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A TYPICAL NUMBER IS EXTREMELY NON-NORMAL

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ABSTRACT. Fix a positive integer $N \geq 2$. For a real number $x \in [0,1]$ and a digit $i \in \{0,1,\ldots,N-1\}$, let $\Pi_i(x,n)$ denote the frequency of the digit i among the first n N-adic digits of x. It is well-known that for a typical (in the sense of Baire) $x \in [0,1]$, the sequence of digit frequencies diverges as $n \to \infty$. In this paper we show that for any regular linear transformation T there exists a residual set of points $x \in [0,1]$ such that the T-averaged version of the sequence $\big(\Pi_i(x,n)\big)_n$ also diverges significantly.

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1. Statement of results

Fix a positive integer $N \geq 2$. Throughout this paper, we will consider the unique, non-terminating, base N expansion of a number $x \in [0,1]$, written as

$$x = \frac{d_1(x)}{N} + \frac{d_2(x)}{N^2} + \cdots + \frac{d_n(x)}{N^n} + \cdots$$

with $d_i(x) \in \{0,1,\ldots,N-1\}$. For each digit $i \in \{0,1,\ldots,N-1\}$, we will write

$$\Pi_i(x,n) := \frac{1}{n}\,|\{1 \leq j \leq n \;:\; d_j(x) = i\}|$$

for the frequency of the digit i among the first n digits of x.

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Let Δ_N denote the family of N-dimensional probability vectors, that is

$$\Delta_N := \left\{ (p_0, p_1, \ldots, p_{N-1}) \;:\; p_i \geq 0,\; \sum_{i=0}^{N-1} p_i = 1 \right\}$$

and write

$$\mathbf{\Pi}(x,n) = \big(\Pi_0(x,n), \Pi_1(x,n), \ldots, \Pi_{N-1}(x,n)\big)$$

for the vector of frequencies of digits. Then for every natural number n the vector of frequencies $\mathbf{\Pi}(x,n)$ clearly lies in the simplex Δ_N, and consequently, the set of accumulation points with respect to $\|.\|_1$ of the sequence $\big(\mathbf{\Pi}(x,n)\big)_n$ is a subset of Δ_N.

Borel's famous normal number theorem [4] states that for Lebesgue almost every point in $[0,1]$ its vector of frequencies converges towards the uniform probability vector $\big(\frac{1}{N}, \ldots, \frac{1}{N}\big)$ with respect to $\|.\|_1$ as $n \to \infty$. More generally, Birkhoff's Ergodic Theorem [3] tells us that for each probability measure μ on $[0,1]$ which is invariant under the transformation $x \to Nx \mod 1$, the sequence $\big(\mathbf{\Pi}(x,n)\big)$ converges to a specific probability vector $p_\mu \in \Delta_N$ for μ almost every $x \in [0,1]$ as $n \to \infty$. Yet still from a topological viewpoint there is a completely different picture.

We recall that in a metric space X, a set R is called residual if its complement is of the first category. Also, recall that we say that a typical element $x \in X$ has property P if the set $R := \{x \in X \;:\; x \text{ has property } P\}$ is residual. We refer the reader to Oxtoby [15] for more details. Recently, the study of the limiting behaviour of the sequence of frequencies $(\Pi_i(x,n))_n$ from a topological viewpoint has enjoyed a great interest from many authors [1, 5, 11, 16–21]. In particular, perhaps unexpectedly, Olsen in [11] showed that for a typical $x \in [0,1]$, the set of accumulation points of $\big(\mathbf{\Pi}(x,n)\big)_n$ is all of Δ_N.

Theorem 1.1. *The set*

$$\Big\{x \in [0,1] \;:\; \textit{The set of the accumulation points of } \big(\mathbf{\Pi}(x,n)\big)_n \textit{ equals } \Delta_N \Big\}$$

is residual.

The result above has inspired a series of papers [8–10, 12, 13] that appeared later in the literature extending Theorem 1.1. These later results were sometimes considering different settings such us continued fraction expansions, iterated function systems, finite or countable Markov partitions and other times enhancing the way in which the vectors of frequencies were diverging. In this paper we offer a generalisation of the second form. Namely, Theorem 1.3 below states that for any regular linear transformation T, there exists a residual set of points x whose set of accumulation points of their T-averaged vector of frequencies $\big(\Pi^T(x,m)\big)_m$ includes the simplex of all N-dimensional probability vectors. The precise definitions are given below.

Given a divergent sequence, by considering an averaged version of this sequence one may succeed in producing a convergent sequence. In this paper we prove a somewhat unexpected result that strengthens Theorem 1.1 considerably. To do that we need the following definition from [7].

Definition 1.2. An $\mathbb{N}\times\mathbb{N}$ real valued matrix $T=[c_{m,n}]$, is called a *linear transformation* if for any real sequence $(s_n)_n$ and any natural number m, we have that $t_m=\sum_{n\geq 1} c_{m,n}s_n$ converges to a real number. The sequence $(t_m)_m$ is called the T-averaged version of $(s_n)_n$ and if $\lim_{m\to\infty} t_m$ exists we call it the T-value of $(s_n)_n$.

Example. The linear transformation $T=[c_{m,n}]_{\mathbb{N}\times\mathbb{N}}$ with

$$c_{m,n}=\begin{cases}\frac{1}{m} & \text{if } 1\leq n\leq m,\\ 0, & \text{otherwise},\end{cases}$$

corresponds to considering the sequence of the natural averages of a sequence.

Namely, we show that for any linear transformation T, the T-averaged version of the sequence $\big(\mathbf{\Pi}(x,n)\big)_n$ accumulates at every point in Δ_N for a residual set of points in $[0,1]$. The special case of Cesàro averages was considered in [8]. To state this result more precisely, we define a regular linear transformation below.

Linear transformations are used to help generalise the sense of a limit of a sequence, in order to assign a *sensible* value of a limit to a divergent sequence. However, if a sequence is already convergent we would like the averaged version of this sequence to also converge to the same limit. Hence we call a linear transformation $T=[c_{m,n}]$ *regular*, if T maps all convergent sequences to their original limit, that is T is regular if

$$t_m\to l \quad \text{as} \quad m\to\infty \qquad \text{whenever} \qquad s_k\to l \quad \text{as} \quad k\to\infty.$$

Fix a regular linear transformation $T=[c_{m,n}]$ and consider for $x\in[0,1]$ the T-averaged version of the sequences of digit frequencies

$$\Pi_i^T(x,m):=\sum_{n=1}^{\infty} c_{m,n}\,\Pi_i(x,n) \qquad \text{for } 0\leq i\leq N-1.$$

Then, the T-averaged version of the vector of digit frequencies is given by the sequence with m-th term

$$\mathbf{\Pi}^T(x,m):=\big(\Pi_0^T(x,m),\Pi_1^T(x,m),\ldots,\Pi_{N-1}^T(x,m)\big).$$

We now state our main theorem.

Theorem 1.3. *For any regular linear transformation T the set*

$$R_T := \left\{x \in [0,1] : \left(\mathbf{\Pi}^T(x,m)\right)_m \textit{ accumulates at every point of } \Delta_N\right\},$$

is residual.

Remark 1. Results similar in nature but in somewhat more general settings were obtained independently by Olsen and West in [13] and Aveni and Leonetti in [2].

The proof of Theorem 1.3 is given in Section 2 below. Before proceeding we make two final remarks concerning our result. Firstly, fix a countable family of regular linear transformations $\mathcal{T} = \{T_1, T_2, \dots\}$ (Examples include Hölder means, Cesàro averages,...). Notice that it clearly follows from Theorem 1.3 that the countable intersection $\mathcal{R}_{\mathcal{T}} := \bigcap_{T\in\mathcal{T}} R_T$ is also residual and hence, in particular, we recover the main theorem of [8].

Additionally, also in terms of dimensions, the size of R_T varies between *very big* and *very small* depending on the exact viewpoint. Below we write $\dim_H$ and $\dim_P$ for the Hausdorff dimension and the packing dimension, respectively. The reader is referred to [6] for the definitions of the Hausdorff and the packing dimensions. It follows from [14] that $\dim_H R_T = 0$. On the other hand, using Theorem 1.3 we can find the packing dimension of the set R_T. We prove the following result whose analogue appeared in [11].

Corollary 1.4. *For any regular linear transformation T the set R_T has full packing dimension, that is*

$$\dim_P R_T = 1.$$

We postpone our proof until the next section.

2. Proof of Theorem 1.3

Throughout the proof, we will work with a subset of $[0,1]$, namely

$$\mathbb{I} := [0,1] \setminus \{x \in [0,1] \;:\; x \text{ has a terminating } N\text{-adic expansion}\}.$$

For brevity, write

$$\mathbb{D} := \mathbb{Q}^N \cap \Delta_N \setminus \{(1,0,\dots,0),(0,\dots,0,1)\}$$

where we excluded two particular vectors for technical reasons, which become apparent in the proof of Claim 2.4.

Let $\psi : \mathbb{N} \to \mathbb{N}$ be any increasing function such that $\psi(n) > n^2$ for all $n \in \mathbb{N}$, and write ψ^m for the composition of ψ with itself m times. For $h \in \mathbb{N}$ we define the property $P_{h,\psi}$ as follows.

Definition 2.1. We say that a sequence $(\mathbf{x}_n)_n$ in $\mathbb{R}^N$ has property $P_{h,\psi}$ if for all $i \in \mathbb{N}$, $m \in \mathbb{N}$ and all $\mathbf{q} \in \mathbb{D}$, there exists $j \in \mathbb{N}$ satisfying the following:

- $j \geq i$,
- $\frac{j}{\psi(j)} \leq \frac{1}{h}$,
- if $j \leq n \leq \psi^m(j)$, then $\|\mathbf{x}_n - \mathbf{q}\|_1 < \frac{1}{h}$.

Our proof of Theorem 1.3 will consist of three lemmas.

(1) First we will prove that for any $h \in \mathbb{N}$ and any increasing $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(n) > n^2$ for all $n \in \mathbb{N}$, the following set is residual:

$$A_{h,\psi} := \left\{ x \in \mathbb{I} \ : \ \big(\mathbf{\Pi}(x,n)\big)_n \text{ has property } P_{h,\psi} \right\}. \tag{1}$$

(2) Then we will show that if T is any regular linear transformation and $h \in \mathbb{N}$, then there exists $\psi_{h,T} : \mathbb{N} \to \mathbb{N}$ such that if $x \in A_{6Mh,\psi_{h,T}}$ with M depending on T, then $\big(\mathbf{\Pi}^T(x,m)\big)_m$ has property $P_{h,\psi_{h,T}}$.

(3) Finally, setting $A_T := \bigcap_{h\in\mathbb{N}} A_{6Mh,\psi_{h,T}}$ we will show that

$$A_T \subset R_T$$

(recall that R_T is defined in the statement of Theorem 1.3).

Lemma 2.2. *For all $h \in \mathbb{N}$ and all increasing $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(n) > n^2$ for all $n \in \mathbb{N}$, the set $A_{h,\psi}$ is residual (recall that $A_{h,\psi}$ is defined in* (1)*).*

Proof. For fixed $i \in \mathbb{N}$, $m \in \mathbb{N}$ and $\mathbf{q} \in \mathbb{D}$, we define property $P_{h,\psi,i,m,\mathbf{q}}$, as follows. We say that a sequence $(\mathbf{x}_n)_n$ has property $P_{h,\psi,i,m,\mathbf{q}}$ if there exists $j \in \mathbb{N}$ satisfying:

(1) $j \geq i$,

(2) $\frac{j}{\psi(j)} < \frac{1}{h}$,

(3) if $j < n < \psi^m(j)$, then $\|\mathbf{x}_n - \mathbf{q}\|_1 < \frac{1}{h}$.

Let $G_{h,\psi,i,m,\mathbf{q}} := \left\{ x \in \mathbb{I} \ : \ \big(\mathbf{\Pi}(x,n)\big)_n \text{ has property } P_{h,\psi,i,m,\mathbf{q}} \right\}$. Clearly,

$$\bigcap_{i\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcap_{\mathbf{q}\in\mathbb{D}} G_{h,\psi,i,m,\mathbf{q}} = A_{h,\psi}.$$

Claim 2.3. *$G_{h,\psi,i,m,\mathbf{q}}$ is open.*

Proof. Let $x \in G_{h,\psi,i,m,\mathbf{q}}$. Then, there exists a positive integer j such that $j \geq i$, $\frac{j}{\psi(j)} < \frac{1}{h}$, and if $j < n < \psi^m(j)$, then $\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 < \frac{1}{h}$. We now choose δ to equal $\frac{1}{N^a}$ where $a > \psi^{m+1}(j) + 1$. Further, since $x \in \mathbb{I}$ we can also insist that $a \in \mathbb{N}$ is such that the $(a-1)$-st and $(a-2)$-nd digits of x are not both either 0 or $N-1$. Then all $y \in B(x,\delta)$ have their first $\psi^m(j)$ digits the same as x, and so $B(x,\delta) \subset G_{\psi,h,i,m,\mathbf{q}}$. This completes the proof of Claim 2.3. $\square$

Claim 2.4. *$G_{h,\psi,i,m,\mathbf{q}}$ is dense.*

Proof. Let $x \in \mathbb{I}$ and $\delta > 0$. We must now find $y \in B(x,\delta) \cap G_{h,\psi,i,m,\mathbf{q}}$. Let $t \in \mathbb{N}$ be such that $\frac{1}{N^t} < \delta$. We can clearly choose a positive integer $s \in \mathbb{N}$ and $z_1, \dots, z_s \in \{0,1,\dots,N-1\}$ such that if $z = \frac{z_1}{N} + \frac{z_2}{N^2} + \cdots + \frac{z_s}{N^s}$, then $\mathbf{\Pi}(z,s) = \mathbf{q}$. Let

$$y = \frac{d_1(x)}{N} + \cdots + \frac{d_t(x)}{N^t} + \sum_{d=0}^{\infty}\Big(\frac{z_1}{N^{t+ds+1}} + \frac{z_2}{N^{t+ds+2}} + \cdots + \frac{z_s}{N^{t+ds+s}}\Big).$$

Then $y \in B(x,\delta)$ (as the first t digits of y are the same as those of x).

Next we show that $y \in G_{h,\psi,i,m,\mathbf{q}}$. All z_i's cannot be 0 or $N-1$, because we excluded the vectors $(1,0,\dots,0)$, $(0,\dots,0,1)$ from $\mathbb{D}$. Therefore, y has a non-terminating N-adic expansion. Choose j big enough such that

$$\frac{j}{\psi(j)} < \frac{1}{h} \quad \text{and} \quad j \geq \max\Big\{i, t, hN \max_{0 \leq \ell \leq N-1}\Big\{N_\ell(z,s)\Big(2 + \frac{t}{s}\Big) + N_\ell(y,t)\Big\}\Big\},$$

where $N_\ell(x,n) = |\{1 \leq k \leq n \ : \ d_k(x) = \ell\}|$. Fix a positive integer n with $j < n < \psi^m(j)$ and observe that we can find integers r and b_ℓ with $0 \leq r < s$ and $0 \leq b_\ell < N_\ell(z,s)$, such that $n = t + \left[\frac{n-t}{s}\right] s + r$, and $N_\ell(y,n) = N_\ell(y,t) + \left[\frac{n-t}{s}\right] N_\ell(z,s) + b_\ell$. We now have

$$\begin{aligned}
\|\mathbf{\Pi}(y,n) - \mathbf{q}\|_1 &= \sum_{\ell=0}^{N-1}\left|\frac{N_\ell(y,n)}{n} - \frac{n\frac{N_\ell(z,s)}{s}}{n}\right| \\
&= \sum_{\ell=0}^{N-1}\left|\frac{N_\ell(y,t) + [\frac{n-t}{s}]N_\ell(z,s) + b_\ell}{n} - \frac{(t + [\frac{n-t}{s}]s + r)\frac{N_\ell(z,s)}{s}}{n}\right| \\
&\leq \sum_{\ell=0}^{N-1}\left(\frac{\frac{N_\ell(z,s)}{s}(t+r) + N_\ell(y,t) + b_\ell}{n}\right) \\
&< \sum_{\ell=0}^{N-1}\left(\frac{\frac{N_\ell(z,s)}{s}(t+r) + N_\ell(y,t) + N_\ell(z,s)}{n}\right) \\
&< \sum_{\ell=0}^{N-1}\left(\frac{N_\ell(z,s)(2+\frac{t}{s}) + N_\ell(y,t)}{j}\right) \leq \frac{1}{h}.
\end{aligned}$$

This shows that $y \in G_{h,\psi,i,m,\mathbf{q}}$ and completes the proof of Claim 2.4. $\square$

It follows from Claim 2.3 and Claim 2.4, that $A_{h,\psi}$ is the countable intersection of open and dense sets, and hence residual. This completes the proof of Lemma 2.2. $\square$

Lemma 2.5. *For any regular linear transformation T and any $h \in \mathbb{N}$ we can:*

(1) *construct an increasing function $\psi_{h,T} : \mathbb{N} \to \mathbb{N}$ such that $\psi(n) > n^2$ for all $n \in \mathbb{N}$ and*
(2) *show that for some M depending on T, we have that for each*

$$x \in A_{6Mh,\psi_{h,T}} \text{ the sequence } \left(\mathbf{\Pi}^T(x,m)\right)_m \text{ has property } P_{h,\psi_{h,T}}.$$

Before proceeding to prove Lemma 2.5, let us first recall a very useful characterisation result for *regular* linear transformations. Given a linear transformation T corresponding to an $\mathbb{N}\times\mathbb{N}$ real valued matrix $[c_{m,n}]$ and a sequence $(s_n)_n$ we are interested in the behaviour of its averaged version, that is the sequence whose mth term is given by $t_m = \sum_{n\geq 1} c_{m,n}s_n$.

Proposition 2.6 ([7]). *A linear transformation $T = [c_{m,n}]_{\mathbb{N}\times\mathbb{N}}$ is regular if and only if the following conditions are satisfied:*

(1) *there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $\sum_{n=1}^{\infty} |c_{m,n}| < M$,*
(2) *for all $n \in \mathbb{N}$, $\lim_{m\to\infty} c_{m,n} = 0$ and*
(3) *we have that $\lim_{m\to\infty} \sum_{n=1}^{\infty} c_{m,n} = 1$.*

Proof of Lemma 2.5. Fix $h \in \mathbb{N}$ and $T = [c_{m,n}]$ a regular linear transformation. Firstly, we construct $\psi_{h,T} : \mathbb{N} \to \mathbb{N}$. To do that recall that from Proposition 2.6 we can associate to T a number $K \in \mathbb{N}$ and sequences of natural numbers $(M_k)_k$ and $(N_k)_k$ such that:

(1) for each $m \in \mathbb{N}$ we have $\sum_{n\geq M_m} |c_{m,n}| \leq \frac{1}{12h}$,
(2) for each $n \in \mathbb{N}$ we have $|c_{m,n}| < \frac{1}{2^{n+10}h}$ whenever $m > N_n$ and
(3) for $m > K$ we have $|1 - \sum_{n=1}^{\infty} c_{m,n}| < \frac{1}{2h}$.

Recall also $M \in \mathbb{N}$ the uniform bound of the absolute row sums of T. Having fixed these constants depending on the linear transformation T and the natural number h, we proceed by defining our function $\psi_{h,T} : \mathbb{N} \to \mathbb{N}$ by $\psi_{h,T}(0) = K+2$ and recursively for $n \geq 1$

$$\psi_{h,T}(n) := M_n + N_n + n^2 + \psi_{h,T}(n-1).$$

This function $\psi_{h,T} : \mathbb{N} \to \mathbb{N}$ is increasing and crucially

$$\psi_{h,T}(n) > \max\{n^2, M_n, N_n, K\} + 1 \quad \text{for all} \quad n \geq 0.$$

Then, fix $i \in \mathbb{N}$, $m \in \mathbb{N}$ and $\mathbf{q} \in \mathbb{D}$. For $x \in A_{6Mh,\psi_{h,T}}$, the sequence $\big(\mathbf{\Pi}(x,n)\big)_n$ satisfies property $P_{6Mh,\psi_{h,T}}$. We can therefore find $j' \geq i$, with $\frac{j'}{\psi_{h,T}(j')} < \frac{1}{6Mh}$ and such that if $j' < n < \psi_{h,T}^{m+2}(j')$, then

$$\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 < \frac{1}{6Mh}. \tag{2}$$

Set $j = \psi_{h,T}(j')$. Then, clearly $j \geq i$ and $\frac{j}{\psi_{h,T}(j)} = \frac{\psi_{h,T}(j')}{\psi_{h,T}(\psi_{h,T}(j'))} < \frac{1}{\psi_{h,T}(j')} < \frac{j'}{\psi_{h,T}(j')} < \frac{1}{h}$. For all $j < r < \psi_{h,T}^m(j)$, that is $\psi_{h,T}(j') < r < \psi_{h,T}^{m+1}(j')$, we have

$$\begin{aligned}
\left\|\mathbf{\Pi}^T(x,r) - \mathbf{q}\right\|_1 &\leq \left\|\sum_{n\geq 1} c_{r,n}\left(\mathbf{\Pi}(x,n) - \mathbf{q}\right)\right\|_1 \\
&\quad + \left\|\mathbf{q}(1 - \sum_{n\geq 1} c_{r,n})\right\|_1 \\
&\leq \sum_{n\geq 1} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 + \frac{1}{2h}
\end{aligned}$$

since $r > \psi_{h,T}(j') > K$ and $\|\mathbf{q}\|_1 = 1$. Now we proceed by splitting our summation into three parts which we bound separately.

$$\begin{aligned}
\sum_{n\geq 1} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 &= \sum_{n=1}^{j'} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 \\
&\quad + \sum_{n=j'+1}^{\psi_{h,T}^{m+2}(j')-1} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 \\
&\quad + \sum_{n\geq \psi_{h,T}^{m+2}(j')} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 .
\end{aligned}$$

Firstly, using the fact that $r > \psi_{h,T}(j')$ we get that $r > \max\{N_1, \ldots, N_{j'}\}$ and hence

$$\sum_{n\leq j'} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 \leq 2\sum_{n\leq j'} |c_{r,n}| \leq \sum_{n\leq j'} \frac{1}{2^{n+9}h} \leq \frac{1}{6h}.$$

Then using (2) we can bound the second summand by

$$\sum_{n=j'+1}^{\psi_{h,T}^{m+2}(j')-1} |c_{r,n}|\,\|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 \leq \frac{1}{6Mh}\sum_{n\geq 1} |c_{r,n}| \leq \frac{1}{6h}.$$

Finally, since $\psi_{h,T}^{m+2}(j') = \psi_{h,T}\big(\psi_{h,T}^{m+1}(j')\big) > \psi_{h,T}(r) > M_r$ we have that

$$\sum_{n \geq \psi_{h,T}^{m+2}(j')} |c_{r,n}| \, \|\mathbf{\Pi}(x,n) - \mathbf{q}\|_1 \leq 2 \sum_{n \geq M_r} |c_{r,n}| \leq \frac{1}{6h}.$$

Putting all these together we get that for all $j < r < \psi_{h,T}^{m}(j)$

$$\big\|\mathbf{\Pi}^T(x,r) - \mathbf{q}\big\|_1 \leq \frac{1}{h}.$$

The choices of the positive integers i and m, and of the vector $\mathbf{q} \in \mathbb{D}$, were arbitrary hence the sequence $\big(\mathbf{\Pi}^T(x,m)\big)_m$ has property $P_{h,\psi_{h,T}}$. This completes the proof of Lemma 2.5. $\square$

Lemma 2.7. *Let $T = [c_{m,n}]$ be a regular linear transformation and recall the set $A_T := \bigcap_{h \in \mathbb{N}} A_{6Mh,\psi_{h,T}}$. Then*

$$A_T \subset R_T = \big\{x \in [0,1] : \big(\mathbf{\Pi}^T(x,m)\big)_m \text{ accumulates at every point of } \Delta_N\big\}.$$

Proof. Let $x \in A_T$. To prove that $x \in R_T$, it suffices to show that each $\mathbf{p} \in \Delta_N$ is an accumulation point of $\big(\mathbf{\Pi}^T(x,m)\big)_m$. Therefore, we start by fixing $\mathbf{p} \in \Delta_N$ and $k \in \mathbb{N}$ and choose a vector $\mathbf{q} \in \mathbb{D}$ such that $\|\mathbf{p} - \mathbf{q}\|_1 < \frac{1}{k}$. We claim that we can find $n_k > k$ such that

$$\big\|\mathbf{q} - \mathbf{\Pi}^T(x,n_k)\big\|_1 \leq \frac{1}{k}. \tag{3}$$

We now prove the claim above. Indeed, for any $x \in A_T$, by using Lemma 2.5 we have that $\big(\mathbf{\Pi}^T(x,m)\big)_m$ has property $P_{k,\psi_{k,T}}$. In particular, we can find $j \in \mathbb{N}$ with $j \geq k$ and such that if $j < n < \psi_{k,T}(j)$, then $\|\mathbf{q} - \mathbf{\Pi}^T(x,n)\|_1 \leq \frac{1}{k}$. Hence if n_k is any integer with $j < n_k < \psi_{k,T}(j)$, then

$$\|\mathbf{q} - \mathbf{\Pi}^T(x,n_k)\|_1 \leq \frac{1}{k},$$

and in particular $n_k > j > k$ as required. Then we get that

$$\big\|\mathbf{p} - \mathbf{\Pi}^T(x,n_k)\big\|_1 \leq \|\mathbf{p} - \mathbf{q}\|_1 + \big\|\mathbf{q} - \mathbf{\Pi}^T(x,n_k)\big\|_1 \leq \frac{2}{k}. \tag{4}$$

Since $n_k > k$, we can extract an increasing subsequence $(n_{k_u})_u$ of $(n_k)_k$. It now follows from (4) that $\mathbf{\Pi}^T(x,n_{k_u}) \to \mathbf{p}$ as $u \to \infty$ with respect to $\|.\|_1$. Hence $\mathbf{p}$ is an accumulation point of $\big(\mathbf{\Pi}^T(x,m)\big)_m$. This completes the proof of Lemma 2.7. $\square$

We can now prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 2.2 it follows that for each $h \in \mathbb{N}$ the set $A_{6Mh,\psi_{h,T}}$ is residual in $\mathbb{I}$. Since $A_T = \bigcap_{h\in\mathbb{N}} A_{6Mh,\psi_{h,T}}$ it follows that the set A_T is residual in $\mathbb{I}$ as a countable intersection of residual sets. Then, by Lemma 2.7 since for each regular linear transformation T, the set A_T is a subset of R_T we get that R_T is residual in $\mathbb{I}$. Finally, it can be easily seen that $[0,1] \setminus \mathbb{I}$ is a countable union of nowhere dense sets and so R_T is residual in $[0,1]$. $\square$

Proof of Corollary 1.4. Assume for the sake of contradiction that the set R_T does not have full packing dimension, that is $\dim_P R_T < 1$. Recall that for $F \subseteq \mathbb{R}$ if we let $\overline{\dim}_B(F)$ denote the upper box dimension of F, then we have that

$$\dim_P(F) = \inf \left\{ \sup_{n\in\mathbb{N}} \overline{\dim}_B(E_n) \ : \ E_1, E_2, \dots \subseteq \mathbb{R} \quad \text{and} \quad F \subseteq \bigcup_{n\in\mathbb{N}} E_n \right\}.$$

Then by assumption we can find a countable family of subsets $E_1, E_2, \dots$ of $\mathbb{R}$ with $R_T \subseteq \bigcup_n E_n$ such that $\sup_n \overline{\dim}_B(E_n) < 1$. Since $R_T \subseteq [0,1]$ and $R_T \subseteq \bigcup_n E_n$ we get that

$$R_T \subseteq \bigcup_n \big([0,1] \cap E_n\big).$$

To arrive at the desired contradiction we show that for all $n \in \mathbb{N}$ the set $[0,1]\cap E_n$ is nowhere dense in $[0,1]$ contradicting Theorem 1.3. Assume this is not the case. Then for some $n \in \mathbb{N}$ and some open set $U \subseteq \mathbb{R}$ we have that

$$[0,1] \cap U \neq \emptyset \quad \text{and moreover} \quad [0,1] \cap U \cap E_n \text{ is dense in } [0,1] \cap U.$$

In particular, we deduce that

$$[0,1] \cap U \subseteq \overline{[0,1] \cap U \cap E_n}$$

and therefore

$$\begin{aligned} \overline{\dim}_B(E_n) &= \overline{\dim}_B(\overline{E_n}) \\ &\geq \overline{\dim}_B(\overline{[0,1] \cap U \cap E_n}) \\ &= \overline{\dim}_B(\overline{[0,1] \cap U}) \\ &\geq \dim_P([0,1] \cap U) = 1, \end{aligned}$$

where we have used that $\dim_P([0,1] \cap U) = 1$ for all open sets U that intersect $[0,1]$. This is a contradiction since by assumption $\sup_n \overline{\dim}_B(E_n) < 1$. This completes the proof. $\square$

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