it can be shown that there exist (uncountably many) real numbers ρ such that the sequence (5) is dense in \mathbb{R} . However, if at least one of the quantities $\mu_\pm(\alpha, \beta)$ is positive and finite, then, it is not clear to the author how one can effectively construct the digits $(e_j)_{j \in \mathbb{N}}$ of the Ostrowski expansion (6) of the real ρ . Note that given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists at most one real number $\beta_+ > 0$ (resp. $\beta_- > 0$) such that $\mu_+(\alpha, \beta_+) \in (0, +\infty)$ (resp. $\mu_-(\alpha, \beta_-) \in (0, +\infty)$).

5. Proof of Theorems 1.2 and 1.5

We are now ready to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. As far as the first part of the theorem is concerned, assume that the sequence $(y_k)_{k \in \mathbb{N}}$ defined in (1) is dense in \mathbb{R}^+ . Then, there exists an increasing sequence of natural numbers $(k_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $F(k_n \alpha) > 0$ and $g(k_n) \cdot F(k_n \alpha) \leq \frac{1}{n}$. By passing to a subsequence if necessary, the continuity of F yields that $\|k_n \alpha - r\| \xrightarrow{n \rightarrow +\infty} 0$ for some root r of F , and the claim follows. Work similarly in the case where $(y_k)_{k \in \mathbb{N}}$ is dense in \mathbb{R}^- . In the special case where the root r is rational, an immediate application of Propositions 3.1 and 2.1 implies the claim.

The second part of the theorem follows from Propositions 4.1 and 2.1. □

Proof of Corollary 1.3. The function $F(x) = \sin(2\pi \cdot x)$ is easily seen to satisfy assumption (3). Moreover, all its roots are rationals. The result now follows upon noticing that, given α irrational, $\beta > 0$ and ρ a rational number, $\mu_+(\alpha, \beta, \rho) = 0$ (resp. $\mu_-(\alpha, \beta, \rho) = 0$) implies $\mu_+(\alpha, \beta) = 0$ (resp. $\mu_-(\alpha, \beta) = 0$). This claim follows from Theorem 1.5. The proof of the corollary is complete. □

We now prove Theorem 1.5.

Proof of Theorem 1.5. We will prove only the case dealing with the quantities

$$\lim_{\substack{j \rightarrow +\infty, \\ q|q_j, j \in 2\mathbb{N}}} q_j^\beta \cdot \{q_j \alpha\}_2$$

and $\mu_+(\alpha, \beta, \rho)$. The other case is similar.

\Rightarrow : Fix $\epsilon' > 0$ and let $(q_n)_{n \in \mathbb{N}}$ be the sequence of denominators of convergents of α . Assume that

$$\lim_{\substack{j \rightarrow +\infty, \\ q|q_j, j \in 2\mathbb{N}}} q_j^\beta \cdot \{q_j \alpha\}_2 = 0.$$

Then, there exists $n \in 2\mathbb{N}$ such that $q|q_n$ and, for this q_n , it holds that

$$0 < q_n^\beta \cdot \{q_n \alpha\}_2 := \epsilon \leq \epsilon'. \tag{24}$$

Since n is even, the theory of continued fractions implies that $\{q_n \alpha\}_2 > 0$. We obtain immediately that

$$\alpha = \frac{p_n}{q_n} + \frac{\epsilon}{q_n \cdot q_n^\beta} \tag{25}$$

for some $p_n \in \mathbb{Z}$ with $(p_n, q_n) = 1$.

Write $q_n = q \cdot q'_n$ for some $q'_n \in \mathbb{N}$ and choose $p'_n \in \{1, \dots, q-1\}$ such that $p'_n \cdot p_n \equiv p \pmod{q}$. Then,

$$\begin{aligned} (p'_n \cdot q'_n)^\beta \cdot \{p'_n q'_n \cdot \alpha - \rho\}_2 &\stackrel{(25)}{=} (p'_n \cdot q'_n)^\beta \cdot \left\{ p'_n \cdot q'_n \cdot \left(\frac{p_n}{q_n} + \frac{\epsilon}{q_n \cdot q_n^\beta} \right) - \frac{p}{q} \right\}_2 \\ &= \left(\frac{p'_n \cdot q'_n}{q_n} \right)^{1+\beta} \cdot \epsilon \stackrel{(p < q)}{\leq} \epsilon. \end{aligned}$$

This implies that $\mu_+ \left(\alpha, \beta, \frac{p}{q} \right) \leq \epsilon'$. Therefore, $\mu_+ \left(\alpha, \beta, \frac{p}{q} \right) = 0$ as ϵ' is chosen arbitrary.

\Leftarrow : Assume that $\mu_+ \left(\alpha, \beta, \frac{p}{q} \right) = 0$. Without loss of generality, assume that $p/q \in [0, 1)$. We prove first the case $q \geq 2$. Fix

$$0 < \epsilon_0 < \frac{1}{2 \cdot q^{2+\beta}}. \quad (26)$$

By assumption, there exists $k \in \mathbb{N}$ such that

$$0 < k^\beta \cdot \{k\alpha - \theta\}_2 \leq \epsilon_0.$$

Set

$$\epsilon = k^\beta \cdot \{k\alpha - \theta\}_2.$$

Then,

$$\{k\alpha\} = \theta + \frac{\epsilon}{k^\beta} = \frac{p}{q} + \frac{\epsilon}{k^\beta}. \quad (27)$$

From inequality (26), one obtains that $q\epsilon/k^\beta \in [-\frac{1}{2}, \frac{1}{2})$. Therefore, equation (27) yields

$$\{qk \cdot \alpha\}_2 = \frac{q\epsilon}{k^\beta}.$$

Let $n \in \mathbb{N}$ be such that $q_n \leq qk < q_{n+1}$. Then, it holds that $\|q_n \alpha\| \leq \frac{q\epsilon}{k^\beta}$. Also,

$$\alpha = \frac{p_n}{q_n} + (-1)^n \cdot \frac{\epsilon'}{q_n^{1+\beta}}$$

for some $\epsilon' > 0$ and $p_n \in \mathbb{N}$ with $(p_n, q_n) = 1$. Indeed, from the theory of continued fractions, we have that $\alpha = \frac{p_n}{q_n} + (-1)^n \cdot \frac{\eta}{q_n^2}$ for some $\eta > 0$. Setting $\epsilon' = q_n^{\beta-1} \cdot \eta$ implies that

$$\frac{\epsilon'}{q_n^\beta} \leq \frac{q\epsilon}{k^\beta}. \quad (28)$$

Let us prove that $q|q_n$ and $n \in 2\mathbb{N}$. Choosing ϵ_0 sufficiently small yields that $q_n \geq q$. Therefore, without loss of generality, assume for the rest of the proof that $q_n \geq q$.

Assume that $q \nmid q_n$. Then, for every $j \in \mathbb{N}$,

$$\left\| \frac{j}{q_n} - \frac{p}{q} \right\| \geq \frac{1}{qq_n}$$

since $\frac{j}{q_n} \not\equiv \frac{p}{q} \pmod{1}$ for all $j \in \mathbb{Z}$. Thus, in order for the relations

$$k \cdot \left(\frac{p_n}{q_n} + \frac{\epsilon'}{q_n^{1+\beta}} \right) = \theta + \frac{\epsilon}{k^\beta} \pmod{1} \quad \text{or} \quad k \cdot \left(\frac{p_n}{q_n} - \frac{\epsilon'}{q_n^{1+\beta}} \right) = \theta + \frac{\epsilon}{k^\beta} \pmod{1} \quad (29)$$

to hold, it is necessary that

$$k \cdot \frac{\epsilon'}{q_n^{1+\beta}} \geq \frac{1}{2q \cdot q_n}. \quad (30)$$

However,

$$\begin{aligned} k \cdot \frac{\epsilon'}{q_n^{1+\beta}} &\stackrel{(28)}{\leq} kq \cdot \frac{\epsilon}{q_n \cdot k^\beta} \stackrel{\beta \geq 1}{\leq} q \cdot \frac{\epsilon}{q_n} \\ &\stackrel{\epsilon < \epsilon_0}{\leq} \frac{1}{2q_n \cdot q^{1+\beta}} \leq \frac{1}{2q_n \cdot q}. \end{aligned}$$

This contradiction establishes that $q|q_n$. Set now $\delta = (-1)^n \cdot \epsilon'$ and write

$$\alpha = \frac{p_n}{q_n} + \frac{\delta}{q_n^{1+\beta}}.$$

If n is odd, then relation (29) holds only if inequality (30) is true. This leads again to a contradiction, establishing this way that n is even and, in particular, that

$$\alpha = \frac{p_n}{q_n} + \frac{\epsilon'}{q_n^{1+\beta}} \quad \text{with} \quad k \cdot \frac{\epsilon'}{q_n^{1+\beta}} = \frac{\epsilon}{k^\beta}.$$

Finally, one has that

$$\begin{aligned} \liminf_{\substack{j \rightarrow +\infty, \\ q|q_j}} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 &\leq q_n^\beta \cdot \{q_n\alpha\}_2 \stackrel{(28)}{\leq} q_n^\beta \cdot \frac{q\epsilon}{k^\beta} \\ &\stackrel{q_n \leq qk}{\leq} q^{1+\beta} \cdot \epsilon \leq q^{1+\beta} \cdot \epsilon_0. \end{aligned}$$

By letting $\epsilon_0 \rightarrow 0$, one obtains that

$$\lim_{\substack{j \rightarrow +\infty, \\ q|q_j}} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 = 0.$$

It remains to establish the case $q = 1$; that is, when $\theta \in \mathbb{Z}$. Assume that $\mu_+(\alpha, \beta) = 0$. The goal is to prove that

$$\lim_{j \rightarrow +\infty} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 = 0.$$

The following lemma immediately implies the claim.

LEMMA 5.1. *Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \geq 1$, the relations*

$$\mu_+(\alpha, \beta) = \liminf_{j \rightarrow +\infty} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 \quad \text{and} \quad \mu_-(\alpha, \beta) = \liminf_{j \rightarrow +\infty} -q_{2j-1}^\beta \cdot \{q_{2j-1}\alpha\}_2$$

hold.

Proof. We prove the first relation as the second one follows in the same way. To this end, fix an even integer n . It is enough to prove that

$$(q_n + l \cdot q_{n+1})^\beta \cdot \{(q_n + l \cdot q_{n+1}) \cdot \alpha\}_2 \geq q_n^\beta \cdot \{q_n \alpha\}_2 \quad \text{for every } l \in \llbracket 1, a_{n+2} - 1 \rrbracket$$

since

$$0 < \{q_{n+2}\alpha\}_2 < \{k\alpha\}_2 \leq \{q_n\alpha\}_2 \quad (31)$$

with $k < q_{n+2}$ if and only if $k = q_n + l \cdot q_{n+1}$ for some $l \in \llbracket 0, a_{n+2} - 1 \rrbracket$. The claim is proved at the end of the proof. Thus, for every $l \in \llbracket 1, a_{n+2} - 1 \rrbracket$, we have

$$\begin{aligned} (q_n + l \cdot q_{n+1})^\beta \cdot \{(q_n + l \cdot q_{n+1}) \cdot \alpha\}_2 &= (q_n + l \cdot q_{n+1})^\beta \cdot (\{q_n \alpha\}_2 + l \{q_{n+1} \alpha\}_2) \\ &\geq (a_{n+2} \cdot l \cdot q_n)^\beta \cdot \left(\frac{a_{n+2} - l}{a_{n+2}} \right) \cdot \{q_n \alpha\}_2 \\ &\geq a_{n+2}^{\beta-1} \cdot l^\beta \cdot (a_{n+2} - l) \cdot q_n^\beta \cdot \{q_n \alpha\}_2 \\ &\geq q_n^\beta \cdot \{q_n \alpha\}_2. \end{aligned}$$

It remains to prove claim (31) in order to complete the proof of the lemma.

Fix $k \in \llbracket 1, q_{n+2} - 1 \rrbracket$ such that $\|k\alpha\| \leq \|q_n \alpha\|$. The Ostrowski expansion of k is of the form $k = \sum_{j=0}^{n+1} e_j \cdot q_j$. Let m be the minimal natural number in $\llbracket 0, n+1 \rrbracket$ such that $e_m \geq 1$. If $m \leq n-2$, then, from equation (15), it can easily be deduced that

$$\left\| \sum_{j=m}^{n+1} e_j \cdot q_j \cdot \alpha \right\| > \|q_{j+2}\alpha\| \geq \|q_n \alpha\|.$$

This is a contradiction. If $m = n-1$, then, it cannot hold that $e_m \geq 2$ as otherwise, from equation (15), one obtains that $\|k\alpha\| \geq \|q_{n-1}\alpha\|$. Thus, $e_{n-1} = 1$, which implies from the definition of the Ostrowski expansion that $e_n \leq a_{n+2} - 1$. However, one has that

$$\|(q_{n-1} + e_n q_n + e_{n+1} q_{n+1}) \alpha\| \geq \|(q_{n-1} + e_n q_n) \alpha\| \underset{(e_n \leq a_{n+2} - 1)}{>} \|q_n \alpha\|,$$

which yields again a contradiction. If $m = n$ then $e_{n+1} \leq a_{n+2} - 1$. We have that $e_n \leq 1$ since otherwise $\|(e_n q_n + e_{n+1} q_{n+1}) \alpha\| > \|q_n \alpha\|$. Therefore, we have proved that if $\|k\alpha\| \leq \|q_n \alpha\|$, then $k \in \bigcup_{l=1}^{a_{n+2}-1} \{q_n + l q_{n+1}\} \cup \{l q_{n+1}\}$.

Finally, one has that for every $l \in \llbracket 1, a_{n+2} - 1 \rrbracket$, $\{lq_{n+1} \cdot \alpha\}_2 < 0$. Therefore, inequality (31) holds if and only if $k = q_n + lq_{n+1}$ with $l \in \llbracket 1, a_{n+2} - 1 \rrbracket$. The claim is established, which completes the proof of Lemma 5.1. \square

This concludes the proof of Theorem 1.5. \square

We end this section by showing that the quantities $\mu_{\pm}(\alpha, \beta, \rho)$ (cf. Definition 1.1) cannot be replaced in the statements of Theorems 1.2 and 1.5 with $\mu(\alpha, \beta, \rho)$ (cf. Definition 1.1).

PROPOSITION 5.2. *Given $\beta \geq 1$ and a rational number ρ , there exists a real α such that $\mu_+(\alpha, \beta, \rho) = 0$ and $\mu_-(\alpha, \beta, \rho) > 0$. Conversely, there exists a real α such that $\mu_-(\alpha, \beta, \rho) = 0$ and $\mu_+(\alpha, \beta, \rho) > 0$.*

Proof. From Theorem 1.5 it is enough to prove the claim when $\rho = 0$.

Let $\alpha = [a_0; a_1, a_2, \dots] \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number whose partial quotients will be defined recursively. Set

$$y_n = [0; a_{n+1}, a_{n+2}, \dots] \in [0, 1), \quad n \in \mathbb{N}_0.$$

Let $(p_n/q_n)_{n \in \mathbb{N}}$ be the sequence of convergents of α . A standard analysis of the continued fraction expansion of α yields that

$$\{q_{n-1} \cdot \alpha\}_2 = \frac{(-1)^{n-1}}{q_n + y_n \cdot q_{n-1}} \asymp \frac{(-1)^{n-1}}{q_n} \asymp \frac{(-1)^{n-1}}{a_n \cdot q_{n-1}}.$$

Therefore,

$$q_{n-1} \cdot \{q_{n-1} \cdot \alpha\}_2 \asymp \frac{(-1)^{n-1}}{a_n}.$$

Define the sequence $(a_n)_{n \in \mathbb{N}}$ as follows: for odd $n \in \mathbb{N}$, choose $a_n = \lfloor n \cdot q_{n-1}^{\beta-1} \rfloor$ and for even $n \in \mathbb{N}$, choose $1 \leq a_n \leq C$ for some arbitrary predefined positive constant $C \geq 1$. Then, $\mu_+(\alpha, \beta) = 0$ and $\mu_-(\alpha, \beta) > 0$. \square

6. Proof of Theorem 1.6

Proof of Theorem 1.6. For each $j \geq 0$, write $e_j = e_j(\rho)$ and set \mathfrak{D} as defined in (9). Also, let $(w_k)_{k \in \mathbb{N}}$ be the sequence defined in (5). We prove that

$$\lim_{\substack{k \rightarrow +\infty, \\ k \notin \mathfrak{D}}} |w_k| = +\infty.$$

This, in turn, implies that the sequence $(w_k)_{k \in \mathbb{N}}$ and the subsequence $(w_k)_{k \in \mathfrak{D}}$ have the same finite limit points.

Fix $h \in \mathbb{R}$. By assumption, given $n_h \in \mathbb{N}$ large enough and given $n \geq n_h$, it holds that $q_n^\beta \cdot \|q_n \alpha\| \geq 4|h|$. Fix $n \geq n_h$ and let $k \in \llbracket \kappa_n, \kappa_{n+1} \rrbracket$. If $k \in \llbracket \kappa_n + q_{n+1} \rrbracket \setminus \mathcal{N}_\rho \left(\kappa_n + q_{n+1}, \alpha, \frac{\|q_n \alpha\|}{1 + e_n^\beta} \right)$, then from the definitions of the inhomogeneous Bohr set (7) and of the sequence (8), one obtains that

$$|w_k| \geq \kappa_n^\beta \cdot \frac{\|q_n \alpha\|}{1 + e_n(\rho)^\beta} \geq e_n(\rho)^\beta q_n^\beta \cdot \frac{\|q_n \alpha\|}{1 + e_n(\rho)^\beta} \geq 2|h|.$$

Similarly, if $k \in \llbracket \kappa_n + q_{n+1}, \kappa_{n+1} \rrbracket \setminus \mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1} \alpha\|)$, then

$$|w_k| \geq q_{n+1}^\beta \cdot \|q_{n+1} \alpha\| \geq 4|h|,$$

whence the claim.

We now prove inclusions (10) in the statement of Theorem 1.6. It follows from the definition of the Bohr set (7) that

$$\mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1} \alpha\|) \cap \llbracket 1, \kappa_n \rrbracket \subseteq \mathcal{N}_\rho(\kappa_n, \alpha, \|q_n \alpha\|)$$

and

$$\mathcal{N}_\rho \left(\kappa_n + q_{n+1}, \alpha, \frac{\|q_n \alpha\|}{1 + e_n^\beta} \right) \cap \llbracket 1, \kappa_n \rrbracket \subseteq \mathcal{N}_\rho(\kappa_n, \alpha, \|q_n \alpha\|).$$

Therefore, it is enough to show, on the one hand that

$$\{\kappa_{n+1}\} \subseteq \mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1} \alpha\|) \cap \llbracket \kappa_n + q_{n+1}, \kappa_{n+1} \rrbracket \quad (32)$$

and

$$\mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1} \alpha\|) \cap \llbracket \kappa_n + q_{n+1}, \kappa_{n+1} \rrbracket \subseteq \kappa_n + \bigcup_{l=0}^2 \{(e_{n+1} - l) \cdot q_{n+1}\} \quad (33)$$

and, on the other, that

$$\mathcal{N}_\rho \left(\kappa_n + q_{n+1}, \alpha, \frac{\|q_n \alpha\|}{1 + e_n^\beta} \right) \cap \llbracket \kappa_n, \kappa_n + q_{n+1} \rrbracket \subseteq \kappa_n + \bigcup_{l=0}^1 \{(l+1)q_n\} \cup \{q_{n+1} - lq_n\}. \quad (34)$$

As far as inclusion (32) is concerned, it is easily seen that, for every $n \in \mathbb{N}_0$, it holds that $\kappa_{n+1} \in \mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1} \alpha\|)$. As for inclusion (33), there is nothing to prove if $e_{n+1} = 0$. Therefore, without loss of generality, assume that $e_{n+1} \geq 1$ and $k \in \mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1} \alpha\|) \cap \llbracket \kappa_n + q_{n+1}, \kappa_{n+1} \rrbracket$. Set

$$s_{n+1} = \sum_{j=n+2}^{+\infty} e_j(\rho) \{q_j \alpha\}_2.$$

Inequality (14) yields that $|s_{n+1}| \leq \|q_{n+1} \alpha\|$. Since from the triangle inequality, $\|k\alpha - e_{n+1} \{q_{n+1} \alpha\}_2\| \leq \|k\alpha - e_{n+1} \{q_{n+1} \alpha\}_2 - s_{n+1}\| + \|s_{n+1}\| \leq 2\|q_{n+1} \alpha\|$,

one obtains that

$$\mathcal{N}_\rho(\kappa_{n+1}, \alpha, \|q_{n+1}\alpha\|) \cap \llbracket \kappa_n + q_{n+1}, \kappa_{n+1} \rrbracket \subseteq \kappa_n + \mathcal{N}_0(e_{n+1}q_{n+1}, \alpha, 2 \cdot \|q_{n+1}\alpha\|).$$

In turn, this easily implies that

$$\mathcal{N}_\rho(e_{n+1}q_{n+1}, \alpha, 2 \cdot \|q_{n+1}\alpha\|) \subseteq \left(\bigcup_{l=0}^2 \{(e_{n+1} - l) \cdot q_{n+1}\} \right) \cup \{q_{n+1} - q_n\}.$$

Furthermore, $q_{n+1} - q_n \notin \llbracket q_{n+1}, e_{n+1}q_{n+1} \rrbracket$, which gives the inclusion in (33).

As for the inclusion in (34), we have that

$$\mathcal{N}_\rho\left(\kappa_n + q_{n+1}, \alpha, \frac{\|q_n\alpha\|}{1 + e_n^\beta}\right) \subseteq \kappa_n + \mathcal{N}_0(q_{n+1}, \alpha, 2 \|q_n\alpha\|).$$

It follows easily that

$$\mathcal{N}_\rho(q_{n+1}, \alpha, 2 \|q_n\alpha\|) \subseteq \{q_n, 2q_n, q_{n+1} - q_n, q_{n+1}\}.$$

Therefore,

$$\{\kappa_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{D}$$

and

$$\mathfrak{D} \subseteq \bigcup_{n=0}^{+\infty} \left(\kappa_n + \bigcup_{l=0}^2 \{(e_n - l) \cdot q_{n+1}\} \right) \cup \left(\kappa_n + \bigcup_{l=0}^1 \{(l + 1)q_n, q_{n+1} - lq_n\} \right).$$

The proof is complete. □

This work leaves open the question of determining the density properties of the oscillating sequence (1) defined by more general growth functions than those of the form (2).

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