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MAHLER'S CONJECTURE ON $\xi(3/2)^n \bmod 1$

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ABSTRACT. K. Mahler's conjecture: There exists no $\xi \in \mathbb{R}^+$ such that the fractional parts $\{\xi(3/2)^n\}$ satisfy $0 \le \{\xi(3/2)^n\} < 1/2$ for all $n = 0, 1, 2, \dots$ Such a ξ, if exists, is called a Mahler's Z-number. In this paper we prove that if ξ is a Z-number, then the sequence $x_n = \{\xi(3/2)^n\}$, $n = 1, 2, \dots$ has asymptotic distribution function $c_0(x)$, where $c_0(x) = 1$ for $x \in (0, 1]$.

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1. Introduction

Let $\{x\} = x \bmod 1$ be the fractional part of x and let x_n, $n = 1, 2, \dots$ be a sequence from the unit interval $[0, 1)$. We use following notions:

- $\#\{n \le N; x_n \in [0, x)\}$ is a number $n \le N$ for which $x_n \in [0, x)$.
- $F_N(x) = \#\{n \le N; x_n \in [0, x)\}/N$ is the step distribution function of the finite sequence $x_1, \dots, x_N$ in $[0, 1)$, while $F_N(1) = 1$.
- The function $g : [0, 1] \to [0, 1]$ is a distribution function (abbreviating d.f.) of the sequence x_n, $n = 1, 2, \dots$ if an increasing sequence of positive integers $N_1, N_2, \dots$ exists such that $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ a.e. on $[0, 1]$.

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- For d.f. $g(x)$ and a Lebesgue measurable $f:[0,1]\to[0,1]$ define

$$g_f(x)=\int_{f^{-1}([0,x))} 1\,\mathrm{d}g(x). \tag{1}$$

- D.f. is called an asymptotic d.f. (abbreviating a.d.f.) if

$$\lim_{N\to\infty} F_N(x)=g(x)$$

a.e. on $[0,1]$.

- D.f. $c_0(x)$ is defined as

$$c_0(x)=\begin{cases}0 & \text{for } x=0,\\ 1 & \text{for } x\in(0,1].\end{cases}$$

- D.f. $c_1(x)$ is defined as

$$c_1(x)=\begin{cases}0 & \text{for } x\in[0,1),\\ 1 & \text{for } x=1.\end{cases}$$

- In the following

$$\begin{aligned} x_n &= \xi(3/2)^n \bmod 1, \qquad n=1,2,\dots,\\ f(x) &= 2x \bmod 1,\\ h(x) &= 3x \bmod 1.\end{aligned}$$

In this case, for (1) and for every $x\in[0,1]$, we have

$$g_f(x)=g\big(f_1^{-1}(x)\big)+g\big(f_2^{-1}(x)\big)-g(1/2), \tag{2}$$

$$g_h(x)=g\big(h_1^{-1}(x)\big)+g\big(h_2^{-1}(x)\big)+g\big(h_3^{-1}(x)\big)-g(1/3)-g(2/3), \tag{3}$$

with inverse functions

$$f_1^{-1}(x)=x/2, \qquad f_2^{-1}(x)=(x+1)/2,$$

$$h_1^{-1}(x)=x/3, \qquad h_2^{-1}(x)=(x+1)/3, \qquad h_3^{-1}(x)=(x+2)/3.$$

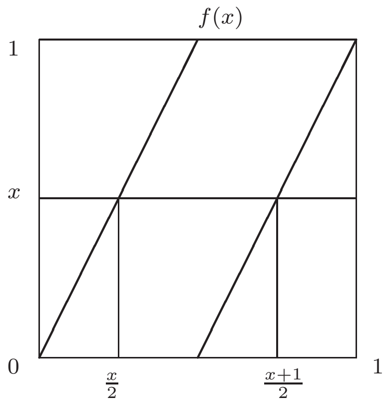
In this paper we prove that if $x_n\in[0,1/2)$ for $n=0,1,2,\dots$, then the sequence $x_n=\xi(3/2)^n \bmod 1$ has an a.d.f $g(x)=c_0(x)$, $x\in[0,1]$. K. Mahler [1] conjectured that such a ξ does not exist.

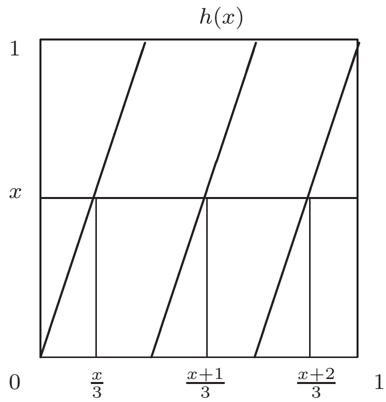
2. Older results

In [2] we have proved the following theorems:

Theorem 1. *Any d.f.* $g(x)$ *of* $\xi(3/2)^n \bmod 1$ *satisfies* $g_f(x) = g_h(x)$ *for all* $x \in [0,1]$. *Here* $g_f(x)$ *and* $g_h(x)$ *are defined by* (2) *and* (3), *respectively.*

For a proof see figures below





Theorem 2. *Denote*

$$F(x,y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|. \quad (4)$$

Then the d.f. $g(x)$ *satisfies* $g_f(x) = g_h(x)$ *on* $x \in [0,1]$ *if and only if*

$$\int_0^1 \int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 0. \quad (5)$$

Theorem 3. *Denote* $F(x,y)$ *by* (4). *Then*

$$2\int_0^1 \big(g_f(x) - g_h(x)\big)^2\,\mathrm{d}x = \int_0^1 \int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) \quad (6)$$

for any d.f. $g(x)$.

Theorem 4. *Denote* $f(x) = 2x \bmod 1$ *and* $h(x) = 3x \bmod 1$ *and let* g_1, g_2 *be any two d.f.s satisfying* $g_{if}(x) = g_{ih}(x)$ *for* $i = 1,2$ *and* $x \in [0,1]$. *Denote*

$$I_1 = [0,1/3], \qquad I_2 = [1/3,2/3], \qquad I_3 = [2/3,1].$$

Let $1 \le i \ne j \le 3$ *be fixed. Suppose that* $g_1(x) = g_2(x)$ *for any* $x \in I_i \cup I_j$. *Then* $g_1(x) = g_2(x)$ *for all* $x \in [0,1]$.

The following Theorem 5 can be used to construct a chain of solutions $g_f(x) = g_h(x)$ but not uniquely.

Theorem 5. *Let $g_1(x)$ be a given absolutely continuous d.f. satisfying $g_{1f}(x) = g_{1h}(x)$ for $x \in [0,1]$, where $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$. Then the absolutely continuous d.f. $g(x)$ satisfies $g_f(x) = g_h(x) = g_1(x)$ for $x \in [0,1]$ if and only if $g(x)$ has a form*

$$g(x) = \begin{cases} \Psi(x), & \text{for } x \in [0,1/6], \\ \Psi(1/6) + \Phi(x-1/6), & \text{for } x \in [1/6,2/6], \\ \Psi(1/6) + \Phi(1/6) + g_1(1/3) - \Psi(x-2/6) \\ +\Phi(x-2/6) - g_1(2x-1/3) + g_1(3x-1), & \text{for } x \in [2/6,3/6], \\ 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) \\ -\Psi(x-3/6) + g_1(2x-1), & \text{for } x \in [3/6,4/6], \\ -\Psi(1/6)+2\Phi(1/6)+g_1(1/3)-g_1(2/3)+g_1(1/2) \\ -\Phi(x-4/6) + g_1(2x-1), & \text{for } x \in [4/6,5/6], \\ -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x-5/6) \\ -\Phi(x-5/6) - g_1(2x-5/3) + g_1(3x-2), & \text{for } x \in [5/6,1], \end{cases}$$

where

$$\Psi(x) = \int_0^x \psi(t)\,\mathrm{d}t, \qquad \Phi(x) = \int_0^x \phi(t)\,\mathrm{d}t, \quad \text{for } x \in [0,1/6],$$

and $\psi(t)$, $\phi(t)$ are Lebesgue integrable functions on $[0,1/6]$ satisfying

$$0 \le \psi(t) \le 2g_1'(2t),$$

$$0 \le \phi(t) \le 2g_1'(2t+1/3),$$

$$2g_1'(2t) - 3g_1'(3t+1/2) \le \psi(t) - \phi(t) \le -2g_1'(2t+1/3) + 3g_1'(3t),$$

for almost all $t \in [0,1/6]$.

3. Examples of d.f. $g(x)$ with $g_f(x) = g_h(x)$

Let $g_1(x)=x$. The next solution of $g_f=g_h$ founded by Theorem 5 is: We put

$$0 \le \psi(t) \le 2,$$

$$0 \le \phi(t) \le 2,$$

$$-1 \le \psi(t) - \phi(t) \le 1,$$

for every $t \in [0,1/6]$ define $\psi(t) = \phi(t) = 0$.

The resulting d.f. is denoted as

$$g_3(x) := \begin{cases} 0 & \text{for } x \in [0, 2/6], \\ x - 1/3 & \text{for } x \in [2/6, 3/6], \\ 2x - 5/6 & \text{for } x \in [3/6, 5/6], \\ x & \text{for } x \in [5/6, 1]. \end{cases}$$

By taking $g_1(x) = g_3(x)$, this $g_3(x)$ can be used as a starting point for a further application of Theorem 5. We used:

$0 \le \psi(t) \le 0$ for $t \in [0, 1/6]$,

$0 \le \phi(t) \le 2$ for $t \in [0, 1/12]$,

$0 \le \phi(t) \le 4$ for $t \in [1/12, 1/6]$,

$-6 \le -\phi(t) \le -2$ for $t \in [0, 1/12]$,

$-6 \le -\phi(t) \le -4$ for $t \in [1/12, 1/9]$,

$-3 \le -\phi(t) \le -1$ for $t \in [1/9, 1/6]$.

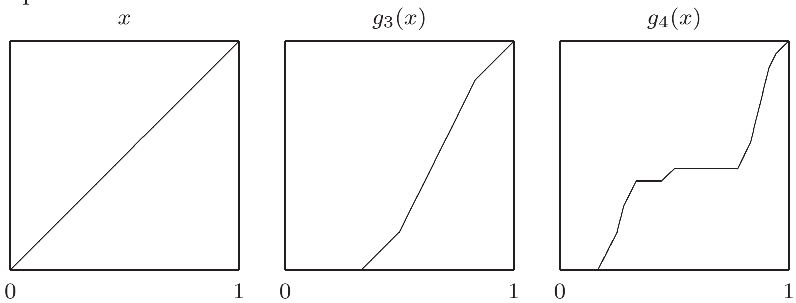
Putting $\Psi(x) = \int_0^x 0 \, \mathrm{d}t = 0$ and

$$\Phi(x) = \int_0^x \phi(t) \, \mathrm{d}t = \begin{cases} 2x & \text{for } x \in [0, 1/12], \\ 4x - 1/6 & \text{for } x \in [1/12, 1/9], \\ 2x + 1/18 & \text{for } x \in [1/9, 1/6] \end{cases}$$

and applying Theorem 5 we find resulting

$$g_4(x) := \begin{cases} 0 & \text{for } x \in [0, 1/6], \\ 2x - 1/3 & \text{for } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{for } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{for } x \in [5/18, 2/6], \\ 7/18 & \text{for } x \in [2/6, 8/18], \\ x - 1/18 & \text{for } x \in [8/18, 3/6], \\ 8/18 & \text{for } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{for } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{for } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{for } x \in [11/12, 17/18], \\ x & \text{for } x \in [17/18, 1]. \end{cases}$$

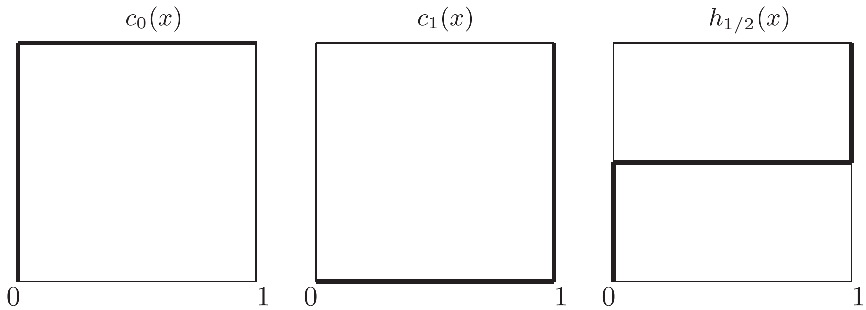
They graphs are



The interval I with maximal length of constat value of $g_4(x)$ is $I = [1/2, 7/9]$. Directly by computation:

$g_{4f}(x) = g_{4h}(x) = g_3(x)$,
$g_{3f}(x) = g_{3h}(x) = x$.

Trivial solutions of $g_f = g_h$ are:



4. New results

Theorem 6. *The d.f. $g(x)$ satisfies $g_f(x) = g_h(x)$ if and only if $g_1(x) := 1 - g(1-x)$ satisfies $g_{1f}(x) = g_{1h}(x)$.*

P r o o f. Using the transformation $x = 1 - y$ then we have:

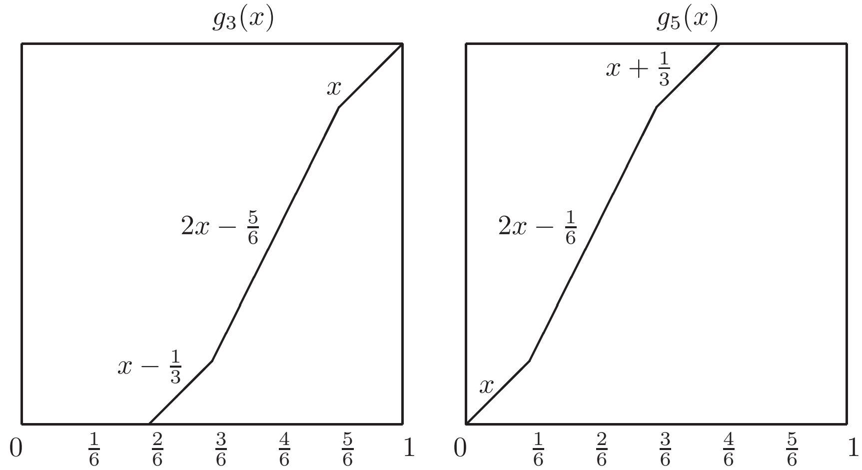
1) $1 - \frac{x}{2} = \frac{y+1}{2}$,
2) $1 - \frac{x+1}{2} = \frac{y}{2}$,
3) $1 - \frac{x}{3} = \frac{y+2}{3}$,
4) $1 - \frac{x+1}{3} = \frac{y+1}{3}$,
5) $1 - \frac{x+2}{3} = \frac{y}{3}$.

Then

$$-\big(g_f(y) - g_h(y)\big) = g_{1f}(x) - g_{1h}(x) \tag{7}$$

for every $x = 1 - y \in [0,1]$.[1] □

Example 1. In the following figures we define $g_5(x)$ as the second graph and we see that $g_5(x) = 1 - g_3(1-x)$. The solution of $g_{3f} = g_{3h}$ for $g_3(x)$ is given in Section 3.



Theorem 7. *Assume that $g_1(x)$ satisfies $g_f = g_h$. Then $g_2(x)$,*

$$g_2(x) = g_{1f}(x) = g_1(x/2) + g_1\big((x+1)/2\big) - g_1(1/2),$$

or

$$g_2(x) = g_{1h}(x) = g_1(x/3) + g_1\big((x+1)/3\big) + g_1\big((x+2)/3\big) - g_1(1/3) - g_1(2/3)$$

also satisfies $g_f = g_h$.

Proof. Assume $g_{1f}(x) = g_{1h}(x) = g_2(x)$. Then $g_{2f} = g_{1hf}$ and $g_{2h} = g_{1fh}$. But the functions f and h commute that is $fh = hf$. □

Example 2. If we start in Theorem 7 with $g_1(x) = g_4(x)$ then we find resulting $g_2(x) = g_3(x)$. Putting $g_3(x)$ in our scheme as $g_1(x)$, then the resulting $g_2(x) = x$.

Also note that $g_{1f^2}(x) = g_{1h^2}(x)$ for $x \in [0,1]$.

[1] Distribution functions (abbreviating d.f.s) $g(x), g_f(x), g_h(x)$ are defined in Section 1.

5. An explicit formula for $F(x,y)=|\{2x\}-\{3y\}|+|\{2y\}-\{3x\}|-|\{2x\}-\{2y\}|-|\{3x\}-\{3y\}|$

We give explicit formulas for:

$$|\{2x\}-\{3y\}|, \qquad |\{2y\}-\{3x\}|, \qquad |\{2x\}-\{2y\}|, \qquad |\{3x\}-\{3y\}|.$$

Put the lines

$$x=0, \quad x=\frac{1}{3}, \quad x=\frac{1}{2}, \quad x=\frac{2}{3}, \quad x=1,$$

and

$$y=0, \quad y=\frac{1}{3}, \quad y=\frac{1}{2}, \quad y=\frac{2}{3}, \quad y=1. \tag{8}$$

Then we start with

$$|\{2x\}-\{3y\}| = |2x-[2x]-(3y-[3y])| = |(2x-3y+[3y]-[2x])|$$

$$= \begin{cases} 2x-3y+[3y]-[2x] & \text{if} \quad 2x-3y+[3y]-[2x] \geq 0, \\ -(2x-3y+[3y]-[2x]) & \text{if} \quad -(2x-3y+[3y]-[2x]) \geq 0. \end{cases} \tag{9}$$

Thus from integer parts $[3y]-[2x]$ in Fig. 1 we have the fractional part $|\{2x\}-\{3y\}|$ in Fig. 2 without straight lines (8).

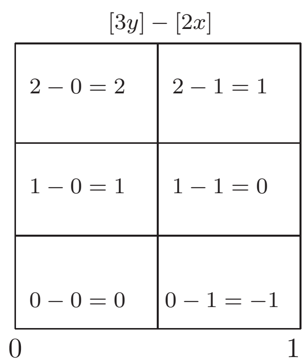


Figure 1.

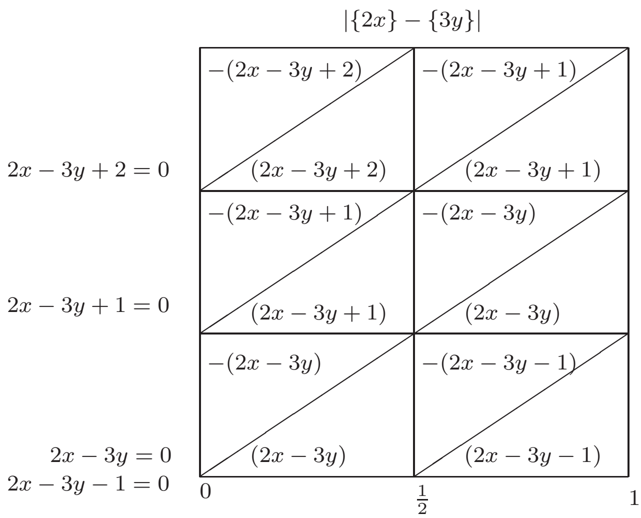


Figure 2.

Then we give explicit formulas in the following figures:

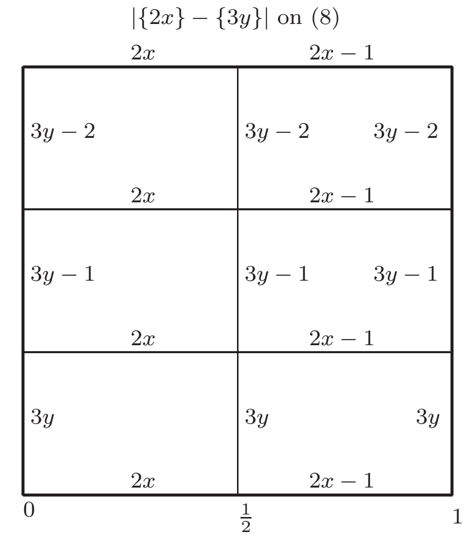


Figure 3.

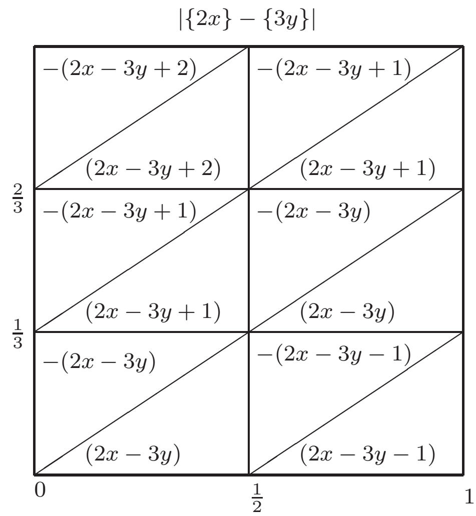


Figure 4.

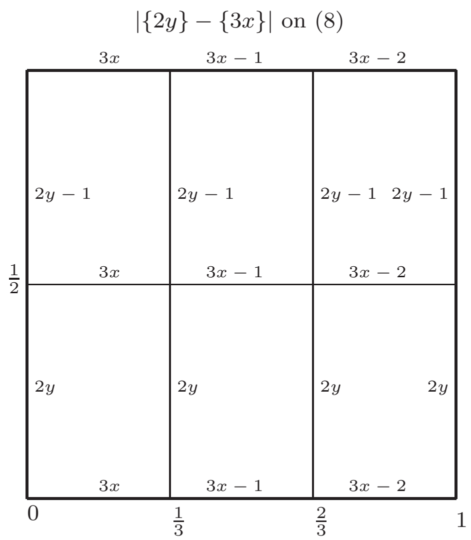


Figure 5.

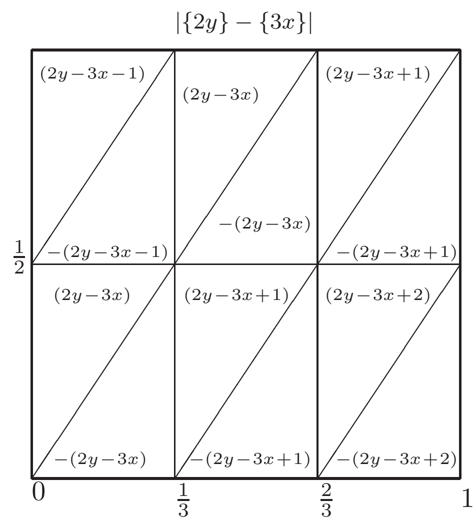


Figure 6.

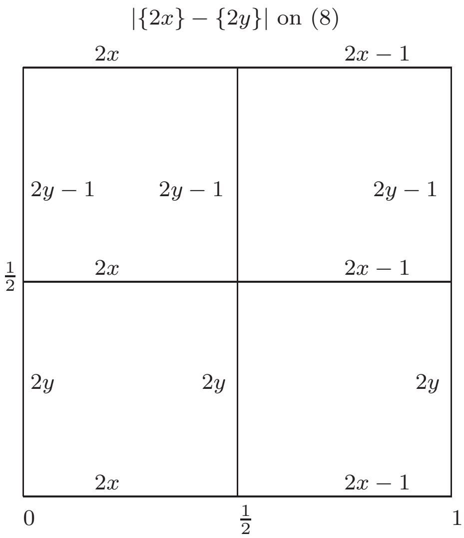


Figure 7.

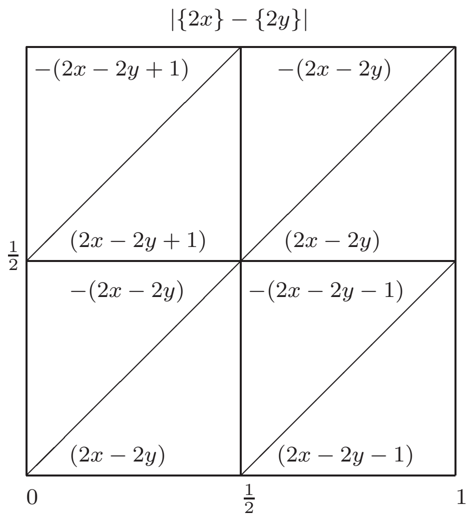


Figure 8.

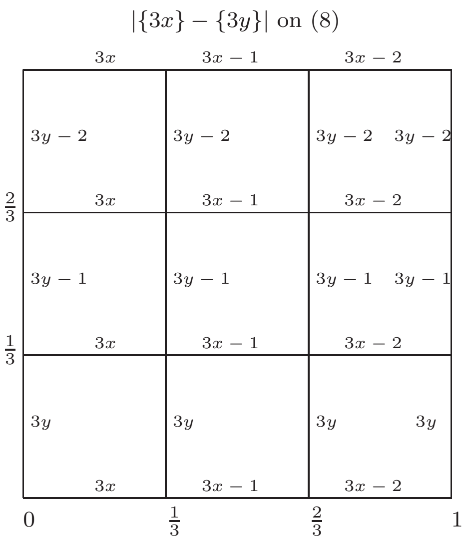


Figure 9.

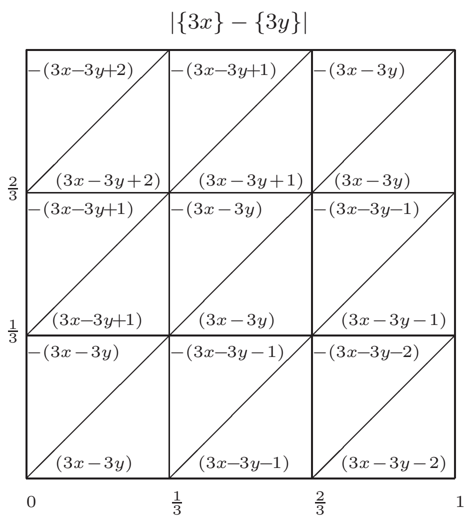


Figure 10.

6. Integral $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)$ where $F(x,y)=|\{2x\}-\{3y\}|+|\{2y\}-\{3x\}|-|\{2x\}-\{2y\}|-|\{3x\}-\{3y\}|$

Using W. G. Young [3] we prove

Theorem 8. *For every d.f. $g(x)$ we have*

$$
\begin{aligned}
2\int_0^1 \big(g_f(x)-g_h(x)\big)^2\,\mathrm{d}x &= \int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= -4g(1/2)g(1/3)-4g(1/2)g(2/3)+2g(1/2)g(1/2)+4g(1/3)g(2/3)\\
&\quad +2g(2/3)g(2/3)+2g(1/3)g(1/3) \qquad (10)\\
&\quad -8\int_0^{1/2} g(x)g\big((2/3)x+2/3\big)\,\mathrm{d}x-8\int_0^1 g(x)g\big((2/3)x+1/3\big)\,\mathrm{d}x\\
&\quad -8\int_0^1 g(x)g\big((2/3)x\big)\,\mathrm{d}x-8\int_{1/2}^1 g(x)g\big((2/3)x-1/3\big)\,\mathrm{d}x\\
&\quad +8\int_0^{1/2} g(x)g(x+1/2)\,\mathrm{d}x+10\int_0^1 g(x)g(x)\,\mathrm{d}x\\
&\quad +12\int_0^{1/3} g(x)g(x+2/3)\,\mathrm{d}x+12\int_0^{2/3} g(x)g(x+1/3)\,\mathrm{d}x \qquad (11)\\
&\quad +\int_0^1 g(x)\,\mathrm{d}x\big(4g(1/2)-4g(1/3)-4g(2/3)\big). \qquad (12)
\end{aligned}
$$

P r o o f. We start by W.G. Young's integral. Let $F(x,y)$ be an arbitrary continuous function on the interval $[x_1,x_2]\times[y_1,y_2]\subset[0,1]^2$ and $g(x)$ be a d.f. Two-times integration by parts given by W.G. Young [3]

$$
\begin{aligned}
\int_{x_1}^{x_2}\int_{y_1}^{y_2} F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) &= g(x_2)g(y_2)F(x_2,y_2)-g(x_1)g(y_2)F(x_1,y_2)\\
&\quad -g(x_2)g(y_1)F(x_2,y_1)+g(x_1)g(y_1)F(x_1,y_1) \qquad (13)\\
&\quad -\int_{x_1}^{x_2} g(x)g(y_2)\,\mathrm{d}F(x,y_2)+\int_{x_1}^{x_2} g(x)g(y_1)\,\mathrm{d}F(x,y_1)\\
&\quad -\int_{y_1}^{y_2} g(x_2)g(y)\,\mathrm{d}F(x_2,y)+\int_{y_1}^{y_2} g(x_1)g(y)\,\mathrm{d}F(x_1,y) \qquad (14)\\
&\quad +\int_{x_1}^{x_2}\int_{y_1}^{y_2} g(x)g(y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y), \qquad (15)
\end{aligned}
$$

where

$$\mathrm{d}_x\,\mathrm{d}_y F(x,y)=F(x,y)+F(x+\mathrm{d}x,y+\mathrm{d}y)-F(x+\mathrm{d}x,y)-F(x,y+\mathrm{d}y). \qquad (16)$$

For example, we shall apply this to continuous $F(x,y) = |\{2x\} - \{2y\}|$ on the sub-square (II) in the following Fig. 11.

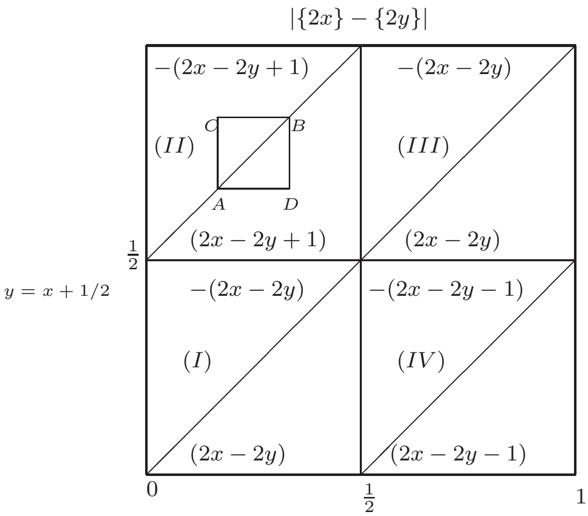


FIGURE 11.

In this case

$$F(x,y) = \begin{cases} -(2x-2y+1) & \text{if } -(2x-2y+1) \geq 0, \\ (2x-2y+1) & \text{if } (2x-2y+1) \geq 0 \end{cases} \tag{17}$$

$$(x_1,y_1) = (0,1/2), \quad (x_2,y_2) = (1/2,1), \quad F(x_1,y_1) = 0, \quad F(x_2,y_2) = 0$$

since (x_1,y_1) and (x_2,y_2) lies on the diagonal of sub-square (II).

Put

$A = (x,y)$, $\quad B = (x+\mathrm{d}x, y+\mathrm{d}y)$, $\quad C = (x, y+\mathrm{d}y)$, $\quad D = (x+\mathrm{d}x, y)$,
where $y = x + 1/2$.

By definition of differential $\mathrm{d}_x\,\mathrm{d}_y F(x,y)$ we have

$$\begin{aligned} \mathrm{d}_x\,\mathrm{d}_y F(x,y) &= F(x,y) + F(x+\mathrm{d}x, y+\mathrm{d}y) - F(x, y+\mathrm{d}y) - F(x+\mathrm{d}x, y) \\ &= (2x-2y+1)(=0) + \big(2(x+\mathrm{d}x) - 2(y+\mathrm{d}y)+1\big)(=0) \\ &\quad - \big(-(2x - 2(y+\mathrm{d}y)+1)\big) - \big(2(x+\mathrm{d}x) - 2y + 1\big) \\ &= -2\,\mathrm{d}y - 2\,\mathrm{d}x = -4\,\mathrm{d}x, \end{aligned} \tag{18}$$

where $\mathrm{d}y = \mathrm{d}x$, since the points A and B lies on the diagonal of sub-square (II).

Now applying (13), (14), (15) to $F(x,y)$ in (17) we find

$$\begin{aligned}
&\int_0^{1/2}\int_{1/2}^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= g(1/2)g(1)F(1/2,1)(=0) - g(0)g(1)F(0,1)(=0)\\
&\quad - g(1/2)g(1/2)\big(F(1/2,1/2)=1\big) + g(0)g(1/2)F(0,1/2)(=0) \qquad (19)\\
&\quad - \int_0^{1/2} g(x)g(1)\,\mathrm{d}F(x,1)\left(=\int_0^{1/2} g(x)g(1)2\,\mathrm{d}x\right)\\
&\quad + \int_0^{1/2} g(x)g(1/2)\,\mathrm{d}F(x,1/2)\left(=\int_0^{1/2} g(x)g(1/2)2\,\mathrm{d}x\right)\\
&\quad - \int_{1/2}^1 g(1/2)g(y)\,\mathrm{d}F(1/2,y)\left(=\int_{1/2}^1 g(1/2)g(y)2\,\mathrm{d}y\right)\\
&\quad + \int_{1/2}^1 g(0)g(y)\,\mathrm{d}F(0,y)(=0) \qquad (20)\\
&\quad + \int_0^{1/2}\int_{1/2}^1 g(x)g(y)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y)\left(=\int_0^{1/2} g(x)g(x+1/2)(-4)\,\mathrm{d}x\right). \qquad (21)
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^{1/2}\int_{1/2}^1 |\{2x\}-\{2y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= -g^2(1/2) + 2\int_0^{1/2} g(x)\,\mathrm{d}x + 2g(1/2)\int_0^1 g(x)\,\mathrm{d}x - 4\int_0^{1/2} g(x)g(x+1/2)\,\mathrm{d}x. \qquad (22)
\end{aligned}$$

$$\int_0^{1/2}\int_0^{1/2} |\{2x\}-\{2y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 4g(1/2)\int_0^{1/2} g(x)\,\mathrm{d}x - 4\int_0^{1/2} g^2(x)\,\mathrm{d}x. \qquad (23)$$

$$\begin{aligned}
&\int_{1/2}^1\int_{1/2}^1 |\{2x\}-\{2y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= -2g(1/2) + 4g(1/2)\int_{1/2}^1 g(x)\,\mathrm{d}x + 4\int_{1/2}^1 g(x)\,\mathrm{d}x - 4\int_{1/2}^1 g^2(x)\,\mathrm{d}x. \qquad (24)
\end{aligned}$$

$$\begin{aligned}
&\int_{1/2}^1\int_0^{1/2} |\{2x\}-\{2y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= -g^2(1/2) + 2g(1/2)\int_0^1 g(x)\,\mathrm{d}x + 2\int_0^{1/2} g(x)\,\mathrm{d}x - 4\int_{1/2}^1 g(x)g(x-1/2)\,\mathrm{d}x. \qquad (25)
\end{aligned}$$

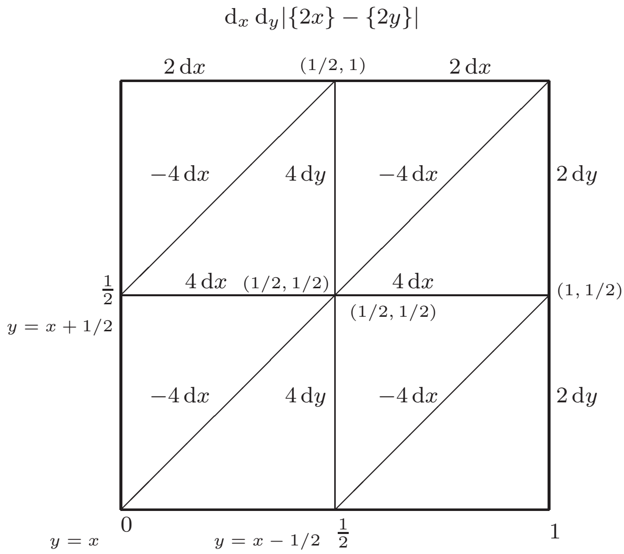


FIGURE 12.

By Fig. 12 or by (22), (23), (24), (25) and for an arbitrary d.f. $g(x)$ we have

$$\int_0^1\int_0^1 |\{2x\}-\{2y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y)$$

$$= -g(1/2)g(1) - g(1)g(1/2) - g(1/2)g(1/2) - g(1/2)g(1/2) \tag{26}$$

$$+\int_0^{1/2} g(x)g(x+1/2)(-4)\,\mathrm{d}x + \int_0^1 g(x)g(x)(-4)\,\mathrm{d}x + \int_{1/2}^1 g(x)g(x-1/2)(-4)\,\mathrm{d}x \tag{27}$$

$$+\int_0^1 g(1/2)g(y)4\,\mathrm{d}y + \int_0^1 g(1)g(y)2\,\mathrm{d}y + \int_0^1 g(x)g(1/2)4\,\mathrm{d}x + \int_0^1 g(x)g(1)2\,\mathrm{d}x. \tag{28}$$

Here $g(1)=1$, (26) are jumps, (27) are integrals over diagonals and (28) integrals over orthogonal lines.

In the following we continue in the proof of Theorem 8 by applying Young's integrals (13), (14), (15) to $F(x,y)$ which is defined on parts of $|\{3x\}-\{3y\}|$, $|\{2x\}-\{3y\}|$, $|\{2y\}-\{3x\}|$, respectively. These parts divide square $[0,1]^2$ on subintervals with continuous $F(x,y)$ considering the Figures 13, 14, and 15. Then using d.f. $g(x)$ in Young's integral can be arbitrary (also discontinuous).

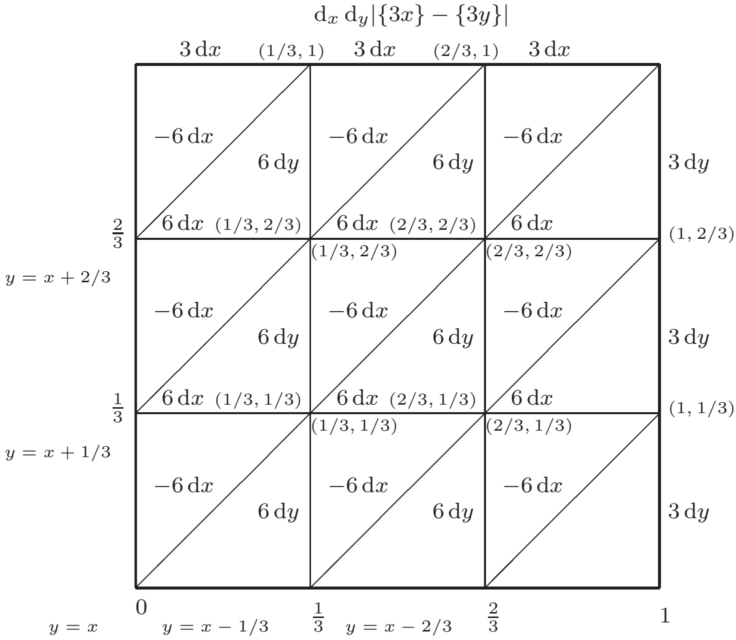


Figure 13.

By Fig. 13 for an arbitrary d.f. $g(x)$ we have

$$
\begin{aligned}
&\int_0^1\int_0^1 |\{3x\}-\{3y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= -g(1/3)g(1)-g(2/3)g(1)-g(1)g(1/3)-g(1)g(2/3)-2\cdot g(1/3)g(2/3)\\
&\quad -2g(2/3)g(2/3)-2\cdot g(1/3)g(1/3)-2g(2/3)g(1/3) \qquad (29)\\
&\quad +\int_0^{1/3} g(x)g(x+2/3)(-6)\,\mathrm{d}x+\int_0^{2/3} g(x)g(x+1/3)(-6)\,\mathrm{d}x+\int_0^1 g(x)g(x)(-6)\,\mathrm{d}x\\
&\quad +\int_{1/3}^1 g(x)g(x-1/3)(-6)\,\mathrm{d}x+\int_{2/3}^1 g(x)g(x-2/3)(-6)\,\mathrm{d}x \qquad (30)\\
&\quad +\int_0^1 g(1/3)g(y)6\,\mathrm{d}y+\int_0^1 g(2/3)g(y)6\,\mathrm{d}y+\int_0^1 g(1)g(y)3\,\mathrm{d}y\\
&\quad +\int_0^1 g(x)g(1)3\,\mathrm{d}x+\int_0^1 g(x)g(2/3)6\,\mathrm{d}x+\int_0^1 g(x)g(1/3)6\,\mathrm{d}x. \qquad (31)
\end{aligned}
$$

Here (29) are jumps, (30) are integrals over diagonals and (31) integrals over orthogonal lines.

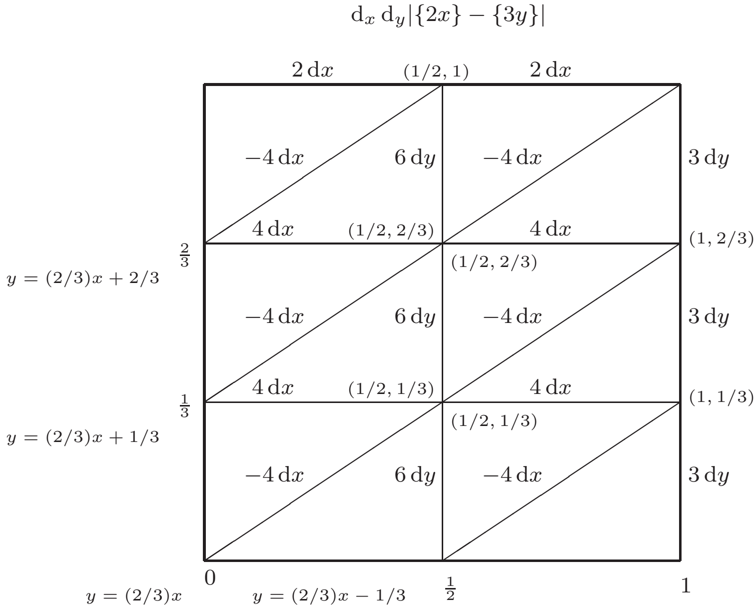


Figure 14.

By Fig. 14 for an arbitrary d.f. $g(x)$ we have

$$\begin{aligned}
&\int_0^1\int_0^1 |\{2x\}-\{3y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y)\\
&= -g(1/2)g(1) - g(1)g(1/3) - g(1)g(2/3) - 2g(1/2)g(1/3) - 2g(1/2)g(2/3) \qquad (32)\\
&\quad + \int_0^{1/2} g(x)g\big((2/3)x+2/3\big)(-4)\,\mathrm{d}x + \int_0^1 g(x)g\big((2/3)x+1/3\big)(-4)\,\mathrm{d}x\\
&\quad + \int_0^1 g(x)g\big((2/3)x\big)(-4)\,\mathrm{d}x + \int_{1/2}^1 g(x)g\big((2/3)x-1/3\big)(-4)\,\mathrm{d}x \qquad (33)\\
&\quad + \int_0^1 g(1/2)g(y)6\,\mathrm{d}y + \int_0^1 g(1)g(y)3\,\mathrm{d}y + \int_0^1 g(x)g(1)2\,\mathrm{d}x\\
&\quad + \int_0^1 g(x)g(1/3)4\,\mathrm{d}x + \int_0^1 g(x)g(2/3)4\,\mathrm{d}x. \qquad (34)
\end{aligned}$$

Here (32) are jumps, (33) are integrals over diagonals and (34) integrals over orthogonal lines.

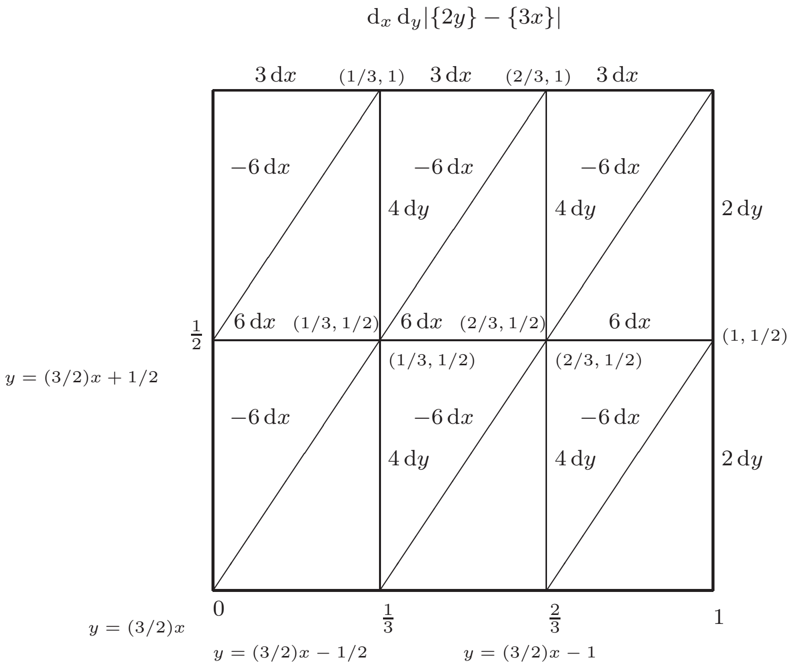


FIGURE 15.

By Fig. 15 for an arbitrary d.f. $g(x)$ we have

$$\int_0^1 \int_0^1 |\{2y\} - \{3x\}|\, dg(x)\, dg(y)$$

$$= -g(1/3)g(1) - g(2/3)g(1) - 2g(1/3)g(1/2) - 2g(2/3)g(1/2) - g(1)g(1/2) \quad (35)$$

$$+ \int_0^{1/3} g(x)g\big((3/2)x + 1/2\big)(-6)\, dx + \int_0^{2/3} g(x)g((3/2)x)(-6)\, dx$$

$$+ \int_{1/3}^1 g(x)g\big((3/2)x - 1/2\big)(-6)\, dx + \int_{2/3}^1 g(x)g\big((3/2)x - 1\big)(-6)\, dx \quad (36)$$

$$+ \int_0^1 g(1/3)g(y)4\, dy + \int_0^1 g(2/3)g(y)4\, dy + \int_0^1 g(1)g(y)2\, dy$$

$$+ \int_0^1 g(x)g(1)3\, dx + \int_0^1 g(x)g(1/2)6\, dx. \quad (37)$$

Here (35) are jumps, (36) are integrals over diagonals and (37) integrals over orthogonal lines.

Note 1. Since

$$\int_0^1\int_0^1 |\{2y\}-\{3x\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y) = \int_0^1\int_0^1 |x-y|\,\mathrm{d}g_f(x)\,\mathrm{d}g_h(y),$$

$$\int_0^1\int_0^1 |\{2x\}-\{3y\}|\,\mathrm{d}g(x)\,\mathrm{d}g(y) = \int_0^1\int_0^1 |x-y|\,\mathrm{d}g_h(x)\,\mathrm{d}g_f(y),$$

then we have (32), (33) and (34) equal to (35), (36) and (37), respectively. For (33)=(36) we give

$$\int_0^{1/2} g(x)g\big((2/3)x+2/3\big)(-4)\,\mathrm{d}x = \int_{2/3}^1 g(x)g\big((3/2)x-1\big)(-6)\,\mathrm{d}x,$$

$$\int_0^1 g(x)g\big((2/3)x+1/3\big)(-4)\,\mathrm{d}x = \int_{1/3}^1 g(x)g\big((3/2)x-1/2\big)(-6)\,\mathrm{d}x,$$

$$\int_0^1 g(x)g\big((2/3)x\big)(-4)\,\mathrm{d}x = \int_0^{2/3} g(x)g\big((3/2)x\big)(-6)\,\mathrm{d}x,$$

$$\int_{1/2}^1 g(x)g\big((2/3)x-1/3\big)(-4)\,\mathrm{d}x = \int_0^{1/3} g(x)g\big((3/2)x+1/2\big)(-6)\,\mathrm{d}x.$$

Summary.

- (10) = (32)+(35)-(26)-(29);
- (11) = (33)+(36)-(27)-(30);
- (12) = (34)+(37)-(28)-(31).

□

7. On d.f. $g(x)$, $g_f = g_h$, $g(x) = 1$ for $x \in [1/2, 1]$

Df. $g(x)$, $g_f(x) = g_h(x)$ and $g(x) = 1$, for $x \in [1/2, 1]$ we shall write as $g_1(x)$. The d.f.s. of $\xi(3/2)^n$ mod 1 for a Z-number ξ is of this type. We shall prove

Theorem 9. *Assume that d.f.* $g_1(x)$ *satisfies* $g_{1f}(x) = g_{1h}(x)$ *for* $x \in [0,1]$ *and* $g_1(x) = 1$ *for* $x \in [1/2, 1]$. *Then* $g_1(x) = c_0(x)$ *for all* $x \in [0,1]$.

By using Theorem 6 that $g(x)$ and $1-g(1-x)$ satisfies $g_f = g_h$ simultaneously, then Theorem 9 can be rewritten by

Theorem 10. *Assume that d.f.* $g_0(x)$ *satisfies* $g_{0f}(x) = g_{0h}(x)$ *for* $x \in [0,1]$ *and* $g_0(x) = 0$ *for* $x \in [0, 1/2]$. *Then* $g_0(x) = c_1(x)$ *for all* $x \in [0,1]$.

For the proof of Theorem 10 we prove

Theorem 11. *Assume that d.f.* $g_0(x)$ *satisfies* $g_0(x) = 0$ *for* $x \in [0, 1/2]$ *and assume that there exists* x_1 *such that* $1/2 < x_1 < 2/3$ *and* $g_0(x_1) > 0$. *Then to the d.f.* $g_0(x)$ *we can construct d.f.* $g_1(x)$ *such that*

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_0(x)\,\mathrm{d}g_0(y) - \int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_1(x)\,\mathrm{d}g_1(y) > 0. \tag{38}$$

Proof. Using Theorem 8 we have

$$\begin{aligned}
&\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_0(x)\,\mathrm{d}g_0(y)\\
&= 2g_0(2/3)g_0(2/3) + 10\int_{1/2}^1 g_0(x)g_0(x)\,\mathrm{d}x + 12\int_{1/2}^{2/3} g_0(x)g_0(x+1/3)\,\mathrm{d}x\\
&\quad - 8\int_{1/2}^1 g_0(x)g_0\big((2/3)x+1/3\big)\,\mathrm{d}x - 8\int_{1/2}^1 g_0(x)g_0\big((2/3)x\big)\,\mathrm{d}x\\
&\quad - 4g_0(2/3)\int_{1/2}^1 g_0(x)\,\mathrm{d}x.
\end{aligned} \tag{39}$$

We shall construct d.f. $g_1(x)$ such that $g_0(x) = g_1(x)$ without the interval $[x_1, x_1+\varepsilon]$, in which $g_1(x)$ has some tooth by following figure

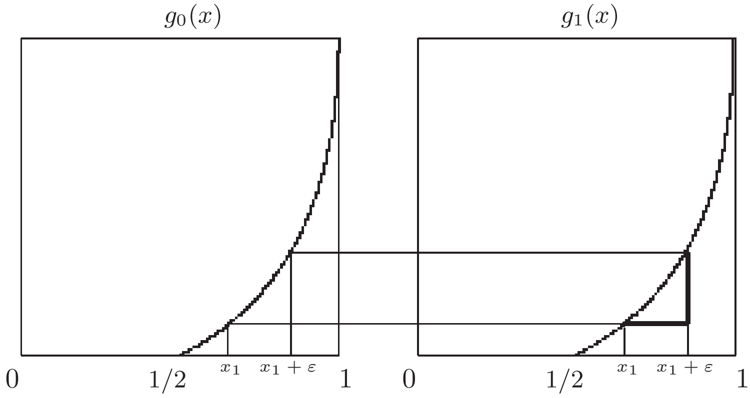


Figure 16.

where $\varepsilon > 0$ is sufficiently small. Increasing $g_0(x)$ implies differentiability $g_0(x)$ for almost all x and then without loss of generality we can also assumed that $g_0(x)$ has in x_1 derivative $0 < g_0'(x_1) < \infty$. [2]

[2]For other type of d.f. $g_0(x)$ as in figure 16, e.g. $g_0(x)$ contains some discontinuous, it can be used another construction of $g_1(x)$.

Similarly as in (39) we can computed $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_1(x)\,\mathrm{d}g_1(y)$ and then, step by step, we find differences with $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_0(x)\,\mathrm{d}g_0(y)$, for example,

$$\begin{aligned}&\int_{1/2}^1\big(g_0(x)-g_1(x)\big)\,\mathrm{d}x=(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2),\\&\int_{1/2}^1\big(g_0^2(x)-g_1^2(x)\big)\,\mathrm{d}x=2g_0(x_1)(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2).\end{aligned}\tag{40}$$

Using (39) we have

$$\begin{aligned}&10\int_{1/2}^1\big(g_0(x)g_0(x)-g_1(x)g_1(x)\big)\,\mathrm{d}x\\&=20g_0(x_1)(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2),\\&12\int_{1/2}^{2/3}\big(g_0(x)g_0(x+1/3)-g_1(x)g_1(x+1/3)\big)\,\mathrm{d}x\\&=12g_0(x_1+1/3)\cdot(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2),\\&\quad-8\int_{1/2}^1\Big(g_1(x)g_1\big((2/3)x+1/3\big)-g_0(x)g_0\big((2/3)x+1/3\big)\Big)\,\mathrm{d}x\\&=-8g_0\big((2/3)x_1+1/3\big)\cdot(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2),\\&\quad-8\int_{1/2}^1\Big(g_0(x)g_0\big((2/3)x\big)-g_1(x)g_1\big((2/3)x\big)\Big)\,\mathrm{d}x\\&=-8g_0((2/3)x_1)\cdot(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2),\\&\quad-4g_0(2/3)\int_{1/2}^1 g_0(x)\,\mathrm{d}x-4g_1(2/3)\int_0^1 g_1(x)\,\mathrm{d}x\\&=-4g_0(2/3)(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2).\end{aligned}\tag{41}$$

Here we assume that $x_1<2/3$ and then by our construction in Fig. 16 $g_0(2/3)=g_1(2/3)$.

Using (41) we find

$$\begin{aligned}&\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_0(x)\,\mathrm{d}g_0(y)-\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_1(x)\,\mathrm{d}g_1(y)\\&=20g_0(x_1)(1/2)g_0'(x_1)\varepsilon^2+12g_0(x_1+1/3)\cdot(1/2)g_0'(x_1)\varepsilon^2\\&\quad-8g_0\big((2/3)x_1+1/3\big)\cdot(1/2)g_0'(x_1)\varepsilon^2-8g_0\big((2/3)x_1\big)\cdot(1/2)g_0'(x_1)\varepsilon^2\\&\quad-4g_0(2/3)(1/2)g_0'(x_1)\varepsilon^2+o(\varepsilon^2).\end{aligned}\tag{42}$$

Since
$x_1 + 1/3 \geq (2/3)x_1 + 1/3$,
$2/3 \leq x_1 + 1/3$ since $x_1 \geq 1/2$ and $2/3 < 5/6$,
$(2/3)x_1 \leq x_1$,
then we have

$$12g_0(x_1 + 1/3) \geq 8g_1\big((2/3)x_1 + 1/3\big) + 4g_0(2/3)$$

and

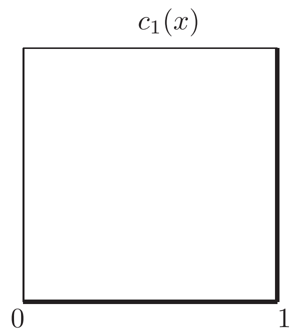
$$20g_0(x_1) \geq 8g_0\big((2/3)x_1\big)$$

and then

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_0(x)\,\mathrm{d}g_0(y) - \int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_1(x)\,\mathrm{d}g_1(y) \geq 12g_0(x_1)(1/2)g_0'(x_1)\varepsilon^2 + o(\varepsilon^2) > 0 \quad (43)$$

assuming as above that $g_0(x_1) > 0$ and $0 < g_0'(x_1) < \infty$. □

P r o o f o f T h e o r e m 10. By Theorem 2 d.f $g_0(x)$ satisfies $g_{0f} = g_{0h}$ if and only if $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_0(x)\,\mathrm{d}g_0(y) = 0$. In any case $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g_1(x)\,\mathrm{d}g_1(y) \geq 0$ (by Theorem 3). Then (43) is contradict to $g_0(x_1) > 0$ and we need $g_0(x) = 0$ for $x \leq 2/3$. But by Theorem 4 we have $g_0(x) = c_1(x) = 0$ for all $x \in [0,1)$. Here



□

8. Mahler's conjecture

The importance of the set of all d.f.s of x_n is reflected in the fact that most properties of a sequence x_n expressed in terms may be characterized using d.f.s. For example K. Mahler [1] conjectured: There exists no $\xi \in \mathbb{R}^+$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for $n = 0, 1, 2, \ldots$ Such a ξ, if exists, is called a Mahler's Z-number. Mahler did not proved the nonexistence of Z-numbers but he showed that there are at most $X^{0.7}$ Z-numbers in $[0, X]$. In this paper, Theorem 9, we have proved that if ξ is a Z-number then the sequence $\xi(3/2)^n \bmod 1$ has the a.d.f. $c_0(x)$. From it

Theorem 12. *If number ξ is a Z-number then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \{\xi(3/2)^n\} = 0. \tag{44}$$

P r o o f.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \{\xi(3/2)^n\} = \int_0^1 x \, \mathrm{d}c_0(x) [x c_0(x)]_0^1 - \int_0^1 c_0(x) \, \mathrm{d}x = 1-1 = 0. \qquad \square$$

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